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Research Paper

# DEDUCTIVE SYSTEMS OF GE-ALGEBRAS

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Abstract. A new sub-structure called (vivid) deductive system is introduced and their properties are examined. Conditions for a subset to be a deductive system are provided. The notion of upper GE-set is also introduced, and an example to show that any upper GE-set may not be a deductive system are supplied. Conditions for an upper GE-set to be a deductive system are provided. An upper GE-set is used to consider conditions for a subset to be a deductive system. The characterization of deductive system is established, and relationship between deductive system and vivid deductive system are created. Conditions for a deductive system to be a vivid deductive system are given, and the extension property for vivid deductive system is constructed.

## 1. Introduction

Following the introduction of Hilbert algebras by L. Henkin in early 50-ties and A. Diego[6], the algebra and related concepts were developed by D. Busneag [3, 4, 5]. Y. B. Jun gave

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characterizations of deductive systems in Hilbert algebras (see [7, 8]), introduced the notion of commutative Hilbert algebras and gave some characterizations of a commutative Hilbert algebra (see [8]). In mathematics, Hilbert algebras occur in the theory of von Neumann algebras in: Commutation theorem and Tomita-Takesaki theory, and it is an important tool for certain investigations in algebraic logic since they can be considered as fragments of any propositional logic containing a logical connective implication (→) and the constant 1 which is considered as the logical value "true". The study of generalization of one known algebraic structure is also an important research task. As a generalization of a Hilbert algebra, Bandaru, Borumand Saeid and Jun[**?**] introduced the notion of a GE-algebra, and investigated several properties. Different new substructures have been introduced in a GE-algebra such as voluntary GE-filters, belligerent GE-filters, imploring GE-filters and prominent GE-filters and studied their properties(see [1, 2, 9]).

In this manuscript, we introduce a new sub-structure called (vivid) deductive system and examine their properties. We provide conditions for a subset to be a deductive system. We also introduce the notion of upper GE-set, and give example to show that any upper GE-set may not be a deductive system. We provide conditions for an upper GE-set to be a deductive system. Using an upper GE-set, we consider conditions for a subset to be a deductive system. We establish characterization of deductive system. We discuss relationship between deductive system and vivid deductive system. We provide conditions for a deductive system to be a vivid deductive system. We build the extension property for vivid deductive system.

## 2. Preliminaries

**Definition 2.1** ([**?**]). By a *GE-algebra* we mean a non-empty set $X$ with a constant 1 and a binary operation "$*$" satisfying the following axioms:

(GE1) $u * u = 1$,

(GE2) $1 * u = u$,

(GE3) $u * (v * w) = u * (v * (u * w))$

for all $u, v, w \in X$.

In a GE-algebra $X$, a binary relation "$\leq$" is defined by

$$(\forall x, y \in X)\,(x \leq y \;\Leftrightarrow\; x * y = 1)\,. \tag{1}$$

**Definition 2.2** ([**?**, 1]). A GE-algebra $X$ is said to be

- *transitive* if it satisfies:

$$(\forall x, y, z \in X)\,(x * y \leq (z * x) * (z * y))\,. \tag{2}$$

- *left exchangeable* if it satisfies:

$$(\forall x, y, z \in X)\,(x * (y * z) = y * (x * z))\,. \tag{3}$$

- *belligerent* if it satisfies:

$$(\forall x, y, z \in X)\,(x * (y * z) = (x * y) * (x * z))\,. \tag{4}$$

**Proposition 2.3** ([?])**.** *Every GE-algebra $X$ satisfies the following items.*

$$(\forall u \in X)\,(u * 1 = 1)\,. \tag{5}$$

$$(\forall u, v \in X)\,(u * (u * v) = u * v)\,. \tag{6}$$

$$(\forall u, v \in X)\,(u \leq v * u)\,. \tag{7}$$

$$(\forall u, v, w \in X)\,(u * (v * w) \leq v * (u * w))\,. \tag{8}$$

$$(\forall u \in X)\,(1 \leq u \;\Rightarrow\; u = 1)\,. \tag{9}$$

$$(\forall u, v \in X)\,(u \leq (v * u) * u)\,. \tag{10}$$

$$(\forall u, v \in X)\,(u \leq (u * v) * v)\,. \tag{11}$$

$$(\forall u, v, w \in X)\,(u \leq v * w \;\Leftrightarrow\; v \leq u * w)\,. \tag{12}$$

