Algebraic Structures and Their Applications Vol. 8 No. 2 (2021) pp 51-65.

Research Paper

ON THE NSE CHARACTERIZATION OF CERTAIN FINITE SIMPLE GROUPS

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Abstract. For a group G, $\pi_e(G)$ and $s_m(G)$ are denoted the set of orders of elements and the number of elements of order m in G, respectively. Let $\mathrm{nse}(G) = \{s_m(G) \mid m \in \pi_e(G)\}$. An arbitrary finite group M is NSE characterization if, for every group G, the equality $\mathrm{nse}(G) = \mathrm{nse}(M)$ implies that $G \cong M$. In this paper, we are going to show that the non-Abelian finite simple groups A_9, A_{10}, A_{12}, $U_4(3)$, $U_5(2)$, $U_6(2)$, $S_6(2)$, $O_8^+(2)$ and HS are characterizable by NSE.

1. Introduction

Given a group G, denote by $\pi_e(G)$ the set of orders of all elements in G. It is clear that the set $\pi_e(G)$ is closed and partially ordered by divisibility, hence, it is uniquely determined by the subset of its maximal elements which is indicated by $\mu(G)$. We also denote by $\pi(n)$ the set of all prime divisors of a positive integer n. For a finite group G, we will write $\pi(G)$

DOI: 10.22034/as.2021.1977

MSC(2010): Primary:20D05

Keywords: Element order, Finite simple group, Prime graph.

Received: 24 July 2020, Accepted: 15 January 2021

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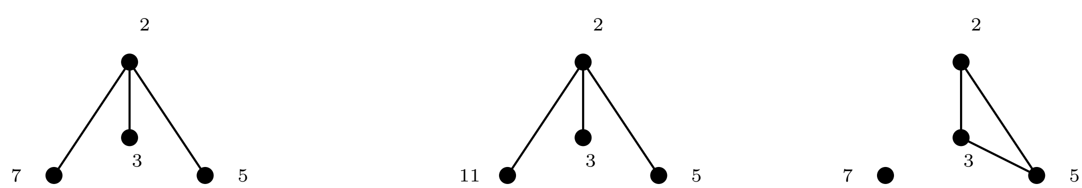
instead of $\pi(|G|)$. The notation $s_m(G)$ is applied for the number of elements of order m in G, and $\mathrm{nse}(G) = \{s_m(G) \mid m \in \pi_e(G)\}$, the set of numbers of all elements in a group G with the same order. A group G is called NSE characterization if for every finite group M, the equality $\mathrm{nse}(G) = \mathrm{nse}(M)$ implies that $G \cong M$.

Up to now, the NSE characterization of many non-Abelian finite simple groups are investigated, for instance, the references [5, 6, 9, 10, 11, 13] introduce some of these groups. It is worth to mention that, in some cases, the order of the groups are included too, in other words, for an arbitrary finite group M, and every finite group G, the equality $\mathrm{nse}(G) = \mathrm{nse}(M)$ and $|G| = |M|$ imply that $G \cong M$. For more information of these results, for instance, see [1, 2, 12].

In what follows we will consider the finite non-Abelian simple groups S with the property $\pi(S) \subseteq \{2, 3, 5, 7, 11\}$. It has been shown that many of these type of simple groups are NSE characterization. In this paper, our goal is to investigate those type of simple groups that have the similar way to prove in the NSE characterization process. These groups are A_9, A_{10}, A_{12}, $U_4(2)$, $U_5(2)$, $U_6(2)$, $S_6(2)$, $O_8^+(2)$, and HS. In fact, the following theorem would be proven.

Main Theorem *All the non-Abelian finite simple groups A_9, A_{10}, A_{12}, $U_4(3)$, $U_5(2)$, $U_6(2)$, $S_6(2)$, $O_8^+(2)$ and HS are characterizable by* NSE.

We conclude the introduction with some further notation and definitions to be used in the rest of this article. To every finite group G we associate a graph known as Gruenberg-Kegel graph (or prime graph) denoted by $\mathrm{GK}(G)$. For this graph, the vertex set is $\pi(G)$, and for two distinct vertices $p, q \in \pi(G)$, p connected to q by an edge if and only if $pq \in \pi_e(G)$. When p and q are adjacent vertices in $\mathrm{GK}(G)$, we will write $p \sim q$. For instance, the prime graphs associated with the simple groups mentioned in Main Theorem, are depicted in Fig. 1.



$\mathrm{GK}(U_4(3))$ $\mathrm{GK}(U_5(2))$ $\mathrm{GK}(S)$, *for* $S \in \{A_9, A_{10}, S_6(2), O_8^+(2)\}$

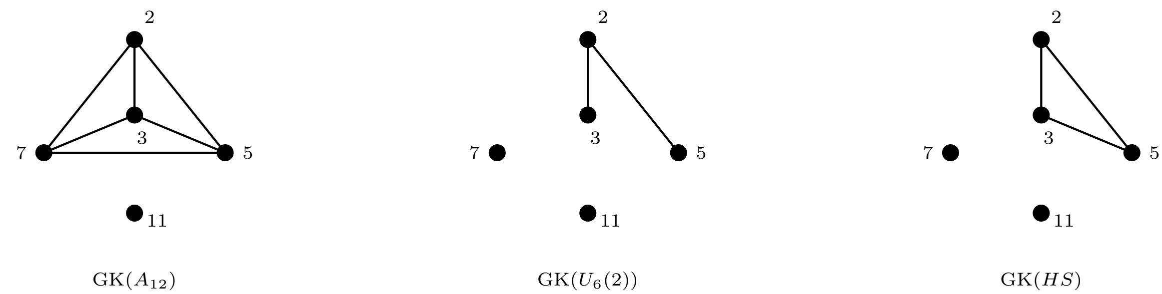


Fig. 1. Prime graphs associated with some simple groups.

