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Research Paper

ON THE LOCAL-GLOBAL PRINCIPLES FOR THE $CD_{<n}$ OF LOCAL COHOMOLOGY MODULES

MARZIYEH HATAMKHANI* AND HAJAR ROSHAN-SHEKALGOURABI

Abstract. The concept of Faltings' local-global principle for $CD_{<n}$ of local cohomology modules over a Noetherian ring R is introduced, and it is shown that this principle holds at levels 1, 2 over local rings. We also establish the same principle at all levels over an arbitrary Noetherian local ring of dimension not exceeding 3. These generalize the main results of Brodmann et al. in [9].

1. Introduction

Throughout this paper, R is a commutative Noetherian ring and $\mathfrak{a}$ is an ideal of R. For an R-module M, the ith local cohomology module of M with respect to $\mathfrak{a}$ is defined as

$$H^i_{\mathfrak{a}}(M) \cong \varinjlim_{n\in\mathbb{N}} \mathrm{Ext}^i_R(R/\mathfrak{a}^n, M).$$

For more details about the local cohomology, we refer the reader to [10].

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*Corresponding author

An important theorem in local cohomology is Faltings' local-global principle for the finiteness of local cohomology modules [12, Satz 1], which states that for a finitely generated R-module M and a positive integer r, the R-module $(H^i_{\mathfrak{a}}(M))_{\mathfrak{p}}$ is finitely generated for all $i \leq r$ and for all $\mathfrak{p} \in \operatorname{Spec}(R)$ if and only if the R-module $H^i_{\mathfrak{a}}(M)$ is finitely generated for all $i \leq r$.

Another formulation of Faltings' local-global principle, particularly relevant for this paper, is in terms of the generalization of the finiteness dimension $f_{\mathfrak{a}}(M)$ of M relative to $\mathfrak{a}$, where

$$f_{\mathfrak{a}}(M) := \inf\{i \in \mathbb{N}_0 | H^i_{\mathfrak{a}}(M) \textit{ is not finitely generated}\};$$

with the usual convention that the infimum of the empty set of integers is interpreted as ∞. In view of [10, 9.6.2 and 9.1.2], it follows that for a finitely generated R-module M,

$$\begin{aligned} f_{\mathfrak{a}}(M) &= \inf\{i \in \mathbb{N}_0 \mid \mathfrak{a}^t H^i_{\mathfrak{a}}(M) \neq 0 \textit{ for all } t \in \mathbb{N}_0\} \\ &= \inf\{f_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \mid \mathfrak{p} \in \operatorname{Spec}(R)\} \\ &= \inf\{f_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \mid \mathfrak{p} \in \operatorname{Supp}_R(M/\mathfrak{a}M) \textit{ and } \dim R/\mathfrak{p} \geq 0\}. \end{aligned}$$

Let n be a non-negative integer. As a generalization of the notion of $f_{\mathfrak{a}}(M)$, Bahmanpour et al. introduced in [7] the notion of the nth finiteness dimension $f^n_{\mathfrak{a}}(M)$ of M relative to $\mathfrak{a}$ by

$$f^n_{\mathfrak{a}}(M) := \inf\{f_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \mid \mathfrak{p} \in \operatorname{Supp}_R(M/\mathfrak{a}M) \textit{ and } \dim R/\mathfrak{p} \geq n\}.$$

Note that $f^n_{\mathfrak{a}}(M)$ is either a positive integer or ∞ and $f^0_{\mathfrak{a}}(M) = f_{\mathfrak{a}}(M)$.

More recently, Asadollahi and Naghipour in [1] introduced the class of in dimension $< n$ modules. If n is a non-negative integer, then M is said to be in dimension $< n$, if there is a finitely generated submodule N of M such that $\dim \operatorname{Supp}_R(M/N) < n$. They showed that if R is a complete local ring, $\mathfrak{a}$ an ideal of R, and M a finitely generated R-module, then for any $n \in \mathbb{N}_0$,

$$f^n_{\mathfrak{a}}(M) := \inf\{i \in \mathbb{N}_0 \mid H^i_{\mathfrak{a}}(M) \textit{ is not in dimension} < n\}.$$

The authors in [14] eliminated the complete local hypothesis in this result.

In [19], Zöschinger defined and investigated coatomic modules over commutative Noetherian rings. A module M is called *coatomic* if every proper submodule of M is contained in a maximal submodule of M. Over Noetherian rings, the class of coatomic modules is a Serre subcategory of the category of R-modules. Moreover, it is clear that every finitely generated R-module is coatomic and that every coatomic, artinian module has finite length. Also, if R is local, then the set of associated primes of any coatomic R-module is finite. ([19, Folgerung 2])

The above definitions motivate us to introduce the notions of $CD_{<n}$ and $C^n_{\mathfrak{a}}(M)$. For a non-negative integer n we say that M is $CD_{<n}$ if $\dim \operatorname{Supp}_R(M/C) < n$ for some coatomic submodule C of M. Also, we define

$$C^n_{\mathfrak{a}}(M) := \inf\{i \in \mathbb{N}_0 | \ H^i_{\mathfrak{a}}(M) \textit{ is not } CD_{<n}\}$$

and we prove the Faltings' Local-global principle for $CD_{<n}$ of local cohomology modules as follows:

Theorem 1.1. *Let $(R,\mathfrak{m})$ be a Noetherian local ring, $\mathfrak{a}$ an ideal of R, and M a finitely generated R-module. Then for any $n \in \mathbb{N}_0$,*

$$f_{\mathfrak{a}}^n(M) = C_{\mathfrak{a}}^n(M) = \inf\{i \in \mathbb{N}_0 |\ \dim(\mathfrak{a}^t H_{\mathfrak{a}}^i(M)) \geq n \ for\ all\ t \in \mathbb{N}\}.$$

In fact, Theorem 1.1 is a generalization of the main result of [1].

Let M be a finitely generated R-module and $\mathfrak{b}$ be a second ideal of R such that $\mathfrak{b} \subseteq \mathfrak{a}$. Based on [10, Definition 9.1.5], the $\mathfrak{b}$-finiteness dimension $f_{\mathfrak{a}}^{\mathfrak{b}}(M)$ of M relative to $\mathfrak{a}$ is defined by

$$f_{\mathfrak{a}}^{\mathfrak{b}}(M) := \inf\{i \in \mathbb{N}_0 \mid \mathfrak{b}^t H_{\mathfrak{a}}^i(M) \neq 0 \ for\ all\ t \in \mathbb{N}_0\}.$$

Note that $f_{\mathfrak{a}}^{\mathfrak{b}}(M)$ is either a positive integer or ∞ and $f_{\mathfrak{a}}^{\mathfrak{a}}(M) = f_{\mathfrak{a}}(M)$.

Brodmann et al. in [9] defined and studied the concept of the local-global principle for annihilation of local cohomology modules at level $r \in \mathbb{N}$ for the ideals $\mathfrak{a}$ and $\mathfrak{b}$ of R. We say that the local-global principle for the annihilation of local cohomology modules holds at level r if for every choice of ideals $\mathfrak{a}, \mathfrak{b}$ of R with $\mathfrak{b} \subseteq \mathfrak{a}$ and every choice of finitely generated R-module M, it is the case that

$$f_{\mathfrak{a}R_{\mathfrak{p}}}^{\mathfrak{b}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) > r \ for\ all\ \mathfrak{p} \in \operatorname{Spec}(R) \Longleftrightarrow f_{\mathfrak{a}}^{\mathfrak{b}}(M) > r.$$

It is shown in [9] that the local-global principle for the annihilation of local cohomology modules holds at levels $1, 2$, over an arbitrary commutative Noetherian ring R and at all levels whenever $\dim R \leq 4$.

