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Research Paper

ANNIHILATORS AND ATTACHED PRIMES OF LOCAL COHOMOLOGY MODULES WITH RESPECT TO A SYSTEM OF IDEALS

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Abstract. Let Φ be a system of ideals of a commutative Noetherian ring, we study the annihilators and attached primes of local cohomology modules with respect to a system of ideals. We prove that if M is a non-zero finitely generated R-module of finite dimension d and Φ is a system of ideals, then

$$\mathrm{Att}_R(H^d_\Phi(M)) = \{\mathfrak{p} \in \mathrm{Ass}_R\, M \mid \mathrm{cd}(\Phi, R/\mathfrak{p}) = d\}.$$

Moreover, if the cohomology dimension of M with respect to Φ is $\dim M - 1$, then

$$\mathrm{Att}_R(H_\Phi^{\dim M-1}(M)) = \{\mathfrak{p} \in \mathrm{Supp}_R M \mid \mathrm{cd}(\Phi, R/\mathfrak{p}) = \dim M - 1\}.$$

1. Introduction

Throughout this paper, R is a commutative Noetherian ring and M is an R-module. An extension of local cohomology theory of Grothendieck which is called local cohomology with

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respect to a system of ideals was introduced by Bijan-Zadeh in [8]. A non-empty set of ideals Φ of R is called to be Φ a system of ideals of R if whenever $\mathfrak{a}, \mathfrak{b} \in \Phi$, then there is an ideal $\mathfrak{c} \in \Phi$ such that $\mathfrak{c} \subseteq \mathfrak{a}\mathfrak{b}$. Let N be an R-module, one can define

$$\Gamma_\Phi(N) = \{x \in N \mid \mathfrak{a}x = 0 \text{ for some } \mathfrak{a} \in \Phi\}.$$

Then Γ_Φ is an additive, covariant, R-linear and left exact functor from the category of R-modules to itself. The functor Γ_Φ is called the general local cohomology functor with respect to Φ. For each integer $i \geq 0$, the ith right derived functor of Γ_Φ is denoted by H^i_Φ. Some basic properties of the module $H^i_\Phi(M)$ were shown in [2, 3, 8, 9, 10, 14]. In particular, if $\Phi = \{\mathfrak{a}^n \mid n \in \mathbb{N}_0\}$, where $\mathfrak{a}$ is an ideal of R, then the functor H^i_Φ coincides with the ordinary local cohomology functor $H^i_\mathfrak{a}$. The determining the annihilators of the ith local cohomology module $H^i_\mathfrak{a}(M)$ is one of important problems in studying the local cohomology module. Many authors have shown a lot of results on it, for example [4, 5, 6, 7, 17, 18, 19, 23, 24]. Base on published results, in this paper, we will investigate the annihilators of the ith local cohomology module $H^i_\Phi(M)$. The first main result is Theorem 3.3.

Theorem 1.1 (See Theorem 3.3)**.** *Let M be a non-zero finitely generated R-module with cohomological dimension* $\operatorname{cd}(\Phi, M) = c$. *Then*

$$\operatorname{Ann}_R H^c_\Phi(M) = \operatorname{Ann}_R(M/T_R(\Phi, M)),$$

where $T_R(\Phi, M)$ is the largest submodule of M such that $\operatorname{cd}(\Phi, T_R(\Phi, M)) < c$.

On the other hand, some properties of the annihilators of the local cohomology modules can be used in studying the attached primes of these modules. We first recall some facts on the attached prime ideals of a module. In [20], Macdonald said that a non-zero R-module N is secondary if for each $x \in R$ the multiplication map induced by x on N is either surjective or nilpotent. If N is secondary, then the ideal $\mathfrak{p} := \sqrt{\operatorname{Ann}_R N}$ is a prime ideal and N is called $\mathfrak{p}$-secondary. A secondary representation of an R-module M is an expression of M as a sum of finitely many secondary submodules of M. An R-module M is said to be representable if it has a secondary representation. A secondary representation of an R-module $M = M_1 + M_2 + \cdots + M_n$ is called minimal if the prime ideals $\mathfrak{p}_i = \sqrt{\operatorname{Ann}_R M_i}, i = 1, 2, \ldots, n$ are all distinct and none of M_i is redundant. The set $\{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n\}$ is independent of the choice of the minimal secondary representation of M. This set is called the set of attached prime ideals of M and denoted by $\operatorname{Att}_R(M)$.

