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Research Paper

VERY TRUE GE-ALGEBRAS

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Abstract. The concept of very true GE-algebra using very true operator is introduced and its properties are studied to expand the scope of research of GE-algebras. The concepts of simple very true GE-algebra and very true GE-filter are introduced. The characterization of simple very true GE-algebra is discussed, and several properties on very true GE-filter are investigated. Using a very true GE-filter, the quotient very true GE-algebra is constructed, and the uniform and topological space are established.

1. Introduction

Hilbert algebras were introduced by Henkin and Skolem in the fifties for investigations in intuitionistic and other nonclassical logics. Diego [7] proved that Hilbert algebras form a variety which is locally finite. As a generalization of a Hilbert algebra, Bandaru et al. [1] introduce the concept of GE-algebra and investigated its properties (see also [2, 3]). Later,

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Song et al. [11, 12, 13] introduced the notions of imploring GE-filters, prominent interior GE-filters, interior GE-filters of GE-algebras and studied the relations between them. Rezaei et al. [10] introduced and studied the properties of prominent GE-filters and GE-morphisms in GE-algebras. Hájek [8] formulated the fuzzy truth value ”Very True” as a unary conjunction (hedge), and presented a complete axiomatization in response to the question “whether natural axiomatizations are possible and how far can even this kind of fuzzy logic be captured by standard methods of mathematical logic?” Hájek [8] introduced the notion of very true operator as a tool for reducing the number of possible logical values in many-valued fuzzy logic. Since then, very true operator has been applied to effect algebras, MTL-algebras, quality algebras, porrims, (pseudo) BCK-algebras, etc. (see [5, 6, 9, 14, 15]).

The purpose of this paper is to enrich the language of GE-algebras by adding a very true operator to obtain an algebra called a very true GE-algebra. We investigate several properties of a very true GE-algebra. We introduce the concept of very true GE-filters and investigate its properties. We also introduce the notion of a simple very true GE-algebra, and discuss its characterization. Using a very true GE-filter, we make a quotient very true GE-algebra and establish the uniform and topological space.

2. Preliminaries

Definition 2.1 ([1])**.** By a *GE-algebra* we mean a non-empty set X with a constant 1 and a binary operation $*$ satisfying the following axioms:

(GE1) $\widetilde{x} * \widetilde{x} = 1$,

(GE2) $1 * \widetilde{x} = \widetilde{x}$,

(GE3) $\widetilde{x} * (\widetilde{y} * \widetilde{z}) = \widetilde{x} * (\widetilde{y} * (\widetilde{x} * \widetilde{z}))$

for all $\widetilde{x}, \widetilde{y}, \widetilde{z} \in X$.

In a GE-algebra X, a binary relation “$\leq$” is defined by

$$(\forall \widetilde{x}, \widetilde{y} \in X)\,(\widetilde{x} \leq \widetilde{y} \;\Leftrightarrow\; \widetilde{x} * \widetilde{y} = 1)\,. \tag{1}$$

Definition 2.2 ([1, 2, 4])**.** A GE-algebra X is said to be

- *transitive* if it satisfies:

$$(\forall \widetilde{x}, \widetilde{y}, \widetilde{z} \in X)\,(\widetilde{x} * \widetilde{y} \leq (\widetilde{z} * \widetilde{x}) * (\widetilde{z} * \widetilde{y}))\,. \tag{2}$$

- *antisymmetric* if the binary relation “$\leq$” is antisymmetric.

Proposition 2.3 ([1])**.** *Every GE-algebra X satisfies the following items.*

(3) $$(\forall \widetilde{x} \in X)\,(\widetilde{x} * 1 = 1)\,.$$

(4) $$(\forall \widetilde{x}, \widetilde{y} \in X)\,(\widetilde{x} * (\widetilde{x} * \widetilde{y}) = \widetilde{x} * \widetilde{y})\,.$$

(5) $$(\forall \widetilde{x}, \widetilde{y} \in X)\,(\widetilde{x} \le \widetilde{y} * \widetilde{x})\,.$$

(6) $$(\forall \widetilde{x}, \widetilde{y}, \widetilde{z} \in X)\,(\widetilde{x} * (\widetilde{y} * \widetilde{z}) \le \widetilde{y} * (\widetilde{x} * \widetilde{z}))\,.$$

(7) $$(\forall \widetilde{x} \in X)\,(1 \le \widetilde{x} \;\Rightarrow\; \widetilde{x} = 1)\,.$$

(8) $$(\forall \widetilde{x}, \widetilde{y} \in X)\,(\widetilde{x} \le (\widetilde{y} * \widetilde{x}) * \widetilde{x})\,.$$

(9) $$(\forall \widetilde{x}, \widetilde{y} \in X)\,(\widetilde{x} \le (\widetilde{x} * \widetilde{y}) * \widetilde{y})\,.$$