*If $X$ is transitive, then*

$$(\forall u, v, w \in X)\,(u \leq v \;\Rightarrow\; w * u \leq w * v,\; v * w \leq u * w)\,. \tag{13}$$

$$(\forall u, v, w \in X)\,(u * v \leq (v * w) * (u * w))\,. \tag{14}$$

**Lemma 2.4** ([?])**.** *In a GE-algebra $X$, the following facts are equivalent each other.*

$$(\forall x, y, z \in X)\,(x * y \leq (z * x) * (z * y))\,. \tag{15}$$

$$(\forall x, y, z \in X)\,(x * y \leq (y * z) * (x * z))\,. \tag{16}$$

**Definition 2.5** ([?])**.** A subset $D$ of a GE-algebra $X$ is called a *GE-filter* of $X$ if it satisfies:

$$1 \in D, \tag{17}$$

$$(\forall x, y \in X)(x * y \in D,\; x \in D \;\Rightarrow\; y \in D). \tag{18}$$

## 3. Deductive systems

In what follows let $X$ denote a GE-algebra unless otherwise specified.

**Definition 3.1.** A nonempty subset $D$ of $X$ is called a *deductive system* of $X$ if it satisfies:

(D1) $X * D := \{x * a \mid x \in X, a \in D\} \subseteq D$.

(D2) $(\forall x, y, z \in X)\ (y, z \in D \;\Rightarrow\; (y * (z * x)) * x \in D)$.

**Example 3.2.** Let $X = \{0, 1, 2, 3, 4\}$ be a set with the Cayley table which is given in Table 1.

TABLE 1. Cayley table for the binary operation "$*$"

| $*$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 2 | 3 | 3 |
| 1 | 0 | 1 | 2 | 3 | 4 |
| 2 | 0 | 1 | 1 | 4 | 4 |
| 3 | 0 | 1 | 1 | 1 | 1 |
| 4 | 0 | 1 | 1 | 1 | 1 |

Then $(X, *, 1)$ is a GE-algebra, and it is routine to verify that $D := \{0, 1\}$ is a deductive system of $X$.

**Lemma 3.3.** *Every deductive system contains the constant* 1.

*Proof.* Let $D$ be a deductive system of $X$. For every $x \in D$, we have

$$1 = x * x \in D * D \subseteq X * D \subseteq D$$

by (GE1). This completes the proof. □

**Lemma 3.4.** *Every deductive system of $X$ satisfies:*

(19) $$(\forall x, y \in X)(y \in D \;\Rightarrow\; (y * x) * x \in D).$$

*Proof.* If we take $z = 1$ in (D2) and use (GE2), then $(y * x) * x = (y * (1 * x)) * x \in D$. □

**Corollary 3.5.** *Every deductive system of $X$ satisfies:*

(20) $$(\forall x, y \in X)(y \in D,\, y \leq x \;\Rightarrow\; x \in D).$$

*Proof.* Let $x, y \in X$ be such that $y \in D$ and $y \leq x$. Then $y * x = 1$, and so $x = 1 * x = (y * x) * x \in D$. □

We consider a subset $D$ of $X$ that satisfies:

(21) $$(\forall x, y, z \in X)(x * (y * z) \in D, y \in D \;\Rightarrow\; x * z \in D).$$

**Lemma 3.6.** *If a subset $D$ of $X$ satisfies two conditions* (17) *and* (21)*, then $D$ satisfies* (20).

*Proof.* Assume that a subset $D$ of $X$ satisfies two conditions (17) and (21). Let $x, y \in X$ be such that $y \in D$ and $y \leq x$. Then $1 * (y * x) = 1 * 1 = 1 \in D$, and so $x = 1 * x \in D$. Hence $D$ satisfies (20). □