In [26], Zöschinger gave another definition of attached prime ideals. Let M be an R-module (not necessarily admitting a secondary representation), a prime ideal $\mathfrak{p}$ of R is said to be an attached prime ideal of M if $\mathfrak{p} = \operatorname{Ann}_R(M/T)$ for some submodule T of M. This definition agrees with the preceding one of attached prime if M admits a secondary representation.

The attached primes of top local cohomology modules with respect to an ideal have been studied by Macdonald and Sharp [21], Dibaei and Yassemi [13], Hellus [16], Atazadeh et. al. [4, 5].

In this paper, we also show some properties of the attached primes of top local cohomology modules with respect to a system of ideals of R. We will extend some results on the attached primes of Dibaei, Yassemi [13] and Atazadeh et. al. [4, 5].

Theorem 1.2 (See Theorem 4.5)**.** *Let M be a non-zero finitely generated R-module of finite dimension d. Then*

$$\mathrm{Att}_R(H^d_\Phi(M)) = \{\mathfrak{p} \in \mathrm{Ass}_R\, M \mid \mathrm{cd}(\Phi, R/\mathfrak{p}) = d\}.$$

Theorem 1.3 (See Theorem 4.11)**.** *Let M be a non-zero finitely generated R-module of finite cohomological dimension* $\mathrm{cd}(\Phi, M) = \dim M - 1$. *Then*

$$\mathrm{Att}_R(H^{\dim M-1}_\Phi(M)) = \{\mathfrak{p} \in \mathrm{Supp}_R M \mid \mathrm{cd}(\Phi, R/\mathfrak{p}) = \dim M - 1\}.$$

The last Section relates to the attached primes of top local cohomology modules with respect to a pair of ideals which were introduced by Takahashi et. al. [25].

2. Cohomological dimension

First we investigate the cohomological dimension with respect to a system of ideals of R. The results of this section will be used in the following section.

Definition 2.1. Let Φ be a system of ideals of R and M an R-module. The cohomological dimension $\mathrm{cd}(\Phi, M)$ of M with respect to Φ is defined as

$$\mathrm{cd}(\Phi, M) := \sup\{i \mid H^i_\Phi(M) \neq 0\},$$

if this supremum exists, otherwise, we define it $-\infty$.

By [8, Proposition 2.3], there is an isomorphism

$$H^i_\Phi(M) \cong \varinjlim_{\mathfrak{a}\in\Phi} H^i_\mathfrak{a}(M)$$

for all $i \geq 0$. Hence, it is clear that $\mathrm{cd}(\Phi, M) \leq \sup\{\mathrm{cd}(\mathfrak{a}, M) \mid \mathfrak{a} \in \Phi\}$ and $\mathrm{cd}(\Phi, M) \leq \dim M$.

The following properties are extensions of the ones in [12] and [15].

Proposition 2.2. *Let M be an R-modules. Then*

$$\mathrm{cd}(\Phi, M) \leq \sup\{\mathrm{cd}(\Phi, N) \mid N \textit{ is a finitely generated submodule of } M\}.$$

Proof. The assertion follows from the facts that H^i_Φ commutes with direct limits and M is a direct limit of all finitely generated submodules of M. □

Proposition 2.3. *Let M be a finitely generated R-module and N an R-module such that* $\mathrm{Supp}_R N \subseteq \mathrm{Supp}_R M$. *Then* $\mathrm{cd}(\Phi, N) \leq \mathrm{cd}(\Phi, M)$.