The degree $\deg_G(p)$ of a vertex $p \in \pi(G)$ is the number of edges incident to p. When there is no ambiguity on the group G, we denote $\deg_G(p)$ simply by $\deg(p)$. If $\pi(G)$ consists of the primes $p_1, p_2, \ldots, p_h$ with $p_1 < p_2 < \cdots < p_h$, then we define

$$\mathrm{D}(G) = \big(\deg_G(p_1), \deg_G(p_2), \ldots, \deg_G(p_h)\big),$$

which is called the degree pattern of G. Given a finite group M, denote by $h_{\mathrm{OD}}(M)$ the number of isomorphism classes of finite groups G such that $|G| = |M|$ and $\mathrm{D}(G) = \mathrm{D}(M)$. A finite group M is called k-fold OD-characterizabale if $h_{\mathrm{OD}}(M) = k$. Usually, a 1-fold OD-characterizabale group is simply called OD-characterizabale. In [8], finite simple groups which are currently known to be k-fold OD-characterizable for $k \in \{1, 2\}$ are listed. Given a group G, we denote by G_p and $\mathrm{Syl}_p(G)$ a Sylow p-subgroup of G and the set of all Sylow p-subgroup of G, respectively, where $p \in \pi(G)$. In addition, we define $n_p(G)$ the number of all Sylow p-subgroups of G, that is $|\mathrm{Syl}_p(G)|$. The p-part of n is defined the largest power of p diving n which noticed by $|n|_p$, and finally, two notation $\varphi(n)$ and $c_m(G)$ are denoted in particular for the Euler's totient function, for positive integer n, and the number of distinct cyclic subgroups of order m of G. Occasionally, when the group we are considering is clear from the context, we will simply write c_m (resp. n_p and s_m) instead of $c_m(G)$ (resp. $n_p(G)$ and $s_m(G)$). All further unexplained notation is standard and refers to [3].

2. **Preliminaries**

In this section, we collect some of the results that will be needed later. We start with a fundamental result due to Frobenius, which is quoted frequently in next section.

Lemma 2.1 ([4]). *Let G be a finite group and m be a positive integer dividing the order of G. Let $L_m(G) = \{x \in G \mid x^m = 1\}$. Then m divides $|L_m(G)|$. Especially, if $p \in \pi(G)$, then $|L_p(G)| = 1 + s_p(G)$ which is divisible by p.*

Corollary 2.2. *Let G be a finite group and $p \in \pi(G)$. If $\mu(G_p) = \{p^t\}$, for some natural number t, then $|G_p| \mid |L_{p^t}(G)|$. Specially, in the case of $t = 1$, we must have $p \mid 1 + s_p(G)$.*

Proof. By Lemma 2.1, the proofs are straightforward. □

Remark 1. The definition of $L_m(G)$, indicates that if $1 = d_1 < d_2 < \cdots < d_k = m$ are all divisors of m , then $|L_m(G)| = \sum_{i=1}^{k} s_{d_i}(G).$, for $p \in \pi(G)$,

Lemma 2.3 ([7])**.** *Let G be a finite group and $p \in \pi(G)$ be odd. Suppose that $n = p^{\alpha}m$ with $(p,m) = 1$. If a Sylow p-subgroup G_p is not cyclic and $\alpha > 1$, then the number of elements of order n is always a multiple of p^{α}.*

Lemma 2.4 ([11])**.** *Let G be a group and P be a cyclic Sylow p-subgroup of G of order p^{α}. If there is a prime r such that $p^{\alpha}r \in \pi_e(G)$, then $s_{p^{\alpha}r} = s_r(C_G(P))s_{p^{\alpha}}$. In particular, $\varphi(r)s_{p^{\alpha}} \mid s_{p^{\alpha}r}$, where $\varphi(r)$ is the Euler function of r.*

Lemma 2.5 ([13])**.** *Let G be a group containing more than two elements. If the maximum number s of* $\mathrm{nse}(G)$ *is finite, then G is a finite group and especially $|G| \leq s(s^2 - 1)$.*

Remark 2. For a group G, it is clear that $s_m(G) = c_m(G)\varphi(m)$, for some natural number $m \in \pi_e(G)$. Hence $\varphi(m) \mid s_m(G)$.

Lemma 2.6. *Let G be a finite group, and $p \in \pi(G)$. If G_p is cyclic and $|G_p| = p^t$, for some natural number t, then $\frac{s_{p^t}}{\varphi(p^t)} \mid |G|$. In particular, $\frac{s_{p^t}}{\varphi(p^t)} \equiv 1 \pmod p$.*

Proof. Since G_p is cyclic, so $c_{p^t}(G) = n_p(G)$. Now, by Remark 2, and Sylow's theorems, we must have $n_p(G) = \frac{s_{p^t}}{\varphi(p^t)} \mid |G|$, and $n_p(G) = \frac{s_{p^t}}{\varphi(p^t)} \equiv 1 \pmod p$, as required. □

Lemma 2.7. *Let G be a finite group. then the following statements hold:*

(1) $\mathrm{nse}(G)$ *contains at most two odd numbers.*

(2) $\sum_m s_m(G) \leq |G|$.

(3) *If G is a 2-group, then $|\pi_e(G)| \leq \max\{|n|_2 + 2 : n \in nse(G)\}$.*

(4) *Let $\{p,q\} \subseteq \pi(G)$. If $pq \notin \pi_e(G)$, then $|G_p| \mid s_q(G)$.*

Proof. (1) Since $\varphi(m)$ is always even, for every natural number $m > 2$, and by Remark 2, it is clear that $\mathrm{nse}(G)$ contains only one odd number if and only if $|G|$ is odd or G has a unique involution(i.e $s_2(G) = 1$), otherwise, $\mathrm{nse}(G)$ has two odd numbers, that is 1 and $s_2(G)$.

(2) The function $m \to s_m$ from $\pi_e(G)$ to $\mathrm{nse}(G)$ is onto which implies $|\mathrm{nse}(G)| \leq |\pi_e(G)|$ and $\sum_m s_m(G) \leq |G|$. If it is injective then $\sum_m s_m(G) = |G|$.