**Theorem 3.7.** *Every deductive system $D$ of $X$ satisfies two conditions* (17) *and* (21)*.*

*Proof.* Let $D$ be a deductive system of a GE-algebra $X$. Then $D$ contains the constant 1 by Lemma 3.3. Let $x, y, z \in D$ be such that $x*(y*z) \in D$ and $y \in D$. Then $(x*(y*z))*(y*(x*z)) = 1$ by (8). It follows from (GE2) and (D2) that

$$x * z = 1 * (x * z) = ((x * (y * z)) * (y * (x * z))) * (x * z) \in D.$$

Hence (21) is valid. □

**Theorem 3.8.** *If a subset $D$ of $X$ satisfies two conditions* (17) *and* (21)*, then $D$ is a deductive system of $X$.*

*Proof.* Let $D$ be a subset of $X$ that satisfies (17) and (21). Let $y \in X * D$. Then $y = x * a$ for some $x \in X$ and $a \in D$. Then $x * (a * a) = x * 1 = 1 \in D$ by (GE1), (5) and (17). It follows from (21) that $y = x * a \in D$. Hence $X * D \subseteq D$. Let $x \in X$ and $y, z \in D$. Using (GE1), (GE2), (GE3), (5) and (17) and we have

$$1 * (y * ((y * (z * x)) * (z * x))) = y * ((y * (z * x)) * (z * x))$$
$$= y * ((y * (z * x)) * (y * (z * x))) = y * 1 = 1 \in D.$$

It follows from (GE2) and (21) that

$$(y * (z * x)) * (z * x) = 1 * ((y * (z * x)) * (z * x)) \in D.$$

Hence

$$1 * (z * ((y * (z * x)) * x)) = z * ((y * (z * x)) * x)$$
$$= z * ((y * (z * x)) * (z * x)) \in D$$

which implies from (GE2) and (21) that $(y * (z * x)) * x \in D$. Therefore $D$ is a deductive system of $X$. □

For any $a, b \in X$, we consider the set

$$X_a^b := \{x \in X \mid a \leq b * x\}, \tag{22}$$

which is called the *upper GE-set* of $a$ and $b$ in $X$.

**Example 3.9.** Let $X = \{0,1,2,3,4\}$ be a set with the Cayley table which is given in Table 2.

TABLE 2. Cayley table for the binary operation "$*$"

| $*$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 3 | 3 |
| 1 | 0 | 1 | 2 | 3 | 4 |
| 2 | 0 | 1 | 1 | 4 | 4 |
| 3 | 1 | 1 | 2 | 1 | 1 |
| 4 | 0 | 1 | 1 | 1 | 1 |

Then $(X, *, 1)$ is a GE-algebra and all upper GE-sets are calculated as follows.

$$X_0^0 = X_0^1 = X_0^2 = X_1^0 = X_2^0 = \{0,1,2\},$$

$$X_1^1 = \{1\},$$

$$X_1^2 = X_2^1 = X_2^2 = \{1,2\},$$

$$X_1^3 = X_3^1 = X_3^3 = \{0,1,3,4\},$$

$$X_1^4 = X_2^4 = X_4^1 = X_4^2 = X_4^4 = \{1,2,3,4\},$$

$$X_0^3 = X_0^4 = X_2^3 = X_3^0 = X_3^2 = X_3^4 = X_4^0 = X_4^3 = X.$$

**Proposition 3.10.** *In a GE-algebra $X$, we have*

(i) $(\forall a, b \in X)$ $(1, a, b \in X_a^b)$.

(ii) $(\forall a, b \in X)$ $(b \leq x$ for all $x \in X \ \Rightarrow\ X_a^b = X = X_b^a)$.

*Proof.* (i) is straightforward by (GE1), (5) and (7). Let $a, b \in X$ be such that $b \leq x$ for all $x \in X$. For any $z \in X$, we have $a * (b * z) = a * 1 = 1$, that is, $a \leq b * z$. Thus $z \in X_a^b = X_b^a$. Therefore (ii) is valid. □

The following example shows that the upper GE-set of $a$ and $b$ in $X$ is not a deductive system of $X$.