Proof. First, assume that K is a finitely generated R-module such that $\mathrm{Supp}_R K \subseteq \mathrm{Supp}_R M$. Using the same method in the proof of [15, 2.2], we can claim that $\mathrm{cd}(\Phi, K) \leq \mathrm{cd}(\Phi, M)$. Hence, the assertion follows from Proposition 2.2. □

The following result is implied immediately from Proposition 2.3.

Corollary 2.4. *Let M, N be two finitely generated R-modules such that* $\mathrm{Supp}_R N = \mathrm{Supp}_R M$. *Then* $\mathrm{cd}(\Phi, N) = \mathrm{cd}(\Phi, M)$.

Lemma 2.5. *Let M be a finitely generated R-module and $\mathfrak{b}$ an ideal such that* $\mathfrak{b} \subseteq \mathrm{Ann}_R M$. *Then*

$$\mathrm{cd}(\Phi, M) \leq \mathrm{cd}(\Phi, R/\mathfrak{b}).$$

Proof. Let $\Phi(R/\mathfrak{b}) = \{\frac{\mathfrak{a}+\mathfrak{b}}{\mathfrak{b}} \mid \mathfrak{a} \in \Phi\}$ and we see that $\Phi(R/\mathfrak{b})$ is a system of ideals of $R/\mathfrak{b}$. Note that M is an $R/\mathfrak{b}$-module. We have by [8, 2.5] that $H^i_\Phi(M) \cong H^i_{\Phi(R/\mathfrak{b})}(M)$ for all $i \geq 0$ and then

$$\begin{aligned}\mathrm{cd}(\Phi, M) &= \mathrm{cd}(\Phi(R/\mathfrak{b}), M)\\ &\leq \mathrm{cd}(\Phi(R/\mathfrak{b}), R/\mathfrak{b})\\ &= \mathrm{cd}(\Phi, R/\mathfrak{b}),\end{aligned}$$

where the inequality is followed by Proposition 2.3. □

Theorem 2.6. *Let M be a finitely generated R-module with* $\mathrm{cd}(\Phi, M) < \infty$. *Then*

$$\mathrm{cd}(\Phi, M) = \sup\{\mathrm{cd}(\Phi, R/\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Supp}_R M\}.$$

Moreover, there is a minimal element $\mathfrak{p}$ of $\mathrm{Supp}_R M$ *such that* $\mathrm{cd}(\Phi, M) = \mathrm{cd}(\Phi, R/\mathfrak{p})$.

Proof. Let $\mathfrak{p} \in \mathrm{Supp}_R M$, it follows from Proposition 2.3 that $\mathrm{cd}(\Phi, R/\mathfrak{p}) \leq \mathrm{cd}(\Phi, M)$ and then

$$\sup\{\mathrm{cd}(\Phi, R/\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Supp}_R M\} \leq \mathrm{cd}(\Phi, M).$$

Now let $m = \sup\{\mathrm{cd}(\Phi, R/\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Supp}_R M\}$ and $n = \mathrm{cd}(\Phi, M)$. Suppose that $m < n$, and we look for a contradiction. It follows from [22, Theorem 6.4] that there is a filtration of submodules of M

$$0 = M_0 \subsetneq M_1 \subsetneq \ldots \subsetneq M_k = M$$

such that $M_i/M_{i-1} \cong R/\mathfrak{p}_i$ for some $\mathfrak{p}_i \in \mathrm{Supp}_R M$ for all $i = 1, 2, \ldots, k$. Let $i \geq 1$, the short exact sequence

$$0 \to M_{i-1} \to M_i \to R/\mathfrak{p}_i \to 0$$

induces a long exact sequence

$$H^{n-1}_\Phi(R/\mathfrak{p}_i) \to H^n_\Phi(M_{i-1}) \to H^n_\Phi(M_i) \to H^n_\Phi(R/\mathfrak{p}_i).$$

Note that $H^n_\Phi(R/\mathfrak{p}_i) = 0$ for all $i \geq 1$ and $H^n_\Phi(M_1) \cong H^n_\Phi(R/\mathfrak{p}_1) = 0$. It follows from the long exact sequence that $H^n_\Phi(M_i) = 0$ for all $1 \leq i \leq k$. In particular, $0 = H^n_\Phi(M_k) = H^n_\Phi(M)$, which is a contradiction.