(10) $$(\forall \widetilde{x}, \widetilde{y}, \widetilde{z} \in X)\,(\widetilde{x} \le \widetilde{y} * \widetilde{z} \;\Leftrightarrow\; \widetilde{y} \le \widetilde{x} * \widetilde{z})\,.$$

If X is transitive, then

(11) $$(\forall \widetilde{x}, \widetilde{y}, \widetilde{z} \in X)\,(\widetilde{x} \le \widetilde{y} \;\Rightarrow\; \widetilde{z} * \widetilde{x} \le \widetilde{z} * \widetilde{y},\; \widetilde{y} * \widetilde{z} \le \widetilde{x} * \widetilde{z})\,.$$

(12) $$(\forall \widetilde{x}, \widetilde{y}, \widetilde{z} \in X)\,(\widetilde{x} * \widetilde{y} \le (\widetilde{y} * \widetilde{z}) * (\widetilde{x} * \widetilde{z}))\,.$$

(13) $$(\forall \widetilde{x}, \widetilde{y}, \widetilde{z} \in X)\,(\widetilde{x} \le \widetilde{y},\, \widetilde{y} \le \widetilde{z} \;\Rightarrow\; \widetilde{x} \le \widetilde{z})\,.$$

Lemma 2.4 ([1])**.** *A GE-algebra X is transitive if and only if X satisfies the condition* (12).

Definition 2.5 ([1])**.** A subset F of a GE-algebra X is called a *GE-filter* of X if it satisfies:

(14) $$1 \in F,$$

(15) $$(\forall \widetilde{x}, \widetilde{y} \in X)(\widetilde{x} * \widetilde{y} \in F,\; \widetilde{x} \in F \;\Rightarrow\; \widetilde{y} \in F).$$

Lemma 2.6 ([1])**.** *In a GE-algebra X, every GE-filter F of X satisfies:*

(16) $$(\forall \widetilde{x}, \widetilde{y} \in X)\,(\widetilde{x} \le \widetilde{y},\; \widetilde{x} \in F \;\Rightarrow\; \widetilde{y} \in F)\,.$$

3. Very true GE-algebras

Definition 3.1. A *very true GE-algebra* is defined to be a pair (X, ℓ) in which X is a GE-algebra and ℓ is a self-map on X such that

(17) $$\ell(1) = 1,$$

(18) $$(\forall x \in X)(\ell(x) \le x),$$

(19) $$(\forall x \in X)(\ell(x) \le \ell^2(x)),$$

(20) $$(\forall x, y \in X)(\ell(x * y) \le \ell(x) * \ell(y)).$$

Definition 3.2. A very true GE-algebra (X, ℓ) is said to be

- *transitive* if X is a transitive GE-algebra.
- *antisymmetric* if X is an antisymmetric GE-algebra.

Denote by $\mathcal{V}_t(X)$ the set of all very true GE-algebras.

Example 3.3. Consider a GE-algebra $X = \{1, a, b, c, d\}$ with the binary operation $*$ which is given in Table 1. Define a self-map ℓ on X by Table 2. Then (X, ℓ_1) and (X, ℓ_2) are very true

Table 1. Cayley table for the binary operation "$*$"

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	c	d
b	1	1	1	c	d
c	1	a	a	1	d
d	1	b	b	1	1

Table 2. Tabular representation of ℓ

x	1	a	b	c	d
$\ell_1(x)$	1	a	a	c	d
$\ell_2(x)$	1	a	a	d	d

GE-algebras.

Example 3.4. Consider a GE-algebra $X = \{1, a, b, c, d\}$ with the binary operation $*$ which is given in Table 3. Define a self-map ℓ on X by Table 4. Then (X, ℓ) is a transitive very true

Table 3. Cayley table for the binary operation "$*$"

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	c	c	d
b	1	1	1	1	d
c	1	1	1	1	d
d	1	a	b	b	1

GE-algebra.

Table 4. Tabular representation of ℓ

x	1	a	b	c	d
$\ell(x)$	1	b	c	c	d

Example 3.5. Consider a GE-algebra $X = \{1, a, b, c, d\}$ with the binary operation $*$ which is given in Table 5. Define a self-map ℓ on X by Table 6. Then (X, ℓ) is antisymmetric very true GE-algebra.

Table 5. Cayley table for the binary operation "$*$"

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	1	d
b	1	a	1	c	d
c	1	a	b	1	1
d	1	a	b	c	1

Table 6. Tabular representation of ℓ

x	1	a	b	c	d
$\ell(x)$	1	a	b	c	c

Proposition 3.6. *Every very true GE-algebra (X, ℓ) satisfies:*

(i) $(\forall x \in X)\ (\ell(x) = 1 \ \Leftrightarrow\ x = 1)$.

(ii) $(\forall x, y \in X)\ (x \leq y \ \Rightarrow\ \ell(x) \leq \ell(y))$.

Proof. (i) For every $x \in X$, if $\ell(x) = 1$, then $1 = \ell(x) \leq x$ and so $x = 1$. The converse is clear by (17).