The above definitions motivate us to introduce the notion of $C_{\mathfrak{a}}^{\mathfrak{b}}(M)^n$ by

$$C_{\mathfrak{a}}^{\mathfrak{b}}(M)^n := \inf\{i \in \mathbb{N}_0 |\ \mathfrak{b}^t H_{\mathfrak{a}}^i(M) \ is\ not\ CD_{<n} \ for\ all\ t \in \mathbb{N}\}.$$

Note that $C_{\mathfrak{a}}^{\mathfrak{b}}(M)^n$ is either a non-negative integer or ∞, and if M is a finitely generated R-module, in view of Corollaries 3.2 and 3.6, $C_{\mathfrak{a}}^{\mathfrak{a}}(M)^n = C_{\mathfrak{a}}^n(M)$ and $C_{\mathfrak{a}}^{\mathfrak{b}}(M)^0 = f_{\mathfrak{a}}^{\mathfrak{b}}(M)$. We say that the local-global principle for the $CD_{<n}$ of local cohomology modules holds at level $r \in \mathbb{N}$ if for every choice of ideals $\mathfrak{a}, \mathfrak{b}$ of R with $\mathfrak{b} \subseteq \mathfrak{a}$ and every choice of finitely generated R-module M, it is the case that

$$C_{\mathfrak{a}R_{\mathfrak{p}}}^{\mathfrak{b}R_{\mathfrak{p}}}(M_{\mathfrak{p}})^n > r \ for\ all\ \mathfrak{p} \in \operatorname{Spec}(R) \Longleftrightarrow C_{\mathfrak{a}}^{\mathfrak{b}}(M)^n > r.$$

Our main result in Section 3 is to show that the local-global principle for the $CD_{<n}$ of local cohomology modules over a commutative Noetherian local ring R holds at levels $1, 2$. We also establish the same principle at all levels over an arbitrary commutative Noetherian local ring of dimension not exceeding 3. Our tools for proving the main result in Section 3 is the following:

Theorem 1.2. *Suppose that $(R,\mathfrak{m})$ is a Noetherian local ring and let $\mathfrak{a},\mathfrak{b}$ be two ideals of R such that $\mathfrak{b}\subseteq\mathfrak{a}$. Assume that M is a finitely generated R-module and let r be a positive integer such that $\operatorname{Ext}^j(R/\mathfrak{b},H^i_{\mathfrak{a}}(M))$ is $CD_{<n}$ for all j and $i<r$. Then*

$$C^{\mathfrak{b}R_{\mathfrak{p}}}_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}})^n > r \ \textit{for all}\ \mathfrak{p}\in\operatorname{Spec}(R) \Longleftrightarrow C^{\mathfrak{b}}_{\mathfrak{a}}(M)^n > r.$$

Pursuing this point of view further we establish the following consequence of Theorem 1.2 which is an extension of the results of Brodmann et al. in [9, Corollary 2.3] and Raghavan in [17] for an arbitrary Noetherian local ring.

Corollary 1.3. *Let $(R,\mathfrak{m})$ be a Noetherian local ring and let $\mathfrak{a},\mathfrak{b}$ be two ideals of R such that $\mathfrak{b}\subseteq\mathfrak{a}$. Let M be a finitely generated R-module, and*

$$r\in\{1,\operatorname{grade}_{\mathfrak{a}}M, f_{\mathfrak{a}}(M), f^1_{\mathfrak{a}}(M),\ldots,f^n_{\mathfrak{a}}(M)\}.$$

Then

$$C^{\mathfrak{b}R_{\mathfrak{p}}}_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}})^n > r \ \textit{for all}\ \mathfrak{p}\in\operatorname{Spec}(R) \Longleftrightarrow C^{\mathfrak{b}}_{\mathfrak{a}}(M)^n > r.$$

2. FALTINGS' LOCAL-GLOBAL PRINCIPLE FOR THE $CD_{<n}$ OF LOCAL COHOMOLOGY MODULES

Recall that a class of R-modules is a *Serre subcategory* of the category of R-modules when it is closed under taking submodules, quotients and extensions. For example, the classes of Noetherian modules, Artinian modules, minimax modules and weakly Laskerian modules are Serre subcategories. Recall that an R-module M is called *minimax* if there is a finitely generated submodule N of M such that M/N is Artinian([20]). Also, M is said to be *weakly Laskerian* if $\operatorname{Ass}_R(M/N)$ is a finite set for each submodule N of M([11]). As in standard notation, we let $\mathcal{S}$ stand for a Serre subcategory of the category of R-modules.

The following lemma is needed in the sequel.

Lemma 2.1. *Suppose that M is a finitely generated R-module and $N\in\mathcal{S}$. Then $\operatorname{Ext}^i_R(M,N)\in\mathcal{S}$ and $\operatorname{Tor}^R_i(M,N)\in\mathcal{S}$ for all $i\geq 0$.*

Proof. It follows from the definition of Ext and Tor modules. □

Lemma 2.2. *Let M be a finitely generated R-module and N be an arbitrary R-module. Suppose that for some $t\geq 0$, $\operatorname{Ext}^i_R(M,N)\in\mathcal{S}$ for all $i\leq t$. Then for any finitely generated R-module L with $\operatorname{Supp}_R(L)\subseteq\operatorname{Supp}_R(M)$, $\operatorname{Ext}^i_R(L,N)\in\mathcal{S}$ for all $i\leq t$.*

Proof. See [2, Lemma 2.2]. □

In [1], Asadollahi and Naghipour introduced the class of in dimension $< n$ modules and they have given some properties of this modules. If n is a non-negative integer, then M is said to be in dimension $< n$, if there is a finitely generated submodule N of M such that $\dim \operatorname{Supp}_R(M/N) < n$. This motivates the following definition.

Definition 2.3. *For a non-negative integer n we say that an R-module M is $CD_{<n}$, if $\dim \operatorname{Supp}_R(M/C) < n$ for some coatomic submodule C of M.*

Remark 2.4. *Let n be a non-negative integer and let M be an R-module.*

(i) *If $n = 0$, then M is $CD_{<n}$ if and only if M is coatomic.*
(ii) *If M is minimax, then M is $CD_{<1}$. In particular, if M is Noetherian or Artinian, then M is $CD_{<1}$.*
(iii) *If M is weakly Laskerian, then M is $CD_{<2}$, by* [5, Theorem 3.3]

Lemma 2.5. *For any non-negative integer n, the class of $CD_{<n}$ modules over a Noetherian ring R consists a Serre subcategory of the category of R-modules.*

Proof. The assertion follows from [18, Corollary 3.5]. □

Definition 2.6. *If T is an arbitrary subset of $\operatorname{Spec}(R)$ and $n \in \mathbb{N}_0$, then we set $(T)_{\geq n} := \{\mathfrak{p} \in T \mid \dim R/\mathfrak{p} \geq n\}$.*

Lemma 2.7. *Let $(R, \mathfrak{m})$ be a local ring and n be a non-negative integer. Then for every $CD_{<n}$ R-module M, the set $(\operatorname{Ass}_R(M))_{\geq n}$ is finite.*