**Example 3.11.** In Example 3.9, we can observe that $X_1^3 = \{0,1,3,4\}$ and it is not a deductive system of $X$ since $3, 4 \in X_1^3$ but $(3 * (4 * 2)) * 2 = 2 \notin X_1^3$.

We provide conditions for the upper GE-set to be a deductive system.

**Theorem 3.12.** *In a belligerent GE-algebra $X$, the upper GE-set of $a$ and $b$ in $X$ is a deductive system of $X$.*

*Proof.* Assume that $X$ is a belligerent GE-algebra. Let $x \in X * X_a^b$. Then $x = y * z$ for some $y \in X$ and $z \in X_a^b$. Hence $a \leq b * z$, i.e., $a * (b * z) = 1$. It follows from (5) and (4) that

$$\begin{aligned} a * (b * (y * z)) &= a * ((b * y) * (b * z)) \\ &= (a * (b * y)) * (a * (b * z)) \\ &= (a * (b * y)) * 1 = 1. \end{aligned}$$

Hence $x = y * z \in X_a^b$, and thus $X * X_a^b \subseteq X_a^b$. Let $x \in X$ and $y, z \in X_a^b$. Then $a \leq b * y$ and $a \leq b * z$, i.e., $a * (b * y) = 1$ and $a * (b * z) = 1$. The combination of (GE1), (GE2) and (4) induces

$$\begin{aligned} & a * (b * ((y * (z * x)) * x)) = a * ((b * (y * (z * x))) * (b * x)) \\ &= (a * (b * (y * (z * x)))) * (a * (b * x)) \\ &= ((a * (b * y)) * (a * (b * (z * x)))) * (a * (b * x)) \\ &= (1 * (a * (b * (z * x)))) * (a * (b * x)) \\ &= (a * (b * (z * x))) * (a * (b * x)) \\ &= (((a * (b * z)) * (a * (b * x)))) * (a * (b * x)) \\ &= ((1 * (a * (b * x)))) * (a * (b * x)) \\ &= (a * (b * x)) * (a * (b * x)) = 1, \end{aligned}$$

that is, $a \leq b * ((y * (z * x)) * x)$. Hence $(y * (z * x)) * x \in X_a^b$. In conclusion, $X_a^b$ is a deductive system of $X$. □

**Theorem 3.13.** *Every deductive system $D$ of $X$ contains the upper GE-set $X_a^b$ for all $a, b \in D$.*

*Proof.* For every $a, b \in D$, let $x \in X_a^b$. Then $a \leq b * x$, i.e., $a * (b * x) = 1$. It follows from (GE2) and (D2) that $x = 1 * x = (a * (b * x)) * x \in D$. Hence $X_a^b \subseteq D$ for all $a, b \in D$. □

**Theorem 3.14.** *If a subset $D$ of $X$ satisfies:*

$$(\forall a, b \in D)(X_a^b \subseteq D), \tag{23}$$

*then $D$ is a deductive system of $X$.*

*Proof.* Let $D$ be a subset of a GE-algebra $X$ that satisfies the condition (23). Then $1 \in X_a^b \subseteq D$. Let $x, y, z \in X$ be such that $x * (y * z) \in D$ and $y \in D$. The condition (8) induces $(x * (y * z)) * (y * (x * z)) = 1$. Hence $x * z \in X_a^b \subseteq D$ for $a := x * (y * z)$ and $b := y$. It follows from Theorem 3.8 that $D$ is a deductive system of $X$. □

By the combination of Theorem 3.13 and Theorem 3.14, we have a characterization of a deductive system as follows.