(ii) Let $x, y \in X$ be such that $x \leq y$. Then $x * y = 1$, and so

$$1 = \ell(1) = \ell(x * y) \leq \ell(x) * \ell(y)$$

by (17) and (20). Hence $\ell(x) * \ell(y) = 1$ by (7), that is $\ell(x) \leq \ell(y)$. □

Proposition 3.7. *Every antisymmetric very true GE-algebra (X, ℓ) satisfies:*

(i) $(\forall x \in X)\ (\ell^2(x) = \ell(x))$.

(ii) $(\forall x, y \in X)\ (\ell(x) \leq y \ \Leftrightarrow\ \ell(x) \leq \ell(y))$.

Proof. (i) By (18) and (19), we have $\ell(x) \leq \ell^2(x) \leq \ell(x)$, and thus $\ell^2(x) = \ell(x)$ for all $x \in X$.

(ii) Let $x, y \in X$ be such that $\ell(x) \leq y$. Then $\ell(x) = \ell^2(x) \leq \ell(y)$ by (i) and Proposition 3.6(ii). The converse is straightforward by (18). □

If (X, ℓ) is a very true GE-algebra which is not antisymmetric, then Proposition 3.7 is not true as seen in the following example.

Example 3.8. Consider a GE-algebra $X = \{1, a, b, c, d\}$ with the binary operation $*$ which is given in Table 7. Define a self-map ℓ on X by Table 8. Then (X, ℓ) is a very true GE-algebra

TABLE 7. Cayley table for the binary operation "$*$"

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	1	1
b	1	a	1	1	a
c	1	1	1	1	1
d	1	1	1	c	1

TABLE 8. Tabular representation of ℓ

x	1	a	b	c	d
$\ell(x)$	1	d	c	a	a

which is not antisymmetric. We can observe that (X, ℓ) does not satisfy Proposition 3.7(i) since $\ell(\ell(a)) = \ell(d) = a \neq d = \ell(a)$. Also, $\ell(a) * b = d * b = 1$ but $\ell(a) * \ell(b) = d * c = c \neq 1$. We can observe that $\ell(a) * \ell(c) = d * a = 1$ but $\ell(a) * c = d * c = c \neq 1$. This shows that Proposition 3.7(ii) is false.

In a very true GE-algebra (X, ℓ), consider the sets

$$\ker(\ell) := \{x \in X \mid \ell(x) = 1\} \text{ and } \mathcal{I}(\ell) := \{x \in X \mid \ell(x) = x\}$$

which is called the *kernel* and the *identity part*, respectively, of (X, ℓ).

Proposition 3.9. *If (X, ℓ) is a very true GE-algebra, then*

(i) $\ker(\ell) = \{1\}$.

(ii) *The image of* ℓ *is the identity part of* (X, ℓ) *when* X *is antisymmetric.*

(iii) *If* ℓ *is surjective, then it is identity when* X *is antisymmetric.*

Proof. (i) If $x \in \ker(\ell)$, then $1 = \ell(x) \leq x$, and so $x = 1$, that is, $\ker(\ell) = \{1\}$.

(ii) If $x \in \ell(X)$, then $\ell(y) = x$ for some $y \in X$. It follows from Proposition 3.7(i) that $\ell(x) = \ell^2(x) = \ell^3(y) = \ell(y) = x$, i.e., $x \in \mathcal{I}(\ell)$. Hence $\ell(X) \subseteq \mathcal{I}(\ell)$. It is clear that $\mathcal{I}(\ell) \subseteq \ell(X)$.

(iii) Assume that ℓ is surjective and let $x \in X$. Then there exists $y \in X$ such that $\ell(y) = x$. Using Proposition 3.7(i), we have $\ell(x) = \ell^2(y) = \ell(y) = x$ which means that ℓ is identity. □

Proposition 3.10. *Every transitive very true GE-algebra* (X, ℓ) *satisfies:*

(i) $(\forall x, y \in X)\ (\ell(x) \leq \ell(y) * x)$.

(ii) $(\forall x, y \in X)\ (\ell(x) \leq \ell(x * y) * y,\ \ell(x) \leq \ell(x * y) * x,\ \ell(x) \leq \ell(y * x) * x)$.

(iii) $(\forall x, y, z \in X)\ (x \leq y\ \Rightarrow\ \ell(z * x) \leq \ell(z * y),\ \ell(y * z) \leq \ell(x * z))$.

Proof. (i) Let $x, y \in X$. Since $x \leq y * x$, it follows from Proposition 3.6(ii), (18) and (11) that $\ell(x) \leq \ell(y * x) \leq y * x \leq \ell(y) * x$.

(ii) Since $x \leq (x * y) * y$, we have

$$\ell(x) \leq \ell((x * y) * y) \leq \ell(x * y) * \ell(y) \leq \ell(x * y) * y$$

by Proposition 3.6(ii), (20), (18) and (11). Similarly, we get $\ell(x) \leq \ell(x * y) * x$ and $\ell(x) \leq \ell(y * x) * x$.