Proof. Since M is $CD_{<n}$, there exists a coatomic submodule C of M such that $\dim \operatorname{Supp}_R(M/C) < n$. From the exact sequence

$$0 \longrightarrow C \longrightarrow M \longrightarrow M/C \longrightarrow 0$$

we get

$$\{\mathfrak{p} \in \operatorname{Ass}_R(M) \mid \dim R/\mathfrak{p} \geq n\} \subseteq \{\mathfrak{p} \in \operatorname{Ass}_R(C) \mid \dim R/\mathfrak{p} \geq n\} \cup \{\mathfrak{p} \in \operatorname{Ass}_R(M/C) \mid \dim R/\mathfrak{p} \geq n\}.$$

Since $\dim \operatorname{Supp}_R(M/C) < n$, we have $\{\mathfrak{p} \in \operatorname{Ass}_R(M/C) \mid \dim R/\mathfrak{p} \geq n\} = \emptyset$. Thus

$$\{\mathfrak{p} \in \operatorname{Ass}_R(M) \mid \dim R/\mathfrak{p} \geq n\} \subseteq \{\mathfrak{p} \in \operatorname{Ass}_R(C) \mid \dim R/\mathfrak{p} \geq n\} \subseteq \operatorname{Ass}_R(C).$$

Now, the proof is complete since $\operatorname{Ass}_R(C)$ is finite by [19, Folgerung 2.2]. □

The following result is a generalization of the main result of Quy [16], Brodmann and Lashgari [8], Assadollahi and Naghipour [1], and Mehrvarz et al. [14] for local rings.

Corollary 2.8. *Let $\mathfrak{a}$ be an ideal of a local ring $(R,\mathfrak{m})$ and M be a $CD_{<n}$ R-module. Let s be a non-negative integer such that $\operatorname{Ext}^j_R(R/\mathfrak{a}, H^i_{\mathfrak{a}}(M))$ is $CD_{<n}$ for all $i < s$ and all $j \geq 0$. Then the set $(\operatorname{Ass}_R(H^s_{\mathfrak{a}}(M)))_{\geq n}$ is finite.*

Proof. Note that $\operatorname{Hom}_R(R/a, H^s_{\mathfrak{a}}(M))$ is $CD_{<n}$ by [4, Theorem 2.2] and Lemma 2.5. Now, the result follows from Lemma 2.7 and the fact that

$$\operatorname{Ass}_R(\operatorname{Hom}_R(R/a, H^s_{\mathfrak{a}}(M))) = \operatorname{Ass}_R(H^s_{\mathfrak{a}}(M)).$$

□

Now, we define the notion $C^n_{\mathfrak{a}}(M)$ as follows:

Definition 2.9. *Let R be a Noetherian ring, $\mathfrak{a}$ be an ideal of R and M be an R-module. For a non-negative integer n we define the notion $C^n_{\mathfrak{a}}(M)$ as follows:*

$$C^n_{\mathfrak{a}}(M) = \inf\{i \in \mathbb{N}_0 |\ H^i_{\mathfrak{a}}(M) \text{ is not } CD_{<n}\}.$$

Corollary 2.10. *Let $\mathfrak{a}$ be an ideal of a local ring R, and M a $CD_{<n}$ R-module. Then $\operatorname{Hom}_R(R/\mathfrak{a}, H^{C^n_{\mathfrak{a}}(M)}_{\mathfrak{a}}(M))$ is $CD_{<n}$ and the set $(\operatorname{Ass}_R(H^{C^n_{\mathfrak{a}}(M)}_{\mathfrak{a}}(M)))_{\geq n}$ is finite.*

Proof. It follows from [4, Theorem 2.2] and Corollary 2.8. □

Lemma 2.11. *Let n be a non-negative integer and M a $CD_{<n}$ R-module. If $\mathfrak{p}$ is a prime ideal of R with $\dim R/\mathfrak{p} \geq n$, then $M_{\mathfrak{p}}$ is a coatomic $R_{\mathfrak{p}}$-module.*

Proof. By definition there is a coatomic submodule C of M such that $\dim \operatorname{Supp}_R(M/C) < n$. Therefore $(M/C)_{\mathfrak{p}} = 0$. Hence $M_{\mathfrak{p}} \cong C_{\mathfrak{p}}$ and so $M_{\mathfrak{p}}$ is a coatomic $R_{\mathfrak{p}}$-module by [19, Folgerung 1.2]. □

Lemma 2.12. *Let $(R,\mathfrak{m})$ be a local ring, $\mathfrak{a}$ be an ideal of R and n be a non-negative integer. If M is a $CD_{<n}$ R-module such that $\operatorname{Supp}_R(M) \subseteq V(\mathfrak{a})$, then there exists an integer t such that $\dim \mathfrak{a}^t M < n$.*

Proof. By Lemma 2.7 the set $(\operatorname{Ass}_R(M))_{\geq n}$ is finite. Assume that $(\operatorname{Ass}_R(M))_{\geq n} = \{\mathfrak{p}_1, \dots, \mathfrak{p}_k\}$. Since M is $CD_{<n}$, Lemma 2.11 follows that $M_{\mathfrak{p}_j}$ is coatomic $R_{\mathfrak{p}_j}$-module for all $1 \leq j \leq k$. By assumption, M is $\mathfrak{a}$-torsion and so by [19, Lemma 1.2] for any integer $1 \leq j \leq k$ there exists an integer t_j such that $(\mathfrak{a}^{t_j} M)_{\mathfrak{p}_j} = 0$. Let $t := \operatorname{Max}\{t_1, \dots, t_k\}$. It is easy to see that $\{\mathfrak{p}_1, \dots, \mathfrak{p}_k\} \cap \operatorname{Supp}_R(\mathfrak{a}^t M) = \emptyset$. We show that $\operatorname{Supp}_R(\mathfrak{a}^t M) \subseteq \{\mathfrak{p} \in \operatorname{Spec}(R) | \dim R/\mathfrak{p} < n\}$. If there exists a prime ideal $\mathfrak{p} \in \operatorname{Supp}_R(\mathfrak{a}^t M)$ such that $\dim R/\mathfrak{p} \geq n$, then there exists

$\mathfrak{q} \in \mathrm{Ass}_R(\mathfrak{a}^t M)$ such that $\mathfrak{q} \subseteq \mathfrak{p}$ and so $\dim R/\mathfrak{q} \geq n$. But $\mathfrak{q} \in \mathrm{Supp}_R(\mathfrak{a}^t M)$ and thus $\mathfrak{q} \in \{\mathfrak{p}_1, \ldots, \mathfrak{p}_k\} \cap \mathrm{Supp}_R(\mathfrak{a}^t M)$ which is a contradiction. This completes the proof. □

Theorem 2.13. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and n a non-negative integer. Let M be a finitely generated R-module and s be a positive integer. Then the following conditions are equivalent.*

(i) $H^i_{\mathfrak{a}}(M)$ *is* $CD_{<n}$ *for all* $i < s$;

(ii) *There exists an integer t such that* $\dim(\mathfrak{a}^t H^i_{\mathfrak{a}}(M)) < n$ *for all* $i < s$;

(iii) *There exists an integer t such that*

$$\mathrm{Supp}_R(\mathfrak{a}^t H^i_{\mathfrak{a}}(M)) \subseteq \{\mathfrak{p} \in \mathrm{Supp}_R(M/\mathfrak{a}M) | \dim R/\mathfrak{p} < n\}$$

for all $i < s$;

(iv) $(H^i_{\mathfrak{a}}(M))_{\mathfrak{p}}$ *is a finitely generated $R_{\mathfrak{p}}$-module for all $i < s$ and for all* $\mathfrak{p} \in \mathrm{Supp}_R(M/\mathfrak{a}M)$ *with* $\dim R/\mathfrak{p} \geq n$.