**Theorem 3.15.** *A subset $D$ of $X$ is a deductive system of $X$ if and only if it satisfies* (23).

**Theorem 3.16.** *Every deductive system $D$ of $X$ is represented by the union of the upper GE-sets for all $a, b \in D$.*

*Proof.* Let $D$ be a deductive system of $X$. If $x \in D$, then clearly $x \in X_x^1$ and thus

$$D \subseteq \bigcup_{x \in D} X_x^1 \subseteq \bigcup_{a,b \in D} X_a^b.$$

If $y \in \bigcup_{a,b \in D} X_a^b$, then $y \in X_a^b$ for some $a, b \in D$ and so $y \in D$ by Theorem 3.13. This shows that $\bigcup_{a,b \in D} X_a^b \subseteq D$. Therefore $D = \bigcup_{a,b \in D} X_a^b$. □

**Corollary 3.17.** *If $D$ is a deductive system of $X$, then $D = \bigcup_{x \in D} X_x^1$.*

**Definition 3.18.** A nonempty subset $D$ of $X$ is called a *vivid deductive system* of $X$ if it satisfies (D1) and

$$(\forall x, y, z \in X)(x \in D,\, x * (y * z) \in D \;\Rightarrow\; ((z * y) * y) * z \in D). \tag{24}$$

**Example 3.19.** Let $X = \{0, 1, 2, 3, 4\}$ be a set with the Cayley table which is given in Table 3. Then $(X, *, 1)$ is a GE-algebra. It is routine to verify that $D := \{0, 1, 2\}$ is a vivid deductive system of $X$.

Table 3. Cayley table for the binary operation "$*$"

| $*$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 2 | 3 | 4 |
| 1 | 0 | 1 | 2 | 3 | 4 |
| 2 | 1 | 1 | 1 | 3 | 3 |
| 3 | 0 | 1 | 1 | 1 | 1 |
| 4 | 0 | 1 | 2 | 1 | 1 |

It is clear that $D := \{1\}$ is a deductive system of $X$, but it is not a vivid deductive system of $X$ as seen in the following example.

Table 4. Cayley table for the binary operation "$*$"

| $*$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 4 |
| 1 | 0 | 1 | 2 | 3 | 4 |
| 2 | 0 | 1 | 1 | 1 | 4 |
| 3 | 0 | 1 | 2 | 1 | 1 |
| 4 | 1 | 1 | 1 | 3 | 1 |

**Example 3.20.** Let $X = \{0, 1, 2, 3, 4\}$ be a set with the Cayley table which is given in Table 4. Then $(X, *, 1)$ is a GE-algebra. The set $D := \{1\}$ is not a vivid deductive system of $X$ since $1 \in D$ and $1 * (0 * 2) = 1 * 1 = 1 \in D$ but

$$((2 * 0) * 0) * 2 = (0 * 0) * 2 = 1 * 2 = 2 \notin D.$$

**Question 3.21.** *If $X$ is a left exchangeable and transitive GE-algebra, then is the set $D := \{1\}$ a vivid deductive system of $X$?*

The answer to Question 3.21 is negative as seen in the following example.

**Example 3.22.** Let $X = \{0, 1, 2, 3, 4\}$ be a set with the Cayley table which is given in Table 5. Then $(X, *, 1)$ is a GE-algebra which is left exchangeable and transitive. We can observe

Table 5. Cayley table for the binary operation "$*$"

| $*$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 2 | 1 | 1 |
| 1 | 0 | 1 | 2 | 3 | 4 |
| 2 | 1 | 1 | 1 | 1 | 1 |
| 3 | 0 | 1 | 2 | 1 | 1 |
| 4 | 0 | 1 | 2 | 1 | 1 |

that $D := \{1\}$ is not a vivid deductive system of $X$ since $1 \in D$ and $1 * (2 * 0) = 1 * 1 = 1 \in D$ but

$$((0 * 2) * 2) * 0 = (2 * 2) * 0 = 1 * 0 = 0 \notin D.$$

We discuss relationship between deductive system and vivid deductive system.