(iii) is obtained by using (11) and Proposition 3.6(ii). □

The following example shows that $\mathcal{V}_t(X)$ is not closed under the composition "$\circ$" of functions, that is, if (X, ℓ) and (X, κ) are very true GE-algebras, then $(X, \ell \circ \kappa)$ may not be a very true GE-algebra.

Example 3.11. Consider a GE-algebra $X = \{1, a, b, c, d, e\}$ with the binary operation $*$ which is given in Table 9. Define a self-maps ℓ and κ on X by Table 10. Then (X, ℓ) and (X, κ) are very true GE-algebras. But $(X, \ell \circ \kappa)$ is not very true GE-algebra, since $(\ell \circ \kappa)(a) * a = \ell(\kappa(a)) * a = \ell(c) * a = d * a = b \neq 1$, that is, $(\ell \circ \kappa)(a) \not\leq a$.

TABLE 9. Cayley table for the binary operation "$*$"

$*$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	1	1	1	d	d	1
b	1	1	1	1	1	1
c	1	1	1	1	1	1
d	1	b	b	1	1	1
e	1	b	b	c	c	1

TABLE 10. Tabular representation of ℓ

x	1	a	b	c	d	e
$\ell(x)$	1	c	c	d	c	d
$\kappa(x)$	1	c	b	b	b	e

Lemma 3.12. *If $\mathcal{V}_t(X)$ is closed under the composition "$\circ$" of functions, then $(\mathcal{V}_t(X), \circ)$ is a semigroup.*

Proof. Straightforward. □

Proposition 3.13. *If there exist $\ell, \kappa \in \mathcal{V}_t(X)$ such that $\ell \circ \kappa = \ell$, then $\ell \leq \kappa$ where $\ell \leq \kappa$ means that $\ell(x) \leq \kappa(x)$ for all $x \in X$.*

Proof. Assume that $\ell \circ \kappa = \ell$ for some $\ell, \kappa \in \mathcal{V}_t(X)$. Then $\ell(x) = (\ell \circ \kappa)(x) = \ell(\kappa(x)) \leq \kappa(x)$ for all $x \in X$, that is, $\ell \leq \kappa$. □

Corollary 3.14. *If $\mathcal{V}_t(X)$ is closed under "$\circ$", then*

$$(\forall \ell, \kappa \in \mathcal{V}_t(X))(\ell \circ \kappa = \ell \ \Rightarrow\ \ell \leq \kappa) \tag{21}$$

where $\ell \leq \kappa$ means that $\ell(x) \leq \kappa(x)$ for all $x \in X$.

Lemma 3.15. *If X is a transitive GE-algebra, then*

$$(\forall x, y, z \in X)(x \leq y,\ y \leq z \ \Rightarrow\ x \leq z).$$

Proof. Let X be a transitive GE-algebra and $x, y, z \in X$ be such that $x \leq y$ and $y \leq z$. Then $x * y = 1$ and $y * z = 1$. Since X is transitive, we have

$$1 = (x * y) * ((y * z) * (x * z)) = 1 * (1 * (x * z)) = x * z.$$

by (12). Therefore $x * z = 1$. Thus $x \leq z$. □

Theorem 3.16. *Let X be a transitive and antisymmetric GE-algebra. If $\mathcal{V}_t(X)$ is closed under "$\circ$", then*

$$(\forall \ell, \kappa \in \mathcal{V}_t(X))(\ell \leq \kappa \ \Rightarrow \ \ell \circ \kappa = \ell). \tag{22}$$

Proof. If $\ell \leq \kappa$, then $\ell(x) = \ell^2(x) \leq \ell(\kappa(x)) = (\ell \circ \kappa)(x)$ for all $x \in X$. Moreover $(\ell \circ \kappa)(x) = \ell(\kappa(x)) \leq \kappa(\kappa(x)) = \kappa(x) \leq x$, and so $(\ell \circ \kappa)(x) \leq x$ by Lemma 3.15. It follows from Proposition 3.6(ii) and Proposition 3.7(i) that

$$(\ell \circ \kappa)(x) = ((\ell \circ \ell) \circ \kappa)(x) = (\ell \circ (\ell \circ \kappa))(x) \leq \ell(x)$$

for all $x \in X$. Hence $\ell \circ \kappa = \ell$. □

4. Very true GE-filters in very true GE-algebras

Given a subset F of X in a very true GE-algebra (X, ℓ), consider the next assertion:

$$(\forall x \in X)(x \in F \ \Rightarrow \ell(x) \in F). \tag{23}$$

The following example shows that there exists a GE-filter F of X which does not satisfy the condition (23) in a very true GE-algebra (X, ℓ).