Proof. The implication $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$ immediately follows from Theorem 2.12 and assumption.

$(iii) \Rightarrow (iv)$: Let $\mathfrak{p} \in \mathrm{Supp}_R(M/\mathfrak{a}M)$ with $\dim R/\mathfrak{p} \geq n$. By assumption (iii), $\mathfrak{p} \notin \mathrm{Supp}_R(\mathfrak{a}^t H^i_{\mathfrak{a}}(M))$ for all $i < s$. Thus $(\mathfrak{a}^t H^i_{\mathfrak{a}}(M))_{\mathfrak{p}} = 0$ for all $i < s$. Therefore [10, Proposition 9.1.2] implies that $(H^i_{\mathfrak{a}}(M))_{\mathfrak{p}}$ is finitely generated $R_{\mathfrak{p}}$-module for all $i < s$.

$(iv) \Rightarrow (i)$: We prove the theorem by induction on s. The case of $s = 1$ is trivial since $H^0_{\mathfrak{a}}(M)$ is finitely generated. We assume that $s > 1$ and the theorem is true for $s - 1$. By induction, $H^0_{\mathfrak{a}}(M), \ldots H^{s-2}_{\mathfrak{a}}(M)$ are $CD_{<n}$. So we have to show that $H^{s-1}_{\mathfrak{a}}(M)$ is $CD_{<n}$. At first, we prove that there exists an integer t such that $\mathrm{Supp}_R(\mathfrak{a}^t H^{s-1}_{\mathfrak{a}}(M)) \subseteq \{\mathfrak{p} \in \mathrm{Supp}_R(M/\mathfrak{a}M) | \dim R/\mathfrak{p} < n\}$. Since $H^i_{\mathfrak{a}}(M)$ is $CD_{<n}$ for all $0 \leq i \leq s-2$, Corollary 2.8 implies that the set $(\mathrm{Ass}_R(H^{s-1}_{\mathfrak{a}}(M)))_{\geq n}$ is finite. Let $(\mathrm{Ass}_R(H^{s-1}_{\mathfrak{a}}(M)))_{\geq n} = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$. By the hypothesis, $(H^{s-1}_{\mathfrak{a}}(M))_{\mathfrak{p}_j}$ is a finitely generated $R_{\mathfrak{p}_j}$-module for all $1 \leq j \leq r$. Thus for any integer $1 \leq j \leq r$, there exists an integer t_j such that $(\mathfrak{a}^{t_j} H^{s-1}_{\mathfrak{a}}(M))_{\mathfrak{p}_j} = 0$.

Let $t := \mathrm{Max}\{t_1, \ldots, t_r\}$. Then $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\} \cap \mathrm{Supp}_R(\mathfrak{a}^t H^{s-1}_{\mathfrak{a}}(M)) = \emptyset$. We claim that $\mathrm{Supp}_R(\mathfrak{a}^t H^{s-1}_{\mathfrak{a}}(M)) \subseteq \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \dim R/\mathfrak{p} < n\}$. If there exists a prime ideal $\mathfrak{p} \in \mathrm{Supp}_R(\mathfrak{a}^t H^{s-1}_{\mathfrak{a}}(M))$ such that $\dim R/\mathfrak{p} \geq n$, then there exists $\mathfrak{q} \in \mathrm{Ass}_R(\mathfrak{a}^t H^{s-1}_{\mathfrak{a}}(M))$ such that $\mathfrak{q} \subseteq \mathfrak{p}$ and so $\dim R/\mathfrak{q} \geq n$. This implies that $\mathfrak{q} \in \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\} \cap \mathrm{Supp}_R(\mathfrak{a}^t H^{s-1}_{\mathfrak{a}}(M))$, a contradiction. Thus $\mathrm{Supp}_R(\mathfrak{a}^t H^{s-1}_{\mathfrak{a}}(M)) \subseteq \{\mathfrak{p} \in \mathrm{Supp}_R(M/\mathfrak{a}M) | \dim R/\mathfrak{p} < n\}$.

By [10, Corollary 2.1.7], $H^i_{\mathfrak{a}}(M) \cong H^i_{\mathfrak{a}}(M/\Gamma_{\mathfrak{a}}(M))$ for all $i > 0$ and by [10, Lemma 2.1.2], $M/\Gamma_{\mathfrak{a}}(M)$ is an $\mathfrak{a}$-torsion-free R-module. Thus we may assume that M is an $\mathfrak{a}$-torsion-free R-module. Then, in view of [10, Lemma 2.1.1], the ideal $\mathfrak{a}$ contains an element x which is

M-regular. Thus $x^t \in \mathfrak{a}^t$ is also M-regular. So $\dim \operatorname{Supp}_R(x^t H^{s-1}_{\mathfrak{a}}(M)) < n$. Moreover, we get the following exact sequence

$$H^{j-1}_{\mathfrak{a}}(M) \longrightarrow H^{j-1}_{\mathfrak{a}}(M/x^tM) \longrightarrow H^{j}_{\mathfrak{a}}(M) \xrightarrow{x^t} H^{j}_{\mathfrak{a}}(M).$$

Then by the hypothesis, $(H^{j-1}_{\mathfrak{a}}(M/x^tM))_{\mathfrak{p}}$ is finitely generated for all $j < s$ and for all $\mathfrak{p} \in \operatorname{Supp}_R(M/\mathfrak{a}M)$ with $\dim R/\mathfrak{p} \geq n$. From this we obtain by the inductive hypothesis that $H^{j-1}_{\mathfrak{a}}(M/x^tM)$ is $CD_{<n}$ for all $j < s$. In particular, $H^{s-2}_{\mathfrak{a}}(M/x^tM)$ is $CD_{<n}$. Therefore the exact sequence

$$H^{s-2}_{\mathfrak{a}}(M/x^tM) \longrightarrow (0 :_{H^{s-1}_{\mathfrak{a}}(M)} x^t) \longrightarrow 0,$$

follows that $(0 :_{H^{s-1}_{\mathfrak{a}}(M)} x^t)$ is $CD_{<n}$. We now consider the short exact sequence

$$0 \longrightarrow (0 :_{H^{s-1}_{\mathfrak{a}}(M)} x^t) \longrightarrow H^{s-1}_{\mathfrak{a}}(M) \longrightarrow x^t H^{s-1}_{\mathfrak{a}}(M) \longrightarrow 0.$$

Since $(0 :_{H^{s-1}_{\mathfrak{a}}(M)} x^t)$ is $CD_{<n}$ and $\dim \operatorname{Supp}_R(x^t H^{s-1}_{\mathfrak{a}}(M)) < n$, we obtain by the above exact sequence that $H^{s-1}_{\mathfrak{a}}(M)$ is $CD_{<n}$, as required. □

Now, we can prove the first main result of this paper, which shows that the least integer i, such that $H^i_{\mathfrak{a}}(M)$ is not $CD_{<n}$, equals to $f^n_{\mathfrak{a}}(M)$.

Corollary 2.14. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, $\mathfrak{a}$ an ideal of R, and M a finitely generated R-module. Then for any $n \in \mathbb{N}_0$,*

$$f^n_{\mathfrak{a}}(M) = C^n_{\mathfrak{a}}(M) = \inf\{i \in \mathbb{N}_0 |\ \dim(\mathfrak{a}^t H^i_{\mathfrak{a}}(M)) \geq n \ for\ all\ t \in \mathbb{N}\}.$$

Proof. The result follows immediately from Theorem 2.13. □

3. THE LOCAL-GLOBAL PRINCIPLE FOR THE $CD_{<n}$ OF LOCAL COHOMOLOGY MODULES

In this section we introduce the local-global principle for the $CD_{<n}$ of local cohomology modules as a generalization of the Faltings' local-global principle for the annihilation and for the in dimension $< n$ of local cohomology modules.