If $\mathfrak{p} \in \mathrm{Supp}_R M$, then there exists $\mathfrak{q}$ is a minimal element of $\mathrm{Supp}_R M$ such that $\mathfrak{q} \subseteq \mathfrak{p}$. According to Proposition 2.3, we have $\mathrm{cd}(\Phi, R/\mathfrak{p}) \leq \mathrm{cd}(\Phi, R/\mathfrak{q})$. This implies that $\mathrm{cd}(\Phi, R/\mathfrak{q}) = \mathrm{cd}(\Phi, M)$, and the proof is complete. □

3. Annihilators of local cohomology modules with respect to a system of ideals

Our aim of this section is generalized some results on the annihilators of local cohomology modules with respect an ideals.

Definition 3.1. Let M be a non-zero finitely generated R-module. We denote by $T_R(\Phi, M)$ the largest submodule of M such that $\mathrm{cd}(\Phi, T_R(\Phi, M)) < \mathrm{cd}(\Phi, M)$. It is easy to check that

$$T_R(\Phi, M) = \bigcup \{N \mid N \text{ is a submodule of } M \text{ and } \mathrm{cd}(\Phi, N) < \mathrm{cd}(\Phi, M)\}.$$

The first result of this section gives a decomposition of $T_R(\Phi, M)$.

Theorem 3.2. *Let M be a non-zero finitely generated R-module with cohomological dimension $c = \mathrm{cd}(\Phi, M)$. Assume that $0 = \bigcap_{i=1}^n N_i$ is a reduced primary decomposition of the zero submodule of M and N_i is a $\mathfrak{p}_i$-primary submodule of M. Then*

$$T_R(\Phi, M) = \bigcap_{\mathfrak{p}_i \in \mathrm{Ass}_R(M), \mathrm{cd}(\Phi, R/\mathfrak{p}_i) = c} N_i.$$

Proof. Let

$$N = \bigcap_{\mathfrak{p}_i \in \mathrm{Ass}_R(M), \mathrm{cd}(\Phi, R/\mathfrak{p}_i) = c} N_i \text{ and } K = \bigcap_{\mathfrak{p}_i \in \mathrm{Ass}_R(M), \mathrm{cd}(\Phi, R/\mathfrak{p}_i) < c} N_i.$$

Then $K \cap N = 0$ and there is an exact sequence

$$0 \to N \to M/K.$$

It follows from Proposition 2.3 that $\mathrm{cd}(\Phi, N) \leq \mathrm{cd}(\Phi, M/K)$. Note that

$$\mathrm{Ass}_R N \subseteq \mathrm{Ass}_R(M/K) = \{\mathfrak{p} \in \mathrm{Ass}_R(M) \mid \mathrm{cd}(\Phi, R/\mathfrak{p}) < c\}.$$

By Theorem 2.6, we have $\mathrm{cd}(\Phi, M/K) < c$ and then $\mathrm{cd}(\Phi, N) < c$. This implies that $N \subseteq T_R(\Phi, M)$.