Example 4.1. Consider a GE-algebra $X = \{1, a, b, c, d\}$ with the binary operation $*$ which is given in Table 11. Define a self-map ℓ on X by Table 12. Then (X, ℓ) is a very true GE-algebra

Table 11. Cayley table for the binary operation "$*$"

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	b	d
b	1	1	1	1	1
c	1	a	1	1	1
d	1	a	1	1	1

and $F := \{1, a\}$ is a GE-filter of X. But it does not satisfy (23) since $a \in F$ but $\ell(a) = b \notin F$.

Table 12. Tabular representation of ℓ

x	1	a	b	c	d
$\ell(x)$	1	b	b	d	c

Definition 4.2. Let (X, ℓ) be a very true GE-algebra. Then a subset F of X is called a *very true GE-filter* of (X, ℓ) if F is a GE-filter of X which satisfies the condition (23).

It is clear that X itself and $\ker(\ell)$ in a very true GE-algebra (X, ℓ) are very true GE-filters of (X, ℓ).

Example 4.3. Consider a GE-algebra $X = \{1, a, b, c, d\}$ with the binary operation $*$ which is given in Table 13. Define a self-map ℓ on X by Table 14. Then (X, ℓ) is a very true GE-algebra

Table 13. Cayley table for the binary operation "$*$"

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	1	1
c	1	a	1	1	1
d	1	a	c	c	1

Table 14. Tabular representation of ℓ

x	1	a	b	c	d
$\ell(x)$	1	a	b	b	b

and $F := \{1, a\}$ is a very true GE-filter of X which is neither $\ker(\ell)$ nor X itself.

Theorem 4.4. *Let (X, ℓ) be a very true GE-algebra.*

(i) *If F is a GE-filter of $\ell(X)$, then $\ell^{-1}(F)$ is a very true GE-filter of (X, ℓ).*
(ii) *If F is a very true GE-filter of (X, ℓ), then $\ell(F)$ is a GE-filter of $\ell(X)$.*

Proof. (i) Let F be a GE-filter of $\ell(X)$. It is clear that $1 \in \ell^{-1}(F)$. Let $x, y \in X$ be such that $x * y \in \ell^{-1}(F)$ and $x \in \ell^{-1}(F)$. Then $\ell(x * y) \in F$ and $\ell(x) \in F$. Since $\ell(x * y) \leq \ell(x) * \ell(y)$ by (20), it follows from Lemma 2.6 that $\ell(x) * \ell(y) \in F$. Hence $\ell(y) \in F$, that is, $y \in \ell^{-1}(F)$. Therefore $\ell^{-1}(F)$ is a GE-filter of X. If $x \in \ell^{-1}(F)$, then $\ell(x) \in F$, and so $\ell^2(x) \in F$ by

Lemma 2.6 and (19). Hence $\ell(x) \in \ell^{-1}(F)$. Consequently, $\ell^{-1}(F)$ is a very true GE-filter of (X, ℓ).

(ii) Assume that F is a very true GE-filter of (X, ℓ). We first show that $\ell(F) = \ell(X) \cap F$. If $y \in \ell(F)$, then $y = \ell(x)$ for some $x \in F$ which implies from (23) that $y = \ell(x) \in F$. Hence $\ell(F) \subseteq F \cap \ell(X)$. If $x \in \ell(X) \cap F$, then $x \in F$ and $x = \ell(y) \leq y$ for some $y \in X$ by (18). Hence $y \in F$ by Lemma 2.6, and so $x = \ell(y) \in \ell(F)$. This shows that $\ell(X) \cap F = \ell(F)$. Since $1 \in F$, we have $1 = \ell(1) \in \ell(F)$. Let $x, y \in \ell(X)$ be such that $x * y \in \ell(F) = \ell(X) \cap F$ and $x \in \ell(F) = \ell(X) \cap F$. Then $y \in F$ by (15), and thus $y \in \ell(X) \cap F = \ell(F)$. Therefore $\ell(F)$ is a GE-filter of $\ell(X)$. □

Theorem 4.5. *The intersection of two very true GE-filters is also a very true GE-filter.*

Proof. Let F and G be very true GE-filters of (X, ℓ). Clearly, $1 \in F \cap G$. Let $x, y \in X$ be such that $x * y \in F \cap G$ and $x \in F \cap G$. Then $x * y \in F$, $x * y \in G$, $x \in F$ and $x \in G$. It follows that $y \in F$ and $y \in G$. Hence $y \in F \cap G$, and so $F \cap G$ is a GE-filter of X. If $x \in F \cap G$, then $x \in F$ and $x \in G$ which implies from (23) that $\ell(x) \in F$ and $\ell(x) \in G$. Thus $\ell(x) \in F \cap G$, and therefore $F \cap G$ is a very true GE-filter of (X, ℓ). □

The following example shows that the union of very true GE-filters may not be a very true GE-filter.

Example 4.6. Consider a GE-algebra $X = \{1, a, b, c, d\}$ with the binary operation $*$ which is given in Table 15. Define a self-map ℓ on X by Table 16. Then (X, ℓ) is a very true GE-algebra.