**Theorem 3.23.** *Every vivid deductive system is a deductive system.*

*Proof.* Let $D$ be a vivid deductive system of $X$. Note that $1 \in D$ by (GE1) and (D1). We first show that

$$(\forall x, y \in X)(x \in D,\, x * y \in D \;\Rightarrow\; y \in D). \tag{25}$$

Let $x, y \in X$ be such that $x \in D$ and $x * y \in D$. Then $x * (1 * y) = x * y \in D$ by (GE2), and so $y = ((y * 1) * 1) * y \in D$ by (GE2), (5) and (24). For every $x \in X$ and $y, z \in D$, we have

$$y * ((y * (z * x)) * (z * x)) = y * ((y * (z * x)) * (y * (z * x))) = y * 1 = 1 \in D$$

by (GE1), (GE3) and (5). It follows from (25) that $(y * (z * x)) * (z * x) \in D$. Hence

$$z * ((y * (z * x)) * x) = z * ((y * (z * x)) * (z * x)) \in D$$

by (GE3) and (D1), and thus $(y * (z * x)) * x \in D$ by (25). Therefore $D$ is a deductive system of $X$. □

The following example shows that any deductive system may not be a vivid deductive system.

**Example 3.24.** Let $X = \{0, 1, 2, 3, 4\}$ be a set with the Cayley table which is given in Table 6.

Table 6. Cayley table for the binary operation "$*$"

| $*$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 |
| 1 | 0 | 1 | 2 | 3 | 4 |
| 2 | 0 | 1 | 1 | 3 | 3 |
| 3 | 0 | 1 | 1 | 1 | 1 |
| 4 | 0 | 1 | 1 | 1 | 1 |

Then $(X, *, 1)$ is a GE-algebra and it is routine to verify that $D := \{1, 2\}$ is a deductive system of $X$. But it is not a vivid deductive system of $X$ since $1 \in D$ and $1 * (0 * 3) = 1 * 1 = 1 \in D$ but

$$((3 * 0) * 0) * 3 = (0 * 0) * 3 = 1 * 3 = 3 \notin D.$$

We provide conditions for a deductive system to be a vivid deductive system.

**Theorem 3.25.** *A deductive system $D$ of $X$ is vivid if and only if it satisfies:*

$$(\forall y, z \in X)(y * z \in D \;\Rightarrow\; ((z * y) * y) * z \in D). \tag{26}$$

*Proof.* Assume that $D$ is a vivid deductive system of $X$. Let $y, z \in X$ be such that $y * z \in D$. Then $1 * (y * z) = y * z \in D$ by (GE2), which implies from Lemma 3.3 and (24) that $((z * y) * y) * z \in D$.

Conversely, let $D$ be a deductive system of $X$ that satisfies (26). Let $x, y, z \in X$ be such that $x \in D$ and $x * (y * z) \in D$. Then

$$y * z = 1 * (y * z) = ((x * (y * z)) * (x * (y * z))) * (y * z) \in D$$

by (GE1), (GE2) and (D2). It follows from (26) that $((z * y) * y) * z \in D$. Therefore $D$ is a vivid deductive system of $X$. □

Given a subset $D$ of $X$, consider the next assertion:

$$(\forall x, y \in X)((x * y) * x \in D \;\Rightarrow\; x \in D). \tag{27}$$

In the following example, we can verify that any deductive system $D$ of $X$ does not satisfy the condition (27).

**Example 3.26.** In Example 3.2, we can observe that the deductive system $D = \{0, 1\}$ of $X$ does not satisfy the condition (27) since $(2 * 4) * 2 = 4 * 2 = 1 \in D$ but $2 \notin D$.

*Proof.* Let $x, y \in X$ be such that $x \in D$ and $x * y = 1$. Then $x * y \in D$ by Lemma 3.3. It follows from (GE1), (GE2) and (D2) that $y = 1 * y = ((x * y) * (x * y)) * y \in D$. □

**Theorem 3.27.** *Let $X$ be a transitive GE-algebra. If a deductive system $D$ of $X$ satisfies the condition* (27)*, then it is a vivid deductive system of $X$.*

*Proof.* Assume that a deductive system $D$ of $X$ satisfies the condition (27). Let $x, y, z \in X$ be such that $x \in D$ and $x * ((y * z) * y) \in D$. Then

$$\begin{aligned}(y * z) * y &= 1 * ((y * z) * y) \\ &= ((x * ((y * z) * y)) * (x * ((y * z) * y))) * ((y * z) * y) \in D\end{aligned}$$

by (GE1), (GE2) and (D2). Thus $y \in D$ by (27). This shows that

$$(\forall x, y, z \in X)(x \in D,\, x * ((y * z) * y) \in D \;\Rightarrow\; y \in D). \tag{28}$$