(3) By Remark 2, if $\mu(G) = \{2^t\}$, for some natural number t, then $\varphi(2^t) = 2^{t-1} \mid s_{2^t}(G)$, therefore $|\pi_e(G)| \leq \max\{|n|_2 + 2 : n \in nse(G)\}$, as required.

(4) we consider the action of Sylow p-subgroup of G on all elements of order q by conjugation. since $pq \notin \pi_e(G)$, the action is fixed-point-free, therefore $|G_p| \mid s_q(G)$. □

3. Proofs

In this section, we are going to prove our Main Theorem. Since the proofs of all the groups A_9, A_{10}, A_{12}, $U_4(3)$, $U_5(2)$, $U_6(2)$, $S_6(2)$, $O_8^+(2)$ and HS are similar, we take HS as an example, and possibility small differences for other cases would be separately noted during the following proofs. To facilitate the proof, the order of HS and the factorization of $s_m(HS)(\neq 1)$ and $s_m(HS)+1$ are given in Table 1, and the rest of them are mentioned in Table 2. As a matter of fact, for every group $\mathcal{S} \in \{A_9, A_{10}, A_{12}, U_4(3), U_5(2), U_6(2), S_6(2), O_8^+(2)\}$, the values of $|\mathcal{S}|$ and the factorization of $s_m(\mathcal{S})(\neq 1)$ and $s_m(\mathcal{S})+1$ would be found in Table 2.

Table 1.

$\|HS\|$	$s_m(HS)$	$s_m(HS)+1$
$\|HS\| = 2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$	$21175 = 5^2 \cdot 7 \cdot 11^2$	$21176 = 2^3 \cdot 26471$
	$123200 = 2^6 \cdot 5^2 \cdot 7 \cdot 11$	$123201 = 3^6 \cdot 13^2$
	$877800 = 2^3 \cdot 3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 19$	$877801 = 29 \cdot 30269$
	$2010624 = 2^9 \cdot 3 \cdot 7 \cdot 11 \cdot 17$	$2010625 = 5^4 \cdot 3217$
	$2956800 = 2^9 \cdot 3 \cdot 5^2 \cdot 7 \cdot 11$	$2956801 = 131 \cdot 22571$
	$3080000 = 2^6 \cdot 5^4 \cdot 7 \cdot 11$	$3080001 = 3 \cdot 1026667$
	$3696000 = 2^7 \cdot 3 \cdot 5^3 \cdot 7 \cdot 11$	$3696001 = 3696001$
	$4435200 = 2^8 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11$	$4435201 = 31 \cdot 173 \cdot 827$
	$6336000 = 2^9 \cdot 3^2 \cdot 5^3 \cdot 11$	$6336001 = 7 \cdot 905143$
	$8064000 = 2^{10} \cdot 3^2 \cdot 5^3 \cdot 7$	$8064001 = 11 \cdot 17 \cdot 29 \cdot 1487$
	$8316000 = 2^5 \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 11$	$8316001 = 8316001$

Let $\mathcal{S} \in \{A_9, A_{10}, A_{12}, U_4(3), U_5(2), U_6(2), S_4(7), S_6(2), O_8^+(2), HS\}$, and G be an arbitrary group that $\mathrm{nse}(G) = \mathrm{nse}(\mathcal{S})$. First of all, it follows, from Lemma 2.5, G is a finite group. We continue the proof in the numbers of lemmas.

Lemma 3.1. *The order of G is even, and G is not a 2-group.*

Proof. By Table 1, 2, and Lemma 2.7 (1), it is easy to see that $s_2(G) = s_2(\mathcal{S})$, and G is even order. If G is a 2-group, and $\mathrm{nse}(G) = \mathrm{nse}(HS)$, then $|\pi_e(G)| \leqslant 12$, by Table 1, and Lemma 2.7 (3). On the other hand, Lemma 2.7 (2) concludes that $|\pi_e(G)| \geqslant 12$, which results

$|\pi_e(G)| = 12$, and

$$|G| = \sum s_m(G) = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11,$$

a contradiction. With the exact same proof, we would also have a contradiction, for other options of $\mathcal{S}$. $\square$

Table 2.