TABLE 15. Cayley table for the binary operation "$*$"

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	c
b	1	a	1	1	a
c	1	a	1	1	a
d	1	1	1	1	1

It is routine to verify that the sets $F := \{1, a\}$ and $G := \{1, b, c\}$ are very true GE-filters of X. But $F \cup G = \{1, a, b, c\}$ is not a very true GE-filter of X since $b * d = a \in F \cup G$ and $b \in F \cup G$ but $d \notin G$.

TABLE 16. Tabular representation of ℓ

x	1	a	b	c	d
$\ell(x)$	1	a	c	b	d

Definition 4.7. A GE-algebra X is said to be *simple* if it has no proper GE-filter, that is, it has only two GE-filters, $\{1\}$ and X itself.

Example 4.8. Consider a GE-algebra $X = \{1, a, b, c, d\}$ with the binary operation $*$ which is given in Table 17. Then $(X, *, 1)$ is a simple GE-algebra.

TABLE 17. Cayley table for the binary operation "$*$"

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	1	1
b	1	1	1	1	1
c	1	a	a	1	1
d	1	1	1	c	1

Definition 4.9. A very true GE-algebra (X, ℓ) is said to be *simple* if it has no proper very true GE-filter, that is, it has only two very true GE-filters, $\{1\}$ and X itself.

Example 4.10. Consider a GE-algebra $X = \{1, a, b, c, d\}$ with the binary operation $*$ which is given in Table 18. Define a self-map ℓ on X by Table 19. Then (X, ℓ) is a simple very true

TABLE 18. Cayley table for the binary operation "$*$"

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	1	1
b	1	a	1	1	1
c	1	1	1	1	1
d	1	1	1	c	1

GE-algebra.

Table 19. Tabular representation of ℓ

x	1	a	b	c	d
$\ell(x)$	1	a	a	a	a

Given a very true GE-algebra (X, ℓ), the following example shows that $(\ell(X), *, 1)$ is not a sub-GE-algebra of X.

Example 4.11. Consider a GE-algebra $X = \{1, a, b, c, d\}$ with the binary operation $*$ which is given in Table 20. Define a self-map ℓ on X by Table 21. Then (X, ℓ) is a very true GE-algebra

Table 20. Cayley table for the binary operation "$*$"

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	1	1
b	1	1	1	1	d
c	1	b	b	1	1
d	1	b	b	c	1

Table 21. Tabular representation of ℓ

x	1	a	b	c	d
$\ell(x)$	1	a	a	a	d

and $\ell(X) = \{1, a, d\}$. We can observe that $d, a \in \ell(X)$ but $d * a = b \notin \ell(X)$. Hence $(\ell(X), *, 1)$ is not a sub-GE-algebra of X.

On the other hand, if we take a very true GE-algebra (X, ℓ) in Example 4.3, then $(\ell(X), *, 1)$ is a sub-GE-algebra of X with $\ell(X) = \{1, a, b\}$.

Theorem 4.12. *Let (X, ℓ) be an antisymmetric very true GE-algebra such that $(\ell(X), *, 1)$ is a sub-GE-algebra of X. Then (X, ℓ) is simple if and only if $(\ell(X), *, 1)$ is simple.*

Proof. Assume that (X, ℓ) is a simple very true GE-algebra such that such that $(\ell(X), *, 1)$ is a GE-algebra. Let F be a GE-filter of $(\ell(X), *, 1)$ and suppose $F \neq \{1\}$. Then $\ell^{-1}(F)$ is a very true GE-filter of (X, ℓ) by Theorem 4.4(i), and thus $\ell^{-1}(F) = \{1\}$ or $\ell^{-1}(F) = X$ since (X, ℓ) is a simple very true GE-algebra. If $x(\neq 1) \in F$, then $\ell(x) = x$ by Proposition 3.9(ii) and thus $x \in \ell^{-1}(F)$, that is, $\ell^{-1}(F) \neq \{1\}$. Hence $\ell^{-1}(F) = X$, and so $\ell(X) \subseteq F$ which implies that $F = \ell(X)$. Therefore $(\ell(X), *, 1)$ is a simple GE-algebra.