The following lemma is needed in the proof of Theorem 3.2.

Lemma 3.1. *Let $\mathcal{S}$ be a Serre subcategory of the category of R-modules, $\mathfrak{a}$ an ideal of R, and M be an arbitrary R-module. Then $\mathfrak{a}M$ belongs to $\mathcal{S}$ if and only if $M/(0 :_M \mathfrak{a})$ belongs to $\mathcal{S}$. In particular, $\mathfrak{a}M$ is $CD_{<n}$ if and only if $M/(0 :_M \mathfrak{a})$ is $CD_{<n}$, where n is a non-negative integer.*

Proof. This follows easily by induction on the number of generators of $\mathfrak{a}$ and the definition of Serre subcategory. □

Theorem 3.2. *Let $\mathcal{S}$ be a Serre subcategory of the category of R-modules, $\mathfrak{a}$ an ideal of R, and s be a positive integer. If M is an arbitrary R-module such that $\operatorname{Ext}_R^{s-1}(R/\mathfrak{a}, M) \in \mathcal{S}$, then the following statements are equivalent:*

(i) $H_{\mathfrak{a}}^i(M) \in \mathcal{S}$ *for all* $i < s$;

(ii) *There exists an integer* $t \geq 1$ *such that* $\mathfrak{a}^t H_a^i(M) \in \mathcal{S}$ *for all* $i < s$.

Proof. The implication $(i) \Rightarrow (ii)$ is obviously true. In order to show $(ii) \Rightarrow (i)$, we proceed by induction on s. If s = 1, then for some integere $t \geq 1$, $\mathfrak{a}^t H_{\mathfrak{a}}^0(M) \in \mathcal{S}$. Moreover, in view of the assumption, $\operatorname{Hom}_R(R/\mathfrak{a}, M) \in \mathcal{S}$. Now, since

$$\operatorname{Hom}_R(R/\mathfrak{a}, H_a^0(M)) \cong \operatorname{Hom}_R(R/\mathfrak{a}, M),$$

it follows that the R-module $(0 :_{H_{\mathfrak{a}}^0(M)} \mathfrak{a}) \in \mathcal{S}$. Therefore, it yields from Lemma 2.2 and Lemma 3.1 that $H_{\mathfrak{a}}^0(M) \in \mathcal{S}$. Suppose that $s > 1$, and the case $s - 1$ is settled. By inductive hypothesis the R-module $H_{\mathfrak{a}}^i(M) \in \mathcal{S}$ for all $i < s - 1$, and so it is enough to show that the R-module $H_{\mathfrak{a}}^{s-1}(M) \in \mathcal{S}$. For this purpose, as there is an integer $t \geq 1$ such that $\mathfrak{a}^t H_{\mathfrak{a}}^{s-1}(M) \in \mathcal{S}$, it follows from Lemma 3.1 that R-module $H_{\mathfrak{a}}^{s-1}(M)/(0 :_{H_{\mathfrak{a}}^{s-1}(M)} \mathfrak{a}^t) \in \mathcal{S}$. On the other hand, by virtue of Lemma 2.1, the R-module $\operatorname{Ext}_R^j(R/\mathfrak{a}, H_{\mathfrak{a}}^i(M)) \in \mathcal{S}$ for all $i < s - 1$ and all $j \geq 0$. Hence, it follows from [4, Theorem 2.2] that $\operatorname{Hom}(R/\mathfrak{a}, H_{\mathfrak{a}}^{s-1}(M)) \in \mathcal{S}$, and so in view of Lemma 2.2, $\operatorname{Hom}(R/\mathfrak{a}^t, H_{\mathfrak{a}}^{s-1}(M)) \in \mathcal{S}$. Consequently, it follows from Lemma 3.1 that the R-module $H_{\mathfrak{a}}^{s-1}(M) \in \mathcal{S}$, as required. □

Corollary 3.3. *Let $\mathfrak{a}$ be an ideal of R and M be a $CD_{<n}$ R-module. Then* $C_{\mathfrak{a}}^n(M) = \inf\{i \in \mathbb{N}_0 | \mathfrak{a}^t H_{\mathfrak{a}}^i(M) \ \textit{ is not } \ CD_{<n} \ \textit{for all} \ t \in \mathbb{N}\}$.

Proof. It follows from Theorem 3.2 and Lemma 2.5. □

In view of Corollary 3.3 it is natural to study about the greatest integer i such that $\mathfrak{b}^t H_{\mathfrak{a}}^i(M)$ is $CD_{<n}$ for some integer $t \in \mathbb{N}$, where $\mathfrak{b} \subseteq \mathfrak{a}$ are two ideals of R. This suggests that we introduce the notion of $\mathfrak{b} - CD_{<n}$ of M relative to $\mathfrak{a}$ (as a generalization of $\mathfrak{b}$-finiteness dimension $f_{\mathfrak{a}}^{\mathfrak{b}}(M)$ [10] and in b-dimension$< n$, $h_{\mathfrak{a}}^{\mathfrak{b}}(M)^n$ [15]).

Definition 3.4. *Let R be a Noetherian ring and M be an R-module. Let $\mathfrak{b} \subseteq \mathfrak{a}$ be two ideals of R. For a non-negative integer n, we define the $\mathfrak{b} - CD_{<n}$ of M relative to $\mathfrak{a}$, denoted by $C_{\mathfrak{a}}^{\mathfrak{b}}(M)^n$, by*

$$C_{\mathfrak{a}}^{\mathfrak{b}}(M)^n := \inf\{i \in \mathbb{N}_0 | \mathfrak{b}^t H_{\mathfrak{a}}^i(M) \ \textit{ is not } \ CD_{<n} \ \textit{for all} \ t \in \mathbb{N}\}.$$

Note that $C_{\mathfrak{a}}^{\mathfrak{b}}(M)^n$ is either a non-negative integer or ∞, and if M is a $CD_{<n}$ R-module then $C_{\mathfrak{a}}^{\mathfrak{a}}(M)^n = C_{\mathfrak{a}}^{n}(M)$ by Corollary 3.3.

Theorem 3.5. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, M be an arbitrary R-module and let $\mathfrak{a}, \mathfrak{b}$ be two ideals of R such that $\mathfrak{b} \subseteq \mathfrak{a}$. Then, for any non-negative integers i and n, the following statements are equivalent.*

(i) *There exists an integer t such that $\dim \operatorname{Supp}_R(\mathfrak{b}^t H_{\mathfrak{a}}^i(M)) < n$;*

(ii) *There exists an integer s such that $\mathfrak{b}^s H_{\mathfrak{a}}^i(M)$ is $CD_{<n}$.*