Now, let $x \in T_R(\Phi, M)$ and it is clear that $\mathrm{cd}(\Phi, Rx) < c$ and $H^c_\Phi(R/\mathfrak{p}) = 0$ for all $\mathfrak{p} \in \mathrm{Ass}_R(Rx)$. Therefore, we obtain

$$\mathrm{Ass}_R(Rx) \subseteq \{\mathfrak{p} \in \mathrm{Ass}_R(M) \mid \mathrm{cd}(\Phi, R/\mathfrak{p}) < c\}.$$

Hence

$$\bigcap_{\mathfrak{p}\in\mathrm{Ass}_R(M),\mathrm{cd}(\Phi,R/\mathfrak{p})<c} \mathfrak{p} \subseteq \bigcap_{\mathfrak{p}\in\mathrm{Ass}_R(Rx)} \mathfrak{p} = \sqrt{\mathrm{Ann}_R(Rx)}.$$

Let $J = \bigcap_{\mathfrak{p}\in\mathrm{Ass}_R(M),\mathrm{cd}(\Phi,R/\mathfrak{p})<c} \mathfrak{p}$, there exists a positive integer m such that

$$J^m x = 0.$$

By the primary decomposition of zero submodule of M, we have

$$J^m x \in N_i$$

for all $1 \leq i \leq n$. Assume that there exists an R-module N_j such that $x \notin N_j$ and $\mathrm{cd}(\Phi, R/\mathfrak{p}_j) = c$. Since N_j is $\mathfrak{p}_j$-primary, we can conclude that $J^m \subseteq \mathfrak{p}_j$. This implies that there is an prime ideal $\mathfrak{p}_k \in \mathrm{Ass}_R(M)$ such that $\mathrm{cd}(\Phi, R/\mathfrak{p}_k) < c$ and $\mathfrak{p}_k \subseteq \mathfrak{p}_j$. Consequently, we have

$$c = \mathrm{cd}(\Phi, R/\mathfrak{p}_j) \leq \mathrm{cd}(\Phi, R/\mathfrak{p}_k) < c,$$

which is a contradiction. Hence $x \in N$ and then $T_R(\Phi, M) = N$. □

We are going to state and prove the first main result of this paper. The following theorem is an extension of [4, Theorem 2.3].

Theorem 3.3. *Let M be a non-zero finitely generated R-module with cohomological dimension $c = \mathrm{cd}(\Phi, M)$. Then*

$$\mathrm{Ann}_R H^c_\Phi(M) = \mathrm{Ann}_R(M/T_R(\Phi, M)).$$

Proof. The short exact sequence

$$0 \to T_R(\Phi, M) \to M \to M/T_R(\Phi, M) \to 0$$

induces the following exact sequence

$$H^c_\Phi(T_R(\Phi, M)) \to H^c_\Phi(M) \to H^c_\Phi(M/T_R(\Phi, M)) \to 0.$$

Since $\mathrm{cd}(\Phi, T_R(\Phi, M)) < c$, there is an isomorphism

$$H^c_\Phi(M) \cong H^c_\Phi(M/T_R(\Phi, M)).$$

The proof is complete by showing that

$$\mathrm{Ann}_R H^c_\Phi(M/T_R(\Phi, M)) = \mathrm{Ann}_R(M/T_R(\Phi, M)).$$

Let $\overline{M} = M/T_R(\Phi, M)$ and it is clear that

$$\mathrm{Ann}_R\overline{M} \subseteq \mathrm{Ann}_R H^c_\Phi(\overline{M}).$$

Now, let $x \in \mathrm{Ann}_R H^c_\Phi(\overline{M})$, we will show $x \in \mathrm{Ann}_R\overline{M}$. The short exact sequence

$$0 \to 0 :_{\overline{M}} x \to \overline{M} \xrightarrow{x} x\overline{M} \to 0$$

deduces the long exact sequence

$$H^c_\Phi(0 :_{\overline{M}} x) \to H^c_\Phi(\overline{M}) \xrightarrow{x} H^c_\Phi(x\overline{M}) \to 0.$$

This implies that $H^c_\Phi(x\overline{M}) = xH^c_\Phi(\overline{M}) = 0$ and then $\mathrm{cd}(\Phi, x\overline{M}) < c$. By the definition of $T_R(\Phi, M)$, we can conclude that $x\overline{M} = 0$. Hence, $x \in \mathrm{Ann}_R\overline{M}$ and the proof is complete. □