$\mathcal{S}$	$\lvert\mathcal{S}\rvert$ & $s_m \in \mathrm{nse}(\mathcal{S})$	$s_m + 1$	$\mathcal{S}$	$\lvert\mathcal{S}\rvert$ & $s_m \in \mathrm{nse}(\mathcal{S})$	$s_m + 1$
$U_4(3)$	$\lvert U_4(3)\rvert = 2^7 \cdot 3^6 \cdot 5 \cdot 7$		A_9	$\lvert A_9\rvert = 2^6 \cdot 3^4 \cdot 5 \cdot 7$	
	$2835 = 3^4 \cdot 5 \cdot 7$	$2836 = 2^2 \cdot 709$		$1323 = 3^3 \cdot 7^2$	$1324 = 2^2 \cdot 331$
	$47600 = 2^4 \cdot 5^2 \cdot 7 \cdot 17$	$47601 = 3^3 \cdot 41 \cdot 43$		$3024 = 2^2 \cdot 331$	$3025 = 5^2 \cdot 53$
	$226800 = 2^4 \cdot 3^4 \cdot 5^2 \cdot 7$	$226801 = 337 \cdot 673$		$5768 = 2^3 \cdot 7 \cdot 103$	$5769 = 3^2 \cdot 641$
	$238140 = 2^2 \cdot 3^5 \cdot 5 \cdot 7^2$	$238141\, is\, prime$		$9072 = 2^4 \cdot 3^4 \cdot 7$	$9073 = 43 \cdot 211$
	$272160 = 2^5 \cdot 3^5 \cdot 5 \cdot 7$	$272161 = 127 \cdot 2143$		$15120 = 2^4 \cdot 3^3 \cdot 5 \cdot 7$	$15121 = 15121$
	$408240 = 2^4 \cdot 3^6 \cdot 5 \cdot 7$	$408241\, is\, prime$		$18900 = 2^2 \cdot 3^3 \cdot 5^2 \cdot 7$	$18901 = 41 \cdot 461$
	$483840 = 2^9 \cdot 3^3 \cdot 5 \cdot 7$	$483841 = 41 \cdot 11801$		$24192 = 2^7 \cdot 3^3 \cdot 7$	$24193 = 13 \cdot 1861$
	$653184 = 2^7 \cdot 3^6 \cdot 7$	$653185 = 5 \cdot 132 \cdot 773$		$25920 = 2^6 \cdot 3^4 \cdot 5$	$25921 = 7^2 \cdot 23^2$
	$933120 = 2^8 \cdot 3^6 \cdot 5$	$933121 = 7 \cdot 133303$		$37800 = 2^3 \cdot 3^3 \cdot 5^2 \cdot 7$	$37801 = 103 \cdot 367$
				$40320 = 2^7 \cdot 3^2 \cdot 5 \cdot 7$	$40321 = 61 \cdot 661$
$S_6(2)$	$\lvert S_6(2)\rvert = 2^9 \cdot 3^4 \cdot 5 \cdot 7$		A_9	$\lvert A_{10}\rvert = 2^7 \cdot 3^4 \cdot 5^2 \cdot 7$	
	$5103 = 3^6 \cdot 7$	$5104 = 2^4 \cdot 11 \cdot 29$		$5355 = 3^2 \cdot 5^7 \cdot 17$	$5356 = 2^2 \cdot 13 \cdot 103$
	$16352 = 2^5 \cdot 7 \cdot 73$	$16353 = 3^2 \cdot 23 \cdot 79$		$31040 = 2^6 \cdot 5 \cdot 97$	$31041 = 3^2 \cdot 3449$
	$48384 = 2^8 \cdot 3^3 \cdot 7$	$48385 = 5 \cdot 9677$		$78624 = 2^5 \cdot 3^3 \cdot 7 \cdot 13$	$78625 = 5^3 \cdot 17 \cdot 37$
	$75600 = 2^4 \cdot 3^3 \cdot 5^2 \cdot 7$	$75601 = 19 \cdot 23 \cdot 173$		$86400 = 2^7 \cdot 3^3 \cdot 5^2$	$86401 = 7 \cdot 12343$
	$96768 = 2^9 \cdot 3^3 \cdot 7$	$96769\, is\, prime$		$90720 = 2^5 \cdot 3^4 \cdot 5 \cdot 7$	$90721 = 257 \cdot 353$
	$145152 = 2^8 \cdot 3^4 \cdot 7$	$145153 = 23 \cdot 6311$		$94500 = 2^2 \cdot 3^3 \cdot 5^3 \cdot 7$	$94501 = 11^3 \cdot 71$
	$161280 = 2^9 \cdot 3^2 \cdot 5 \cdot 7$	$161281\, is\, prime$		$120960 = 2^7 \cdot 3^3 \cdot 5 \cdot 7$	$120961 = 73 \cdot 1657$
	$181440 = 2^6 \cdot 3^4 \cdot 5 \cdot 7$	$181441 = 13 \cdot 17 \cdot 821$		$172800 = 2^8 \cdot 3^3 \cdot 5^2$	$172801\, is\, prime$
	$207360 = 2^9 \cdot 3^4 \cdot 5$	$207361 = 7 \cdot 11 \cdot 2693$		$201600 = 2^7 \cdot 3^2 \cdot 5^2 \cdot 7$	$201601 = 449^2$
	$241920 = 2^8 \cdot 3^3 \cdot 5 \cdot 7$	$241921\, is\, prime$		$226800 = 2^4 \cdot 3^4 \cdot 5^2 \cdot 7$	$226801 = 337 \cdot 673$
	$272160 = 2^5 \cdot 3^5 \cdot 5 \cdot 7$	$272161 = 127 \cdot 2143$		$302400 = 2^6 \cdot 3^3 \cdot 5^2 \cdot 7$	$302401 = 11 \cdot 37 \cdot 743$
				$403200 = 2^8 \cdot 3^2 \cdot 5^2 \cdot 7$	$403201 = 191 \cdot 2111$
$O_8^+(2)$	$\lvert O_8^+(2)\rvert = 2^{12} \cdot 3^5 \cdot 5^2 \cdot 7$		$U_5(2)$	$\lvert U_5(2)\rvert = 2^{10} \cdot 3^5 \cdot 5 \cdot 11$	
	$69615 = 3^2 \cdot 5 \cdot 7 \cdot 13 \cdot 17$	$69616 = 2^4 \cdot 19 \cdot 229$		$3135 = 3 \cdot 5 \cdot 11 \cdot 19$	$3136 = 2^6 \cdot 7^2$
	$365120 = 2^6 \cdot 5 \cdot 7 \cdot 163$	$365121 = 3^3 \cdot 13523$		$56672 = 2^5 \cdot 7 \cdot 11 \cdot 23$	$56673 = 3^3 \cdot 2099$
	$1741824 = 2^{10} \cdot 3^5 \cdot 7$	$1741825 = 5^2 \cdot 19^2 \cdot 193$		$190080 = 2^7 \cdot 3^3 \cdot 5 \cdot 11$	$190081 = 131 \cdot 1451$

Table 2. (Continued)