Proof. The implication $(i) \Rightarrow (ii)$ is clear. In order to show $(ii) \Rightarrow (i)$, since $\mathfrak{b}^s H_{\mathfrak{a}}^i(M)$ is $CD_{<n}$, it follows from Lemma 2.7 that the set $(\operatorname{Ass}_R(\mathfrak{b}^s H_{\mathfrak{a}}^i(M)))_{\geq n}$ is finite. Let $(\operatorname{Ass}_R(\mathfrak{b}^s H_{\mathfrak{a}}^i(M)))_{\geq n} = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$. Then, by Lemma 2.11 for all j with $1 \leq j \leq r$, the $R_{\mathfrak{p}_j}$-module $(\mathfrak{b}^s H_{\mathfrak{a}}^i(M))_{\mathfrak{p}_j}$ is coatomic. Hence, by [19, Lemma 1.2] there exists $t_j \in \mathbb{N}$ such that $(\mathfrak{a}^{t_j}(\mathfrak{b}^s H_{\mathfrak{a}}^i(M)))_{\mathfrak{p}_j} = 0$. Since $\mathfrak{b} \subseteq \mathfrak{a}$, it follows that $(\mathfrak{b}^{s+t_j} H_{\mathfrak{a}}^i(M))_{\mathfrak{p}_j} = 0$. Set $t := \max\{s + t_1, \ldots, s + t_r\}$. Then, for all $1 \leq j \leq r$, $(\mathfrak{b}^t H_{\mathfrak{a}}^i(M))_{\mathfrak{p}_j} = 0$. We show that $\dim \operatorname{Supp}_R(\mathfrak{b}^t H_{\mathfrak{a}}^i(M)) < n$. For all $\mathfrak{p} \in \operatorname{Ass}_R(\mathfrak{b}^t H_{\mathfrak{a}}^i(M))_{\geq n}$ we have $\mathfrak{p} \in \operatorname{Ass}_R(\mathfrak{b}^s H_{\mathfrak{a}}^i(M))$ and so there exists j such that $\mathfrak{p} = \mathfrak{p}_j$, a contradiction. This yeilds that $\dim \operatorname{Supp}_R(\mathfrak{b}^t H_{\mathfrak{a}}^i(M)) < n$, as required. □

Corollary 3.6. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and let $\mathfrak{a}, \mathfrak{b}$ be two ideals of R such that $\mathfrak{b} \subseteq \mathfrak{a}$. Then, for any non-negative integer n and R-module M,*

$$C_{\mathfrak{a}}^{\mathfrak{b}}(M)^n = \inf\{i \in \mathbb{N}_0 | \ \dim \operatorname{Supp}_R(\mathfrak{b}^t H_{\mathfrak{a}}^i(M)) \geq n \ for \ all \ t \in \mathbb{N}\}.$$

In particular, if M is finitely generated then $C_{\mathfrak{a}}^{\mathfrak{b}}(M)^0 = f_{\mathfrak{a}}^{\mathfrak{b}}(M)$.

Proof. The assertion follows from Theorem 3.5 and the definition of $C_{\mathfrak{a}}^{\mathfrak{b}}(M)^n$. □

We also introduce the local-global principle for the $CD_{<n}$ of local cohomology modules as follows:

Definition 3.7. *Let R be a commutative Noetherian ring and let r be a positive integer. For any non-negative integer n, we say that the local-global principle for the $CD_{<n}$ of local cohomology modules holds at level r (over the ring R) if, for every choice of ideals $\mathfrak{a}, \mathfrak{b}$ of R and for every choice of finitely generated R-module M, it is the case that*

$$C_{\mathfrak{a}R_{\mathfrak{p}}}^{\mathfrak{b}R_{\mathfrak{p}}}(M_{\mathfrak{p}})^n > r \ for \ all \ \mathfrak{p} \in \operatorname{Spec}(R) \Longleftrightarrow C_{\mathfrak{a}}^{\mathfrak{b}}(M)^n > r.$$

Theorem 3.8. *Suppose that* $(R, \mathfrak{m})$ *is a Noetherian local ring and let* $\mathfrak{a}, \mathfrak{b}$ *be two ideals of* R *such that* $\mathfrak{b} \subseteq \mathfrak{a}$*. Assume that* M *is a finitely generated* R*-module and let* r *be a positive integer such that* $\operatorname{Ext}^j(R/\mathfrak{b}, H^i_{\mathfrak{a}}(M))$ *is* $CD_{<n}$ *for all* j *and* $i < r$*. Then*

$$C^{\mathfrak{b}R_{\mathfrak{p}}}_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}})^n > r \text{ for all } \mathfrak{p} \in \operatorname{Spec}(R) \Longleftrightarrow C^{\mathfrak{b}}_{\mathfrak{a}}(M)^n > r.$$

Proof. Let $C^{\mathfrak{b}R_{\mathfrak{p}}}_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}})^n > r$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$ and let i be an arbitrary non-negative integer such that $i \leq r$. It is sufficient for us to show that there is a non-negative integer t_0 such that $\mathfrak{b}^{t_0} H^i_{\mathfrak{a}}(M)$ is $CD_{<n}$. Since $\mathfrak{b} \subseteq \mathfrak{a}$, it follows from Lemma 2.2 that $\operatorname{Ext}^j(R/\mathfrak{a}, H^i_{\mathfrak{a}}(M))$ is $CD_{<n}$ for all j and $i < r$, and so in view of [4, Theorem 2.2], the R-module $\operatorname{Hom}_R(R/\mathfrak{a}, H^i_{\mathfrak{a}}(M))$ is $CD_{<n}$ for all $i = 0, 1, \ldots, r$. Hence the set $(\operatorname{Ass}_R(H^i_{\mathfrak{a}}(M)))_{\geq n}$ is finite, by Lemma 2.7, and so $(\operatorname{Ass}_R(\mathfrak{b}^t H^i_{\mathfrak{a}}(M)))_{\geq n}$ is finite, for all $t \in \mathbb{N}_0$. Thus for all $t \in \mathbb{N}_0$, the set $(\operatorname{Supp}_R(\mathfrak{b}^t H^i_{\mathfrak{a}}(M)))_{\geq n}$ is a closed subset of $\operatorname{Spec}(R)$ (in the Zariski topology), and so the descending chain

$$\cdots \supseteq (\operatorname{Supp}_R(\mathfrak{b}^t H^i_{\mathfrak{a}}(M)))_{\geq n} \supseteq (\operatorname{Supp}_R(\mathfrak{b}^{t+1} H^i_{\mathfrak{a}}(M)))_{\geq n} \supseteq \cdots$$

is eventually stationary. Therefore there is a non-negative integer t_0 such that for each $t \geq t_0$,

$$(\operatorname{Supp}_R(\mathfrak{b}^t H^i_{\mathfrak{a}}(M)))_{\geq n} = (\operatorname{Supp}_R(\mathfrak{b}^{t_0} H^i_{\mathfrak{a}}(M)))_{\geq n}.$$

Now, consider $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $\dim R/\mathfrak{p} \geq n$. Since $C^{\mathfrak{b}R_{\mathfrak{m}}}_{\mathfrak{a}R_{\mathfrak{m}}}(M_{\mathfrak{m}})^n > r$, we deduce that there exists an integer $s \geq t_0$ such that $(\mathfrak{b}R_{\mathfrak{m}})^s H^i_{\mathfrak{a}R_{\mathfrak{m}}}(M_{\mathfrak{m}})$ is $CD_{<n}$. Hence in the light of the Lemma 2.11 and the isomorphism

$$((\mathfrak{b}R_{\mathfrak{m}})^s H^i_{\mathfrak{a}R_{\mathfrak{m}}}(M_{\mathfrak{m}}))_{\mathfrak{p}R_{\mathfrak{m}}} \cong (\mathfrak{b}^s H^i_{\mathfrak{a}}(M))_{\mathfrak{p}},$$

it follows that $(\mathfrak{b}^s H^i_{\mathfrak{a}}(M))_{\mathfrak{p}}$ is a coatomic $R_{\mathfrak{p}}$-module. Now, as $(\mathfrak{b}^s H^i_{\mathfrak{a}}(M))_{\mathfrak{p}}$ is $\mathfrak{a}R_{\mathfrak{p}}$-torsion, we infer by [19, Lemma 1.2] that there is an integer $u \geq 1$ such that $(\mathfrak{b}^{s+u} H^i_{\mathfrak{a}}(M))_{\mathfrak{p}} = 0$, and so $\mathfrak{p} \notin (\operatorname{Supp}_R(\mathfrak{b}^{s+u} H^i_{\mathfrak{a}}(M)))_{\geq n} = (\operatorname{Supp}_R(\mathfrak{b}^{t_0} H^i_{\mathfrak{a}}(M)))_{\geq n}$. Therefore,

$$\operatorname{Supp}_R(\mathfrak{b}^{t_0} H^i_{\mathfrak{a}}(M))) \subseteq \{\mathfrak{p} \in \operatorname{Spec}(R) \mid \dim R/\mathfrak{p} < n\}.$$