Let $y, z \in X$ be such that $y * z \in D$. Since $X$ is transitive, the combination of (7) and (13) induces $(((z * y) * y) * z) * y \leq z * y$, and so

$$\begin{aligned} y * z &\leq ((z * y) * y) * ((z * y) * z) \\ &\leq (z * y) * (((z * y) * y) * z) \\ &\leq ((((z * y) * y) * z) * y) * (((z * y) * y) * z). \end{aligned}$$

It follows from (GE2) and Corollary 3.5 that

$$\begin{aligned} &1 * (((((z * y) * y) * z) * y) * (((z * y) * y) * z)) \\ &= ((((z * y) * y) * z) * y) * (((z * y) * y) * z) \in D. \end{aligned}$$

Hence $((z * y) * y) * z \in D$ by (28). Therefore $D$ is a vivid deductive system of $X$ by Theorem 3.25. □

**Theorem 3.28.** *The intersection of two vivid deductive systems is also a vivid deductive system.*

*Proof.* Let Let $D_1$ and $D_2$ be vivid deductive systems of $X$. Then

$$\begin{aligned} X * (D_1 \cap D_2) &= \{x * a \mid x \in X,\, a \in D_1 \cap D_2\} \\ &= \{x * a \mid x \in X,\, a \in D_1\} \cap \{x * a \mid x \in X,\, a \in D_2\} \\ &\subseteq D_1 \cap D_2. \end{aligned}$$

Let $x, y, z \in X$ be such that $y, z \in D_1 \cap D_2$. Then $y, z \in D_1$ and $y, z \in D_2$. It follows from (D2) that $(y * (z * x)) * x \in D_1$ and $(y * (z * x)) * x \in D_2$. Hence $(y * (z * x)) * x \in D_1 \cap D_2$, and therefore $D_1 \cap D_2$ is a vivid deductive system of $X$. □

The following example shows that the union of vivid deductive systems may not be a vivid deductive system.

**Example 3.29.** Let $X = \{0, 1, 2, 3, 4, 5\}$ be a set with the Cayley table which is given in Table 7. Then $(X, *, 1)$ is a GE-algebra. Let $D_1 := \{1, 3\}$ and $D_2 := \{1, 4\}$. Then we can observe that $D_1$ and $D_2$ are vivid deductive systems of $X$. But $D_1 \cup D_2 := \{1, 3, 4\}$ is not a vivid deductive system of $X$ since $3 \in D_1 \cup D_2$ and $3 * (0 * 2) = 3 * 2 = 4 \in D_1 \cup D_2$ but

$$((2 * 0) * 0) * 2 = (0 * 0) * 2 = 1 * 2 = 2 \notin D_1 \cup D_2.$$

**Question 3.30.** *Consider deductive systems $D_1$ and $D_2$ of $X$ with $D_1 \subseteq D_2$. If $D_1$ is a vivid deductive system of $X$, is $D_2$ also a vivid deductive system of $X$?*

Table 7. Cayley table for the binary operation “$*$”

| $*$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 2 | 3 | 4 | 2 |
| 1 | 0 | 1 | 2 | 3 | 4 | 5 |
| 2 | 0 | 1 | 1 | 1 | 1 | 1 |
| 3 | 0 | 1 | 4 | 1 | 4 | 4 |
| 4 | 0 | 1 | 3 | 3 | 1 | 3 |
| 5 | 0 | 1 | 1 | 1 | 1 | 1 |

The answer to Question 3.30 is negative as seen in the following example.