Conversely, suppose that $(\ell(X), *, 1)$ is a simple GE-algebra and let F be a very true GE-filter of (X, ℓ) with $F \neq \{1\}$. Then $\ell(F)$ is a GE-filter of $(\ell(X), *, 1)$ by Theorem 4.4(ii), and so $\ell(F) = \{1\}$ or $\ell(F) = X$. Since $F \neq \{1\}$, there exists $x(\neq 1) \in F$ and so $x = \ell(x) \in \ell(F)$. This shows that $\ell(F) \neq \{1\}$ and thus $X = \ell(F) \subseteq F$. Hence $F = X$, and therefore (X, ℓ) is simple. □

Let F be a very true GE-filter of a very true GE-algebra (X, ℓ) and consider a mapping:

$$\ell_F : X/F \to X/F,\ [x] \mapsto [\ell(x)]. \tag{24}$$

Theorem 4.13. *If F is a very true GE-filter of a very true GE-algebra (X, ℓ), then $(X/F, \ell_F)$ is a very true GE-algebra which is called the quotient very true GE-algebra.*

Proof. Let $x, y \in X$ be such that $[x] = [y]$. Then $x * y \in F$ and $y * x \in F$. It follows from (4.2) that $\ell(x * y) \in F$ and $\ell(y * x) \in F$ Since $\ell(x * y) \leq \ell(x) * \ell(y)$ and $\ell(y * x) \leq \ell(y) * \ell(x)$, we have $\ell(x) * \ell(y) \in F$ and $\ell(y) * \ell(x) \in F$ by Lemma 2.6. Thus $[\ell(x)] = [\ell(y)]$, and so ℓ_F is well defined. For any $x \in X$, we get $[x] * [x] = [x * x] = [1]$ and $[1] * [x] = [1 * x] = [x]$. Let $x, y, z \in X$. Then

$$\begin{aligned}
[x] * ([y] * [z]) &= [x] * [y * z] = [x * (y * z)] \\
&= [x * (y * (x * z))] = [x] * [y * (x * z)] \\
&= [x] * ([y] * [x * z]) = [x] * ([y] * ([x] * [z])).
\end{aligned}$$

Therefore $(X/F, \ell_F)$ is a GE-algebra. Moreover, $\ell_F([1]) = [\ell(1)] = [1]$, $\ell_F([x]) = [\ell(x)] \leq [x]$, $\ell_F([x]) = [\ell(x)] \leq [\ell^2(x)] = \ell_F^2([x])$ and

$$\ell_F([x] * [y]) = [\ell(x * y)] \leq [\ell(x) * \ell(y)] = [\ell(x)] * [\ell(y)] = \ell_F([x]) * \ell_F([y])$$

for all $x, y \in X$. Therefore $(X/F, \ell_F)$ is a very true GE-algebra. □

Theorem 4.14. *Given a very true GE-filter F of a very true GE-algebra (X, ℓ), consider the following set.*

$$\begin{aligned}
G_F &:= \{(x, y) \in X \times X \mid \ell(x) \sim_F \ell(y)\} \\
&= \{(x, y) \in X \times X \mid \ell(x) * \ell(y) \in F,\ \ell(y) * \ell(x) \in F\}.
\end{aligned}$$

If $\mathcal{K}^* = \{G_F \mid F$ *is a very true GE-filter of* $(X, \ell)\}$*, then* $\mathcal{K}^*$ *satisfies the following conditions.*

$$\Delta := \{(x, x) \in X \times X \mid x \in X\} \subseteq G_F \text{ for every } G_F \in \mathcal{K}^*. \tag{25}$$

$$G_F \in \mathcal{K}^* \Rightarrow G_F^{-1} \in \mathcal{K}^*. \tag{26}$$

$$G_F \in \mathcal{K}^* \Rightarrow (\exists G_J \in \mathcal{K}^*)\ (G_J \circ G_J \subseteq G_F). \tag{27}$$

$$G_F, G_J \in \mathcal{K}^* \Rightarrow G_F \cap G_J \in \mathcal{K}^*. \tag{28}$$

where $G_F^{-1} := \{(x, y) \in X \times X \mid (y, x) \in G_F\}$.

Proof. Let $x \in X$ be such that $(x, x) \in \Delta$. Then $\ell(x) * \ell(x) = 1 \in F$, and so $(x, x) \in G_F$. Thus $\Delta \subseteq G_F$. Note that

$$\begin{aligned}(x, y) \in G_F &\Leftrightarrow \ell(x) * \ell(y) \in F,\ \ell(y) * \ell(x) \in F \\ &\Leftrightarrow (y, x) \in G_F \Leftrightarrow (x, y) \in G_F^{-1}.\end{aligned}$$

Hence (25) and (26) are true. Given $G_F \in \mathcal{K}^*$, let

$$\mathcal{F} := \{F_i \mid F_i \text{ is a very true GE-filter of } X \text{ such that } F_i \subseteq F \text{ for } i \in \Lambda\}.$$