Consequently, $\dim \operatorname{Supp}_R(\mathfrak{b}^{t_0} H^i_{\mathfrak{a}}(M))) < n$, and hence $\mathfrak{b}^{t_0} H^i_{\mathfrak{a}}(M)$ is $CD_{<n}$, as required. □

In the sequel, we mention some important consequences of Theorem 3.8

Corollary 3.9. *The local-global principle (for the* $CD_{<n}$ *of local cohomology modules) holds over any commutative Noetherian local ring* R *at level 1.*

Proof. The result follows immediately from Theorem 3.8 and the fact that finitely generated R-modules are $CD_{<n}$. □

Corollary 3.10. *Let $(R, \mathfrak{m})$ be a Noetherian local ring with $\dim R \leq 2$, then the local-global principle (for the $CD_{<n}$ of local cohomology modules) holds over R at all levels $r \in N$.*

Proof. The result follows from [13, Theorem 7.10] and Theorem 3.8. □

Corollary 3.11. *Suppose that $(R, \mathfrak{m})$ is a Noetherian local ring and let $\mathfrak{a}, \mathfrak{b}$ be two ideals of R such that $\mathfrak{b} \subseteq \mathfrak{a}$. Assume that M is a finitely generated R-module and let r be a positive integer such that $\operatorname{Ext}^j(R/\mathfrak{b}, H^i_{\mathfrak{a}}(M))$ is coatomic for all j and $i < r$. Then*

$$f^{\mathfrak{b}R_{\mathfrak{p}}}_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) > r \ for\ all\ \mathfrak{p} \in \operatorname{Spec}(R) \Longleftrightarrow f^{\mathfrak{b}}_{\mathfrak{a}}(M) > r.$$

Proof. The assertion follows from Theorem 3.8 and the fact that $C^{\mathfrak{b}}_{\mathfrak{a}}(M)^0 = f^{\mathfrak{b}}_{\mathfrak{a}}(M)$. □

Corollary 3.12. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, M be a finitely generated R-module and r a non-negative integer such that $H^i_{\mathfrak{a}}(M)$ is in dimension < 2 (or weakly Laskerian) R-module for all $i < r$. Then*

$$C^{\mathfrak{b}R_{\mathfrak{p}}}_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}})^n > r \ for\ all\ \mathfrak{p} \in \operatorname{Spec}(R) \Longleftrightarrow C^{\mathfrak{b}}_{\mathfrak{a}}(M)^n > r.$$

Proof. The result follows from [3, Theorem 3.4] and Theorem 3.8. □

Corollary 3.13. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and M be a finitely generated R-module such that $\dim M/\mathfrak{a}M \leq 1$. Then for any integer r,*

$$C^{\mathfrak{b}R_{\mathfrak{p}}}_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}})^n > r \ for\ all\ \mathfrak{p} \in \operatorname{Spec}(R) \Longleftrightarrow C^{\mathfrak{b}}_{\mathfrak{a}}(M)^n > r.$$

Proof. The result follows from [6, Corollary 2.7] and Theorem 3.8. □

Corollary 3.14. *Let $(R, \mathfrak{m})$ be a local ring and M be a finitely generated R-module such that $\dim M/\mathfrak{a}M \leq 2$. Then for any integer r and $n \geq 2$,*

$$C^{\mathfrak{b}R_{\mathfrak{p}}}_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}})^n > r \ for\ all\ \mathfrak{p} \in \operatorname{Spec}(R) \Longleftrightarrow C^{\mathfrak{b}}_{\mathfrak{a}}(M)^n > r.$$

Proof. The result follows from [6, Corollary 3.2] and Theorem 3.8. □

Corollary 3.15. *Let $(R, \mathfrak{m})$ be a local ring and M be a finitely generated R-module such that $M \neq \mathfrak{a}M$. Then*

$$C^{\mathfrak{b}R_{\mathfrak{p}}}_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}})^n > \operatorname{grade}_{\mathfrak{a}} M \ for\ all\ \mathfrak{p} \in \operatorname{Spec}(R) \Longleftrightarrow C^{\mathfrak{b}}_{\mathfrak{a}}(M)^n > \operatorname{grade}_{\mathfrak{a}} M.$$

Proof. The assertion follows from [10, Theorem 6.2.7] and Theorem 3.8. □

Corollary 3.16. *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring,* M *be a finitely generated* R*-module, and* $r \in \{f_{\mathfrak{a}}(M), f^1_{\mathfrak{a}}(M), \ldots, f^n_{\mathfrak{a}}(M)\}$*. Then*

$$C^{\mathfrak{b}R_{\mathfrak{p}}}_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}})^n > r \ for \ all \ \mathfrak{p} \in \operatorname{Spec}(R) \Longleftrightarrow C^{\mathfrak{b}}_{\mathfrak{a}}(M)^n > r.$$

Proof. The result follows from Theorem 3.8 and Corollary 2.14. □

We are now ready to state and prove the main theorem of this section, which shows that local-global principle for the $CD_{<n}$ of local cohomology modules is valid at level 2 over any commutative Noetherian local ring R. This generalizes the main result of Brodmann et al. in [9] for local rings.

Theorem 3.17. *The local-global principle (for the* $CD_{<n}$ *of local cohomology modules) holds over any commutative Noetherian local ring* R *at level 2.*

Proof. Let M be a finitely generated R-module and let $\mathfrak{a}, \mathfrak{b}$ be two ideals of R such that $\mathfrak{b} \subseteq \mathfrak{a}$ and $C^{\mathfrak{b}R_{\mathfrak{p}}}_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}})^n > 2 \ for \ all \ \mathfrak{p} \in \operatorname{Spec}(R)$. If we prove that $C^{\mathfrak{b}}_{\mathfrak{a}}(M)^n > 2$, the assertion follows. To this end, by Corollary 3.9, we only need to show that there exists a non-negative integer u such that the R-module $\mathfrak{b}^u H^2_{\mathfrak{a}}(M)$ is $CD_{<n}$. Since $C^{\mathfrak{b}R_{\mathfrak{p}}}_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}})^n > 2$, similar to that in the proof of Theorem 3.8, for each $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\dim R/\mathfrak{p} \geq n$, there is $t_{\mathfrak{p}} \in \mathbb{N}_0$ such that $(\mathfrak{b}^{t_{\mathfrak{p}}} H^i_{\mathfrak{a}}(M))_{\mathfrak{p}} = 0$ for all $i = 1, 2$. Furthermore, there exists a non-negative integer s such that $\mathfrak{b}^s H^i_{\mathfrak{a}}(\Gamma_{\mathfrak{b}}(M)) = 0$ for all $i \geq 0$. Now, let $\overline{M} = M/\Gamma_{\mathfrak{b}}(M)$. Then from the short exact sequence

$$0 \longrightarrow \Gamma_{\mathfrak{b}}(M) \longrightarrow M \longrightarrow \overline{M} \longrightarrow 0,$$

we obtain the long exact sequence

$$(1) \qquad (H^1_{\mathfrak{a}}(M))_{\mathfrak{p}} \longrightarrow (H^1_{\mathfrak{a}}(\overline{M}))_{\mathfrak{p}} \longrightarrow (H^2_{\mathfrak{a}}(\Gamma_{\mathfrak{b}}(M)))_{\mathfrak{p}} \longrightarrow (H^2_{\mathfrak{a}}(M))_{\mathfrak{p}} \longrightarrow (H^2_{\mathfrak{a}}(\overline{M}))_{\mathfrak{p}}.$$