$\mathcal{S}$	$\lvert\mathcal{S}\rvert$ & $s_m \in \mathrm{nse}(\mathcal{S})$	$s_m + 1$	$\mathcal{S}$	$\lvert\mathcal{S}\rvert$ & $s_m \in \mathrm{nse}(\mathcal{S})$	$s_m + 1$
	$5821200 = 2^4 \cdot 3^3 \cdot 5^2 \cdot 7^2 \cdot 11$	$5821201 = 19 \cdot 151 \cdot 2029$		$855360 = 2^6 \cdot 3^5 \cdot 5 \cdot 11$	$855361 = 13 \cdot 19 \cdot 3463$
	$10886400 = 2^8 \cdot 3^5 \cdot 5^2 \cdot 7$	$10886401\, is\, prime$		$912384 = 2^{10} \cdot 3^4 \cdot 11$	$912385 = 5 \cdot 251 \cdot 727$
	$19353600 = 2^{12} \cdot 3^3 \cdot 5^2 \cdot 7$	$19353601 = 773 \cdot 25037$		$1082400 = 2^5 \cdot 3 \cdot 5^2 \cdot 11 \cdot 41$	$1082401 = 601 \cdot 1801$
	$19857600 = 2^6 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 197$	$19857601\, is\, prime$		$1520640 = 2^{10} \cdot 3^3 \cdot 5 \cdot 11$	$1520641 = 113 \cdot 13457$
	$24883200 = 2^{12} \cdot 3^5 \cdot 5^2$	$24883201 = 7 \cdot 1789 \cdot 1987$		$1824768 = 2^{11} \cdot 3^4 \cdot 11$	$1824769 = 109 \cdot 16741$
	$26127360 = 2^{10} \cdot 3^6 \cdot 5 \cdot 7$	$26127361 = 13 \cdot 71 \cdot 28307$		$2488320 = 2^{11} \cdot 3^5 \cdot 5$	$2488321 = 11 \cdot 47 \cdot 4813$
	$30240000 = 2^8 \cdot 3^3 \cdot 5^4 \cdot 7$	$30240001 = 11 \cdot 19 \cdot 41 \cdot 3529$		$3231360 = 2^7 \cdot 3^3 \cdot 5 \cdot 11 \cdot 17$	$3231361 = 7 \cdot 97 \cdot 4759$
	$348364480 = 2^6 \cdot 5 \cdot 1088639$	$348364481\, is\, prime$			
$U_6(2)$	$\lvert U_6(2)\rvert = 2^{15} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11$		A_{12}	$\lvert A_{12}\rvert = 2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$	
	$312543 = 3^2 \cdot 7 \cdot 11^2 \cdot 41$	$312544 = 2^5 \cdot 9767$		$63855 = 3^3 \cdot 5 \cdot 11 \cdot 43$	$63856 = 2^4 \cdot 13 \cdot 307$
	$5046272 = 2^{16} \cdot 7 \cdot 11$	$5046273 = 3^3 \cdot 31 \cdot 6029$		$570240 = 2^7 \cdot 3^4 \cdot 5 \cdot 11$	$570241 = 7 \cdot 81463$
	$67525920 = 2^5 \cdot 3^3 \cdot 5 \cdot 7^2 \cdot 11 \cdot 29$	$67525921 = 17 \cdot 3972113$		$3825360 = 2^4 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	$3825361 = 29 \cdot 131909$
	$306561024 = 2^{14} \cdot 3^5 \cdot 7 \cdot 11$	$306561025 = 5^2 \cdot 47 \cdot 139 \cdot 1877$		$776600 = 2^3 \cdot 5^2 \cdot 11 \cdot 353$	$776601 = 3^3 \cdot 7^2 \cdot 587$
	$354816000 = 2^{12} \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$	$354816001 = 4483 \cdot 79147$		$3991680 = 2^7 \cdot 3^4 \cdot 5 \cdot 7 \cdot 11$	$3991681\, is\, prime$
	$1313832960 = 2^{15} \cdot 3^6 \cdot 5 \cdot 11$	$1313832961 = 7 \cdot 187690423$		$4809024 = 2^6 \cdot 3^3 \cdot 11^2 \cdot 23$	$4809025 = 5^2 \cdot 13 \cdot 14797$
	$1005903360 = 2^9 \cdot 3^6 \cdot 5 \cdot 7^2 \cdot 11$	$1005903361 = 67 \cdot 15013483$		$6652800 = 2^7 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11$	$6652801\, is\, prime$
	$681246720 = 2^{16} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11$	$681246721 = 43 \cdot 15842947$		$8553600 = 2^7 \cdot 3^5 \cdot 5^2 \cdot 11$	$8553601 = 7 \cdot 17 \cdot 71879$
	$919683072 = 2^{14} \cdot 3^6 \cdot 7 \cdot 11$	$919683073 = 101 \cdot 127 \cdot 71699$		$11404800 = 2^9 \cdot 3^4 \cdot 5^2 \cdot 11$	$11404801 = 29^2 \cdot 71 \cdot 191$
	$1672151040 = 2^{16} \cdot 3^6 \cdot 5 \cdot 7$	$1672151041 = 11 \cdot 1973 \cdot 77047$		$11975040 = 2^7 \cdot 3^5 \cdot 5 \cdot 7 \cdot 11$	$11975041 = 13 \cdot 921157$
	$613122048 = 2^{15} \cdot 3^5 \cdot 7 \cdot 11$	$613122049 = 277 \cdot 281 \cdot 7877$		$21621600 = 2^5 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	$21621601 = 19 \cdot 31 \cdot 36709$
	$1021870080 = 2^{15} \cdot 3^4 \cdot 5 \cdot 7 \cdot 11$	$1021870081 = 31 \cdot 32963551$		$25530120 = 2^3 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 307$	$25530121 = 29 \cdot 880349$
	$1234759680 = 2^{12} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 29$	$1234759681 = 1217 \cdot 1014593$		$13685760 = 2^{10} \cdot 3^5 \cdot \cdot 5 \cdot 11$	$13685761 = 1283 \cdot 10667$
				$25945920 = 2^6 \cdot 3^4 \cdot 5 \cdot 7 \cdot 11 \cdot 13$	$25945921 = 4691 \cdot 5531$
				$26611200 = 2^9 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11$	$26611201 = 73 \cdot 364537$
				$29937600 = 2^6 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$	$29937601\, is\, prime$
				$43545600 = 2^{10} \cdot 3^5 \cdot 5^2 \cdot 7$	$43545601 = 11^2 \cdot 23 \cdot 15647$