Corollary 3.4. *Let R be a ring with cohomological dimension $c = \mathrm{cd}(\Phi, R)$. Then*

$$\mathrm{Ann}_R H^c_\Phi(R) = T_R(\Phi, R) = \bigcap_{\mathrm{cd}(\Phi, R/\mathfrak{p}_i)=c} \mathfrak{q}_i$$

where $0 = \bigcap_{\mathfrak{p}_i \in \mathrm{Ass}_R(R), \mathrm{cd}(\Phi, R/\mathfrak{p}_i)=c} \mathfrak{q}_i$ is a reduced primary decomposition of the zero ideal of R, $\mathfrak{q}_i$ is a $\mathfrak{p}_i$-primary ideal of R.

Proof. It follows from Theorem 3.2 and Theorem 3.3. □

Corollary 3.5. *Let R be a ring of finite cohomological dimension $\mathrm{cd}(\Phi, R) = c$. Then the following conditions are equivalent:*

(i) $\mathrm{Ann}_R(H^c_\Phi(R)) = 0$.
(ii) $\mathrm{Ass}_R R = \{\mathfrak{p} \in \mathrm{Spec}\, R \mid \mathrm{cd}(\Phi, R/\mathfrak{p}) = c\}$.

Proof. It follows from Corollary 3.4. □

Corollary 3.6. *Let R be a domain and $\mathrm{cd}(\Phi, R) = \dim R$. Then*

$$\mathrm{Ann}_R H^{\dim R}_\Phi(R) = 0.$$

Proof. If R is a domain, then we have $\mathrm{Ass}_R R = \{0\}$. The assertion follows from Corollary 3.5.

□

Corollary 3.7. [4, Corollary 2.10] *Let R be a domain, $\mathfrak{a}$ an ideal of R and $\mathrm{cd}(\mathfrak{a}, R) = \dim R$. Then*

$$\mathrm{Ann}_R H^{\dim R}_{\mathfrak{a}}(R) = 0.$$

4. Attached primes of local cohomology modules with respect to a system of ideals

We will generalize some results on the attached primes of Dibaei and Yassemi [13], Atazadeh et. al. in [4, 5]. First, we recall the concept of attached prime ideals which were introduced by Zöschinger [26].

Definition 4.1 (See [26])**.** Let M be an R-module. A prime ideal $\mathfrak{p}$ of $\mathrm{Spec}\, R$ is called attached to M if there is a submodule N of M such that $\mathfrak{p} = \mathrm{Ann}_R(M/N)$.

The set of attached prime ideals of M is denoted by $\mathrm{Att}_R M$. In the case, where M is a representable R-module, this definition is agree with the one of Macdonald [20].

Lemma 4.2 (See [1])**.** *The following statements hold true.*

(i) *If $0 \to A \to B \to C \to 0$ is an exact sequence of R-modules, then*

$$\mathrm{Att}_R C \subseteq \mathrm{Att}_R B \subseteq \mathrm{Att}_R C \cup \mathrm{Att}_R A.$$

(ii) *If N is a finitely generated R-module, then*

$$\mathrm{Att}_R(M \otimes_R N) = \mathrm{Att}_R M \cap \mathrm{Supp}_R N$$

for all R-module M.

Lemma 4.3. *Let R be a ring of finite cohomological dimension $\mathrm{cd}(\Phi, R) = c$. Then*

$$\mathrm{Att}_R(H^c_\Phi(R)) \subseteq \{\mathfrak{p} \in \mathrm{Spec}\, R \mid \mathrm{cd}(\Phi, R/\mathfrak{p}) = c\}.$$

Proof. Let $\mathfrak{p} \in \mathrm{Att}_R(H^c_\Phi(R))$, we have by the right exactness of H^c_Φ that

$$0 \neq H^c_\Phi(R)/\mathfrak{p}H^c_\Phi(R) \cong H^c_\Phi(R/\mathfrak{p}).$$

Hence $\mathrm{cd}(\Phi, R/\mathfrak{p}) = c$, and the proof is complete. □

Theorem 4.4. *Let M a non-zero finitely generated R-module of finite cohomological dimension $c := \mathrm{cd}(\Phi, M) < \infty$. Then*

$$\mathrm{Att}_R(H^c_\Phi(M)) \subseteq \{\mathfrak{p} \in \mathrm{Supp}_R M \mid \mathrm{cd}(\Phi, R/\mathfrak{p}) = c\}.$$