**Example 3.31.** Let $X = \{0, 1, 2, 3, 4, 5\}$ be a set with the Cayley table which is given in Table 8.

Table 8. Cayley table for the binary operation “$*$”

| $*$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 2 | 3 | 2 | 1 |
| 1 | 0 | 1 | 2 | 3 | 4 | 5 |
| 2 | 5 | 1 | 1 | 1 | 5 | 5 |
| 3 | 0 | 1 | 1 | 1 | 0 | 0 |
| 4 | 1 | 1 | 1 | 3 | 1 | 1 |
| 5 | 1 | 1 | 2 | 3 | 2 | 1 |

Then $(X, *, 1)$ is a GE-algebra. Clearly $D_1 = \{1\}$ and $D_2 = \{0, 1, 5\}$ are deductive systems of $X$ and $D_1 \subseteq D_2$. We can observe that $D_1$ is a vivid deductive system of $X$. But $D_2$ is not a vivid deductive system of $X$ since $5 \in D_2$ and $5 * (3 * 4) = 5 * 0 = 1 \in D_2$ but $((4 * 3) * 3) * 4 = (3 * 3) * 4 = 1 * 4 = 4 \notin D_2$.

We explore conditions in which the answer to Question 3.30 can be positive.

**Theorem 3.32.** (Extension property) *Assume that $X$ is a transitive GE-algebra. Let $D_1$ and $D_2$ be deductive systems of $X$ with $D_1 \subseteq D_2$. If $D_1$ is a vivid deductive system of $X$, then so is $D_2$.*

*Proof.* Assume that $D_1$ is a vivid deductive system of $X$ and let $y, z \in X$ be such that $y * z \in D_2$. Using (GE1), (GE3) and (5), we get

$$y * ((y * z) * z) = y * ((y * z) * (y * z)) = y * 1 = 1 \in D_1,$$

and so $((((y*z)*z)*y)*y)*((y*z)*z) \in D_1 \subseteq D_2$ by Theorem 3.25. Hence

$$\begin{aligned} &(y*z)*(((((y*z)*z)*y)*y)*z) \\ &= (y*z)*(((((y*z)*z)*y)*y)*((y*z)*z)) \in D_2 \end{aligned}$$

by (GE3) and (D1). It follows from (GE1), (GE2) and (D2) that

$$a*z = 1*(a*z) = (((y*z)*(a*z))*((y*z)*(a*z)))*(a*z) \in D_2$$

where $a := ((((y*z)*z)*y)*y)$. Since $X$ is transitive, the combination of (7) and (13) induces $(a*z)*(((z*y)*y)*z) = 1$. Hence $((z*y)*y)*z \in D_2$ by Corollary 3.5. Therefore $D_2$ is a vivid deductive system of $X$ by Theorem 3.25. □

**Corollary 3.33.** *Let $X$ be a transitive GE-algebra. Then $\{1\}$ is a vivid deductive system of $X$ if and only if all deductive systems of $X$ are vivid.*

The following example describes Theorem 3.32.

**Example 3.34.** Let $X = \{0, 1, 2, 3, 4\}$ be a set with the Cayley table which is given in Table 9.

Table 9. Cayley table for the binary operation "$*$"

| $*$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 2 | 3 | 4 |
| 1 | 0 | 1 | 2 | 3 | 4 |
| 2 | 1 | 1 | 1 | 1 | 1 |
| 3 | 0 | 1 | 4 | 1 | 4 |
| 4 | 1 | 1 | 3 | 3 | 1 |

Then $(X, *, 1)$ is a transitive GE-algebra in which $\{1\}$ is not a vivid deductive system of $X$ since $2*0 = 1 \in \{1\}$ but $((0*2)*2)*0 = (2*2)*0 = 1*0 = 0 \notin \{1\}$. Let $D_1 = \{0, 1\}$ and $D_2 = \{0, 1, 4\}$. Then $D_1$ and $D_2$ are deductive systems of $X$ with $D_1 \subseteq D_2$, and $D_1$ is a vivid deductive system of $X$. We can verify that $D_2$ is also a deductive system of $X$.

## 4. Conclusion

We have introduced the concepts of a deductive system, a vivid deductive system of a GE-algebra and investigated the relation between them. We have observed that every vivid deductive system of a GE-algebra is a deductive system of a GE-algebra but not vice-versa.

We have provided conditions for a deductive system to be a vivid deductive system of a GE-algebra. We have introduced the notion of upper GE-set of $a$ and $b$ in a GE-algebra $X$ and characterized deductive system in terms of upper GE-set. We have established the extension property of the vivid deductive system.

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