Lemma 3.2. *$\pi(G) \subseteq \pi(\mathcal{S})$.*

Proof. Let $\mathcal{S} = HS$. Suppose that there exist a prime divisor of $|G|$ as p which $p \notin \pi(HS) = \{2, 3, 5, 7, 11\}$. By Corollary 2.2 and Table 1, p could be the following prime numbers:

$$p \in \{13, 17, 29, 31, 131, 173, 827, 1487, 3217, 22571, 30269, 905143, 1026667, 3696001, 8316001\}.$$

If $p = 13$, then by Table 1, and Corollary 2.2, $s_{13} = 123200$. On the other hand, Remark 2 indicates that $\varphi(13) = 12 \mid s_{13}$, which is a contradiction. With the same way, it would be easily concluded

$$p \notin \{131, 173, 827, 1487, 3217, 22571, 30269, 905143, 1026667\}.$$

Now, assume that $p = 29$, then Corollary 2.2 forces $s_{29} = 877800$ or 8064000. By Remark 2, if $29^t \in \pi_e(G)$, then $t = 1$, and $|G_{29}| \mid |L_{|G_{29}|}(G)| = 1 + s_{29}$, by Corollary 2.2, which results $|G_{29}| = 29$. Now, by Lemma 2.6, $\frac{s_{29}}{\varphi(29)} \mid |G|$. If $s_{29} = 877800$, then $19 \mid |G|$, which is a contradiction, by the above discussion, so $s_{29} = 8064000$, and $2^8 \cdot 3^2 \cdot 5^3 \mid |G|$. We claim that $5 \cdot 29 \notin \pi_e(G)$. Otherwise, by Lemma 2.4, $s_{5 \cdot 29} \geqslant \varphi(5) \cdot s_{29} = 32256000$, which is a contradiction, by Table 1. Therefore, by Lemma 2.7 (4), we have $|G_{29}| \mid s_5$. On the other hand, by Corollary 2.2, $s_5 = 2010624$, which is a contradiction. With the similar argument, one can straightly get that p also can not be 17, or 31. Finally, we are going to show that $p \notin \{3696001, 8316001\}$. By the contrary, assume that $p \in \{3696001, 8316001\}$, so $s_p = p - 1$. If $2p \in \pi_e(G)$, then by Lemma 2.1

$$2p \mid |L_{2p}(G)| = 1 + s_2 + s_p + s_{2p},$$

which is a contradiction, by Table 1. Therefore by Lemma 2.7 (4), $p \mid s_2 = 21175$, again, we have a contradiction. So $\pi(G) \subseteq \pi(HS)$, as required. With the same argument, the result would be concluded for other cases of $\mathcal{S}$. □

Lemma 3.3. *If p is the largest prime divisor of the order of $\mathcal{S}$, then $p \mid |G|$.*

Proof. Assume that p and q are the largest prime divisor of $|\mathcal{S}|$ and $|G|$, respectively. We are going to show that $p = q$. In order to avoid confusion, because of the long proof, we dividing the proof to several steps, and use symbols p and q as we just mentioned.

Step 1. At the first step, we are going to show $q \neq 3$. As we mentioned before, because of the same discussion, we only illustrate the proof for the case $\mathcal{S} = HS$. By the contrary, suppose that $q = 3$, hence G is a $\{2, 3\}$-group. Therefore Table 1 and Sylow's theorem straightly conclude that G_2 and G_3 are not cyclic. At the following, we are going to find the maximum value of $|G_2|$ and $|G_3|$. We get start with finding the upper bound of $|G_2|$. By Remark 2 and Table 1, if $2^t \in \pi_e(G)$, then $t \leqslant 11$. Therefore, Corollary 2.1 concludes that:

$$|G_2| \mid \sum_{t=0}^{11} s_{2^t}.$$

We know $1 + s_2 = 1 + 21175 = 2^3 \cdot 2647$, so for maximizing the order of G_2, we must have $|s_{2^{k_1}}|_2 = 3$, for some $2 \leqslant k_1 \leqslant 4$. By Table 1, the only option is 877800. Hence we have $1 + s_2 + s_{2^{k_1}} = 2^5 \cdot 28093$. With the same above reason, $|s_{2^{k_2}}|_2 = 5$, or $\sum |s_{2^{k_i}}|_2 = 5$, that concludes

$$2^5 \cdot 28093 + 8316000 = 2^{10} \cdot 8999.$$

Again, we must have $|s_{2^{k_3}}|_2 = 10$, or $\sum |s_{2^{k_i}}|_2 = 10$. With applying these processes, simple calculation, and also with the fact that $\varphi(2^t) \mid s_{2^t}$, finally we must have:

$$|G_2| \leqslant 2^{14} \tag{1}$$

Now, we are going to estimate the upper bound of $|G_3|$. By Remark 2, if $3^t \in \pi_e(G)$, then $t \leqslant 3$. On the other hand, Table 1 and Corollary 2.2 force that $s_3 \in \{123200, 3080000\}$. Again, Corollary 2.2 concludes

$$|G_3| \mid 1 + s_3 + s_{3^2} + s_{3^3} = \begin{cases} 3^6 \cdot 13^2 + s_{3^2} + s_{3^3} & \text{if} \quad s_3 = 123200, \\ 3 \cdot 1026667 + s_{3^2} + s_{3^3} & \text{if} \quad s_3 = 3080000. \end{cases}$$

With the same discussion, we just mentioned, and simple calculations, we have:

$$|G_2| \leqslant 3^6 \tag{2}$$

Therefore, by the equation (1) and (2), $|G| \leqslant 2^{14} \cdot 3^6 \lneqq \sum s_m = 39916800$, which is a contradiction, so G is not a $\{2,3\}$-group, as required.

Step 2. In the second step, we are going to show that $q \in \pi(s_2(G))$. In this step, the proof would be divided in several cases, and in every case, the groups with similar arguments would be discussed.

Case 1. If $\mathcal{S} \in \{U_4(3), O_8^+(2), U_5(2), A_{10}, A_{12}\}$, then by Lemma 3.2 and Table 2, there is nothing to prove, since for the case $U_4(3)$, for instance, we have $s_2 = 2835 = 3^4 \cdot 5 \cdot 7$, so, clearly $q \in \pi(s_2(G))$.