Hence, it follows from [10, Lemma 9.1.1] that $(\mathfrak{b}R_{\mathfrak{p}})^{k_{\mathfrak{p}}} H^1_{\mathfrak{a}R_{\mathfrak{p}}}(\overline{M}_{\mathfrak{p}}) = 0$, for some integer $k_{\mathfrak{p}} \in \mathbb{N}_0$. Moreover, by [10, Lemma 2.1.1], there exists $x \in \mathfrak{b}$ which is a non-zerodivisor on $\overline{M}$. Then $x^{k_{\mathfrak{p}}} H^1_{\mathfrak{a}R_{\mathfrak{p}}}(\overline{M}_{\mathfrak{p}}) = 0$. Now, the short exact sequence

$$0 \longrightarrow \overline{M}_{\mathfrak{p}} \xrightarrow{x^{k_{\mathfrak{p}}}} \overline{M}_{\mathfrak{p}} \longrightarrow \overline{M}_{\mathfrak{p}}/x^{k_{\mathfrak{p}}}\overline{M}_{\mathfrak{p}} \longrightarrow 0,$$

induces the exact sequence

$$H^0_{\mathfrak{a}R_{\mathfrak{p}}}(\overline{M}_{\mathfrak{p}}/x^{k_{\mathfrak{p}}}\overline{M}_{\mathfrak{p}}) \longrightarrow H^1_{\mathfrak{a}R_{\mathfrak{p}}}(\overline{M}_{\mathfrak{p}}) \xrightarrow{x^{k_{\mathfrak{p}}}} H^1_{\mathfrak{a}R_{\mathfrak{p}}}(\overline{M}_{\mathfrak{p}}).$$

Hence the $R_{\mathfrak{p}}$-module $H^1_{\mathfrak{a}R_{\mathfrak{p}}}(\overline{M}_{\mathfrak{p}})$ is a homomorphic image of $H^0_{\mathfrak{a}R_{\mathfrak{p}}}(\overline{M}_{\mathfrak{p}}/x^{k_{\mathfrak{p}}}\overline{M}_{\mathfrak{p}})$, and so it is a finitely generated $R_{\mathfrak{p}}$-module, for all $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\dim R/\mathfrak{p} \geq n$. It therefore follows from Theorem 2.13 that $H^1_{\mathfrak{a}}(\overline{M})$ is $CD_{<n}$. Therefore in view of Theorem [4, Theorem 2.2], the R-module $\operatorname{Hom}_R(R/\mathfrak{a}, H^2_{\mathfrak{a}}(\overline{M}))$ is also $CD_{<n}$, and so by Lemma 2.7 the set

$(\mathrm{Ass}_R(\mathrm{Hom}_R(R/\mathfrak{a}, H^2_{\mathfrak{a}}(\overline{M}))))_{\geq n}$ is finite. Consequently the set $(\mathrm{Ass}_R(H^2_{\mathfrak{a}}(\overline{M})))_{\geq n}$ is finite, and so for every non-negative integer t, the set $(\mathrm{Ass}_R(\mathfrak{b}^t H^2_{\mathfrak{a}}(\overline{M})))_{\geq n}$ is also finite. Thus for all $t \in \mathbb{N}_0$, the set $(\mathrm{Supp}_R(\mathfrak{b}^t H^2_{\mathfrak{a}}(\overline{M})))_{\geq n}$ is a closed subset of $\mathrm{Spec}(R)$ (in the Zariski topology), and so the descending chain

$$\ldots \supseteq (\mathrm{Supp}_R(\mathfrak{b}^t H^2_{\mathfrak{a}}(\overline{M})))_{\geq n} \supseteq (\mathrm{Supp}_R(\mathfrak{b}^{t+1} H^2_{\mathfrak{a}}(\overline{M})))_{\geq n} \supseteq \ldots$$

is eventually stationary. Therefore there is a non-negative integer t_0 such that for each $t > t_0$,

$$(\mathrm{Supp}_R(\mathfrak{b}^t H^2_{\mathfrak{a}}(\overline{M})))_{\geq n} = (\mathrm{Supp}_R(\mathfrak{b}^{t_0} H^2_{\mathfrak{a}}(\overline{M})))_{\geq n}.$$

Now, as $(\mathfrak{b}^{t_{\mathfrak{p}}} H^2_{\mathfrak{a}}(M))_{\mathfrak{p}} = 0$, it follows from the exact sequence (1) and [10, Lemma 9.1.1] that there is a non-negative integer $v_{\mathfrak{p}} \geq t_0$ such that $(\mathfrak{b}^{v_{\mathfrak{p}}} H^2_{\mathfrak{a}}(\overline{M}))_{\mathfrak{p}} = 0$. Hence $(\mathfrak{b}^{t_0} H^2_{\mathfrak{a}}(\overline{M}))_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$ with $\dim R/\mathfrak{p} \geq n$, and so

$$\mathrm{Supp}_R(\mathfrak{b}^{t_0} H^2_{\mathfrak{a}}(\overline{M})) \subseteq \{\mathfrak{p} \in \mathrm{Spec}(R) |\, \dim R/\mathfrak{p} < n\}.$$

Now, let $u := s + t_0$. Then, it easily follows from the exact sequence (1) and [10, Lemma 9.1.1] that $\mathrm{Supp}_R(\mathfrak{b}^u H^2_{\mathfrak{a}}(M)) \subseteq \mathrm{Supp}_R(\mathfrak{b}^{t_0} H^2_{\mathfrak{a}}(\overline{M}))$. Consequently,

$$\mathrm{Supp}_R(\mathfrak{b}^u H^2_{\mathfrak{a}}(M)) \subseteq \{\mathfrak{p} \in \mathrm{Spec}(R) |\, \dim R/\mathfrak{p} < n\},$$

and so $\mathfrak{b}^u H^2_{\mathfrak{a}}(M)$ is $CD_{<n}$, as required. □

Corollary 3.18. *The local-global principle (for the $CD_{<n}$ of local cohomology modules) holds over any commutative Noetherian local ring R with $\dim R \leq 3$.*

Proof. The assertion follows from Corollary 3.9, Theorem 3.17 and [10, Exercise 7.1.7]. □

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Marziyeh Hatamkhani

Department of Mathematics, Faculty of Science, Arak University, Arak, 38156-8-8349, Iran.

m-hatamkhani@araku.ac.ir

Hajar Roshan-Shekalgourabi

Department of Basic Sciences, Arak University of Technology, P. O. Box 38135-1177, Arak, Iran.

hrsmath@gmail.com