Then $\mathcal{F}$ is nonempty since $F \in \mathcal{F}$. Let J be the very true GE-filter of (X, ℓ) generated by $\cup_{i\in\Lambda} F_i$. Then $G_J \in \mathcal{K}^*$. We claim that $G_J \circ G_J \subseteq G_F$. If $(x, y) \in G_J \circ G_J$, then there exists $z \in X$ such that $(x, z) \in G_J$ and $(z, y) \in G_J$, that is, $\ell(x) \sim_J \ell(z)$ and $\ell(z) \sim_J \ell(y)$. Hence $\ell(x) \sim_J \ell(y)$, i.e., $\ell(x) * \ell(y) \in J$ and $\ell(y) * \ell(x) \in J$. Since $\cup_{i\in\Lambda} F_i \subseteq F$, we get $J \subseteq F$. Hence $\ell(x) * \ell(y) \in F$ and $\ell(y) * \ell(x) \in F$, that is, $\ell(x) \sim_F \ell(y)$. Thus $(x, y) \in G_F$, and so $G_J \circ G_J \subseteq G_F$. Hence (27) is valid. For every $G_F, G_J \in \mathcal{K}^*$, we have

$$\begin{aligned}&(x, y) \in G_F \cap G_J \Leftrightarrow (x, y) \in G_F,\ (x, y) \in G_J \\ &\Leftrightarrow \ell(x) \sim_F \ell(y),\ \ell(x) \sim_J \ell(y) \\ &\Leftrightarrow \ell(x) * \ell(y) \in F,\ \ell(y) * \ell(x) \in F,\ \ell(x) * \ell(y) \in J,\ \ell(y) * \ell(x) \in J \\ &\Leftrightarrow \ell(x) * \ell(y) \in F \cap J,\ \ell(y) * \ell(x) \in F \cap J \\ &\Leftrightarrow \ell(x) \sim_{F\cap J} \ell(y) \\ &\Leftrightarrow (x, y) \in G_{F\cap J}\end{aligned}$$

and so $G_F \cap G_J = G_{F\cap J}$. Since $F \cap J$ is a very true GE-filter of (X, ℓ) by Theorem 4.5, we have $G_F \cap G_J = G_{F\cap J} \in \mathcal{K}^*$. □

Theorem 4.15. *Given a very true GE-filter F of a very true GE-algebra (X, ℓ), let*

$$\mathcal{K} := \{G \subseteq X \times X \mid G_F \subseteq G \text{ for some } G_F \in \mathcal{K}^*\}.$$

Then $\mathcal{K}$ satisfies (25), (26), (27), (28) *and*

$$G \in \mathcal{K},\, G \subseteq H \subseteq X \times X \;\Rightarrow\; H \in \mathcal{K}. \tag{29}$$

Proof. Using Theorem 4.14, we can verify that $\mathcal{K}$ satisfies (25) – (28). Suppose that $G \in \mathcal{K}$ and $G \subseteq H \subseteq X \times X$. Then there exists $G_F \in \mathcal{K}^*$ such that $G_F \subseteq G \subseteq H$, which means that $H \in \mathcal{K}$. □

Theorem 4.15 states that very true GE-algebra forms a uniform space, that is, $((X, \ell), \mathcal{K})$ is a uniform space.

Theorem 4.16. *Let (X, ℓ) be a very true GE-algebra. For any $x \in X$ and $G \in \mathcal{K}$, the set*

$$T := \{Q \subseteq X \mid \ell(x) \in Q, G[x] \subseteq Q \text{ for some } G \in \mathcal{K}\}$$

is a topology on (X, ℓ) where $G[x] := \{y \in X \mid (x, y) \in G\}$, and the collection

$$\mathcal{G}_x := \{G[x] \mid G \in \mathcal{K}\}$$

forms a neighborhood base at x, making (X, ℓ) a topological space.

Proof. It is clear that $\emptyset \in T$ and $X \in T$. Let $x \in X$ and $P, Q \in T$ be such that $x \in P \cap Q$. Then there exist $G, H \in \mathcal{K}$ such that $G[x] \subseteq Q$ and $H[x] \subseteq P$. Let $W = G \cap H$. Then $W \in \mathcal{K}$ and $W[x] \subseteq G[x] \cap H[x] \subseteq Q \cap P$. Hence $Q \cap P \in T$. It is also clear that T is closed under arbitrary union by the definition. Therefore T is a topology on (X, ℓ). Now, we note that $x \in G[x]$ for each $x \in X$. Since $G_1[x] \cap G_2[x] = (G_1 \cap G_2)[x]$, the intersection of neighborhoods is a neighborhood. Finally, if $G[x] \in \mathcal{G}_x$, then there exists $P \in \mathcal{K}$ such that $P \circ P \subseteq G$ by (27). Hence for any $y \in P[x]$, $P[y] \subseteq P[x]$, proving the theorem. □

5. Conclusion

Very true operator developed by Hájek is a tool that is very useful for reducing the number of possible logical values in many-valued fuzzy logic, and it is the same as the concept of hedge introduced formerly by Zadeh. In this manuscript, we have applied very true operator to GE-algebra. We have introduced the concepts of (simple) very true GE-algebra and very true GE-filter, and investigated its properties. We have discussed the characterization of simple very true GE-algebra, and investigated several properties on very true GE-filter. We have constructed the quotient very true GE-algebra using a very true GE-filter and established the uniform and topological spaces. In our future work, we would like to extend the concept of very true operator to the generalizations of GE-algebras such as (weak) eGE-algebras, pseudo eGE-algebras etc.

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