Case 2. If $\mathcal{S} = HS$ and $q \nmid s_2$, then $q = 3$. However, by Step 1, obviously $q \neq 3$, so $q \in \pi(s_2(G)) = \pi(s_2(HS))$, as required.

Case 3. If $\mathcal{S} \in \{S_6(2), U_6(2)\}$, and $q \nmid s_2$, then $q = 5$. We illustrate the proof for the case $U_6(2)$ only, because of the same discussion. By Lemma 2.1, and Table 2,

$$s_5(G) = s_5(U_6(2)) = 306561024.$$

By Remark 2 and Table 2, if $5^t \in \pi_e(G)$, then $t \leqslant 4$. In the following, we are going to consider the various possibility of t, separately.

- Let $t \geqslant 3$. If G_3 is not cyclic, then by Lemma 2.3 and Table 2, $t \neq 4$, and $s_{5^i} = 354816000$, for $i \geq 2$. Therefore by Corollary 2.2, we have

  $$5^4 \mid 1 + s_5 + s_{5^2} + s_{5^3} = 1016193025,$$

  which is a contradiction. Hence G_5 is cyclic. By applying Remark 2 and Table 2, $s_{5^i} = 354816000$, for $i \geq 2$. On the other hand, Lemma 2.6 indicates that

  $$n_5(G) = \frac{s_{5^i}}{\varphi(5^i)} \mid |G|,$$

  for $i = 3$ or $i = 4$, which contradicts with $q = 5$.
- Let $t = 2$. If G_5 is not cyclic, then Lemma 2.3 results $s_{5^2} = 354816000$. By applying Lemma 2.1, we have

  $$5^3 \mid 1 + s_5 + s + 5^2 = 661377025,$$

  which is a contradiction, so G_5 is cyclic, and $|G_5| = 5^2$. By Lemma sylow, we have $n_5(G) = \frac{s_{5^2}}{\varphi(5^2)} \mid |G|$. However, for every value of $s_{5^2} \in \mathrm{nse}(G)(=\mathrm{nse}(U_6(2)))$, we conclude that 7, or 11 divides the order of G, which contradicts with $q = 5$.
- Finally, let $t = 1$. By Corollary 2.2, $s_5 = 30656102$, and

  $$|G_5| \mid 1 + s_5 = 306561025 = 5^2 \cdot 47 \cdot 139 \cdot 1877,$$

  which concludes $|G_5| = 5$, or 5^2. If $|G_5| = 5$, then

  $$n_5(G) = \frac{s_5}{\varphi(5)} = \frac{306561024}{4} = 26^{11} \cdot 3^5 \cdot 7 \cdot 11 \mid |G|,$$

  by Lemma 2.6, which is a contradiction. So $|G_5| = 5^2$. By Lemma 2.1, we have

  $$2 \cdot 5^2 \mid |L_{2 \cdot 5^2}(G)| = 1 + s_2 + s_5 + s_{2 \cdot 5} = 1 + 312543 + 306561024 + s_{2 \cdot 5},$$

  however, we have a contradiction, for every choices of $s_{2 \cdot 5} \in \mathrm{nse}(G)(=\mathrm{nse}(U_6(2)))$.

At the end, by above discussion, it would be concluded $q \mid s_2(G)$, as required.

Step 3. Now, we are going to show that $p = q$. We illustrate the proof for the case HS, only, the other proofs of $\mathcal{S}$ are similar.

By Step 1, 2, we have $q \in \{7, 11\}$. We claim that $q \neq 7$. By the contrary, assume that $q = 7$. By Lemma 2.1, it is easy to see that $s_7 = 6336000$. If $7^t \in \pi_e(G)$, then $t \leqslant 2$, by Remark 2 and Table 1. For $t = 2$, Lemma 2.3 forces that G_7 is cyclic. Therefor, by Corollary 2.2 and Table 1, we have $s_{7^2} = 8064000$. However, Lemma 2.6 concludes that $\frac{s_{7^2}}{\varphi(7^2)} \equiv 1 \pmod 7$, which is a contradiction, and so $t = 1$. In this case, by Lemma 2.6, $\frac{s_7}{\varphi(7)} = 2^8 \cdot 3 \cdot 5^3 \cdot 11 \mid |G|$, which contradicts with $q = 7$. Therefore $q = 11 = p$, as required. □

Lemma 3.4. *$\pi(G) = \pi(S)$.*

Proof. Since the proof of this lemma for the case $U_5(2)$ has more complexity than the others, we are going to represent it for $U_5(2)$. The other proofs of $\mathcal{S}$, are the same and more simple.

By Lemma 3.3, the largest prime divisor of $|G|$ is 11. By Lemma 2.1, $s_{11} = 2488320$. Our initial goal is to show that G_{11} is cyclic and $|G_{11}| = 11$. Suppose that $11^t \in \pi_e(G)$, then by Table 2 and Remark 2, $t \leqslant 2$. Let $t = 2$, then

$$s_{11^2} \in \{190080, 855360, 1082400, 1520640, 3231360\}.$$

If G_{11} is not cyclic, then by Lemma 2.3, $11^2 | s_{11^2}$, which contradicts with every above value of s_{11^2}. So G_{11} is cyclic and $|G_{11}| = 11^2$. Therefore $s_{11^2} \notin \{855360, 1520640, 1082400, 3231360\}$, since by Lemma 2.6 we have $\frac{s_{11^2}}{\varphi(11^2)} \equiv 1 \pmod{11}$, and $\frac{s_{11^2}}{\varphi(11^2)} \mid |G|$. If $s_{11^2} = 190080$, then

$$2488320 = s_{11} \leqslant \varphi(11) \cdot n_{11}(G) = 10 \cdot 1728 = 2^7 \cdot 3^3 \cdot 5,$$

which is a contradiction. Therefore, $t = 1$, and by Lemma 2.1,

$$|G_{11}| \mid 1 + s_{11} = 1 + 2488320 = 11 \cdot 47 \cdot 481311,$$

which implies $|G_{11}| = 11$, as desired. So, by Lemma 2.6, we have $\frac{s_{11}}{\varphi(11)} = 2^{10} \cdot 3^5 \mid |G|$, which implies $\pi(G) = \{2, 3, 11\}$, or $\pi(G) = \{2, 3, 5, 11\}$, by Lemma 3.2 (for other option of $\mathcal{S}$, here, we straightly conclude $\pi(G) = \pi(\mathcal{S})$). We claim that $\pi(G) \subsetneqq \{2, 3, 11\}$. By the contrary, assume that $\pi(G) = \{2, 3, 11\}$. We are going to find an uppet bound for $|G_2|$, and $|G_3|$.

If $2 \cdot 11 \in \pi_e(G)$, then by Lemma 2.4, $s_{2\cdot 11} \geqslant s_{11} \cdot \varphi(2) = 2488320$. Therefore $s_{2\cdot 11} \in \{2488320, 3231360\}$. Since

$$2 \cdot 11 \mid 1 + s_2 + s_{11} + s_{2\cdot 11} = 1 + 3135 + 2 \cdot 2488320,$$

by Lemma 2.1, we must have $s_{2\cdot 11} \neq 2488320$, which is a contradiction. If $s_{2\cdot 11} = 3231360$, then by Lemma 2.4,

$$s_2(C_G(G_{11})) = \frac{s_{2\cdot 11}}{s_{11}} = \frac{3231360}{2488320},$$

which is not an integer number, again we have a contradiction. Therefore, $2 \cdot 11 \notin \pi_e(G)$, and so by Lemma 2.6, we have $|G_2| \mid s_{11} = 2488320$, which implies $|G_2| \leqslant 2^{11}$.

Now, if $3 \cdot 11 \in \pi_e(G)$, then, by Lemma 2.4,

$$s_{3\cdot 11} \geqslant s_{11} \cdot \varphi(3) = 2 \cdot 2488320,$$

which is a contradiction, by Table 2. Therefore $3 \cdot 11 \notin \pi_e(G)$, and so by Lemma 2.6, we have $|G_3| \mid s_{11} = 2488320$, which implies $|G_3| \leqslant 3^5$.

By above argument, we have

$$|G| \leqslant 2^{11} \cdot 3^5 \cdot 11 \lneqq \sum s_m(G) = 12165120,$$

a contradiction. So $\pi(G) = \{2, 3, 5, 11\}$, as it was claimed. $\square$

Lemma 3.5. $G \cong \mathcal{S}$.

Proof. At the end, we are going to prove that $G \cong \mathcal{S}$, which complete the NSE characterizability of $\mathcal{S}$. Initially, we show that $\mathrm{D}(G) = \mathrm{D}(\mathcal{S})$. We illustrate the proof for the case HS, only, the other cases of $\mathcal{S}$ have the same argument.

By Lemma 2.1, and Table 1, we must have $s_2 = 21175$, $s_3 \in \{123200, 6336000\}$, $s_5 = 2010624$, $s_7 = 6336000$, and $s_{11} = 8064000$. By the same argument mentioned in the proof of Lemma 3.4, we conclude that $r \cdot 11 \notin \pi_e(G)$, for $r \in \{2, 3, 5, 7\}$, so $\deg(11) = 0$, and

$$|G_r| \mid s_{11} = 8064000 = 2^{10} \cdot 3^2 \cdot 5^3 \cdot 7,$$

for $r \in \{2, 3, 5, 7\}$, by Lemma 2.6. Therefore

$$|G| = |G_2| \cdot |G_3| \cdot |G_5| \cdot |G_7| \cdot |G_{11}| \leqslant 2^{10} \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11,$$

and, G_7 and G_{11} are cyclic. With the exact argument represented in the proof of Lemma 3.4, we have $\deg(7) = \deg(11) = 0$.

Now, we are going to discuss on the degree of the vertex 2 in $\mathrm{GK}(G)$. By Lemma 2.1, we have

$$2 \cdot 3 \mid 1 + s_2 + s_3 + s_{2\cdot 3} = \begin{cases} 1 + 21175 + 123200 + s_{2\cdot 3} & \text{if} \quad s_3 = 123200, \\ 1 + 21175 + 3080000 + s_{2\cdot 3} & \text{if} \quad s_3 = 3080000. \end{cases}$$

In both cases we have a contradiction, if $s_{2\cdot 3} = 0$, so $2 \sim 3$ in $\mathrm{GK}(G)$. In addition, if $2 \cdot 5 \notin \pi_e(G)$, then by Lemma 2.6, $|G_5| \leqslant 5^2$. Therefore

$$|G| \leqslant 2^{10} \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11 \lneqq \sum s_m(G),$$

which is a contradiction, so $2 \sim 5$ in $\mathrm{GK}(G)$, and $\deg(2) = 2$.

With the same mentioned argument, one can easily conclude $3 \sim 5$ in $\mathrm{GK}(G)$, so the degree pattern of G is

$$\mathrm{D}(G) = \big(\deg_G(2), \deg_G(3), \deg_G(5), \deg_G(7), \deg_G(11)\big) = \big(2, 2, 2, 0, 0\big),$$

which is equal to the degree pattern of HS in $\mathrm{GK}(HS)$ (see Fig. 1. and [8, Table 2]).

Now, it would be shown that $|G| = |HS|$. Since $2 \cdot 7 \notin \pi_e(G)$, so $|G| \leqslant 2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$, by Lemma 2.6. On the other hand, we know that $39916800 = \sum s_m(G) \leqslant |G|$, which implies

$$|G| = |HS| = 2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11.$$

Finally, by the fact that the simple group HS is OD-characterizabale (infact, all non-abelian simple groups introduced in $\mathcal{S}$ are OD-characterizabale, see [8]), our proof would be completed, so $G \cong HS$, as it was desired. $\square$

4. Acknowledgments

The authors wish to sincerely thank the referees for several useful comments.

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