Proof. Let $\overline{R} = R/\mathrm{Ann}_R M$ and it follows from Corollary 2.4 that $\mathrm{cd}(\Phi, \overline{R}) = c$. By [8, 2.5], there is an isomorphism

$$H^c_\Phi(M) \cong H^c_{\Phi\overline{R}}(M)$$

where $\Phi\overline{R} = \{\mathfrak{a}\overline{R} \mid \mathfrak{a} \in \Phi\}$ is a system of ideals in $\overline{R}$. On the other hand

$$\begin{aligned} H^c_{\Phi\overline{R}}(M) &\cong H^c_{\Phi\overline{R}}(\overline{R} \otimes_{\overline{R}} M) \\ &\cong H^c_{\Phi\overline{R}}(\overline{R}) \otimes_{\overline{R}} M \end{aligned}$$

since $H^c_{\Phi\overline{R}}$ is a right exact functor. It follows from Lemma 4.2(ii) that

$$\mathrm{Att}_{\overline{R}}(H^c_{\Phi\overline{R}}(M)) = \mathrm{Att}_{\overline{R}}(H^c_{\Phi\overline{R}}(\overline{R})) \cap \mathrm{Supp}_{\overline{R}} M = \mathrm{Att}_{\overline{R}}(H^c_{\Phi\overline{R}}(\overline{R})).$$

By Lemma 4.3, we obtain

$$\mathrm{Att}_{\overline{R}}(H^c_{\Phi\overline{R}}(\overline{R})) \subseteq \{\mathfrak{p} \in \mathrm{Spec}\,\overline{R} \mid \mathrm{cd}(\Phi\overline{R}, \overline{R}/\mathfrak{p}) = c\}.$$

Consequently, one gets

$$\mathrm{Att}_R(H^c_\Phi(M) \subseteq \{\mathfrak{p} \in \mathrm{Supp}_R M \mid \mathrm{cd}(\Phi, R/\mathfrak{p}) = c\},$$

and the proof is complete. □

The following result is a generalization of [13, Theorem A] and R is not a local ring.

Theorem 4.5. *Let M be a non-zero finitely generated R-module of finite dimension d. Then*

$$\mathrm{Att}_R(H^d_\Phi(M)) = \{\mathfrak{p} \in \mathrm{Ass}_R\, M \mid \mathrm{cd}(\Phi, R/\mathfrak{p}) = d\}.$$

Proof. It follows from Theorem 4.4 that

$$\mathrm{Att}_R H^d_\Phi(M) \subseteq \{\mathfrak{p} \in \mathrm{Supp}_R M \mid \mathrm{cd}(\Phi, R/\mathfrak{p}) = d\}.$$

Let $\mathfrak{p} \in \mathrm{Supp}_R M$ such that $\mathrm{cd}(\Phi, R/\mathfrak{p}) = d$, this implies that $\dim R/\mathfrak{p} = d$. Therefore $\mathfrak{p} \in \mathrm{Ass}_R\, M$ and

$$\mathrm{Att}_R H^d_\Phi(M) \subseteq \{\mathfrak{p} \in \mathrm{Ass}_R\, M \mid \mathrm{cd}(\Phi, R/\mathfrak{p}) = d\}.$$

Let $\mathfrak{q} \in \mathrm{Ass}_R\, M$ such that $\mathrm{cd}(\Phi, R/\mathfrak{q}) = d$. There exists a submodule K of M such that K is $\mathfrak{q}$-primary and $\mathrm{Ass}_R(M/K) = \{\mathfrak{q}\}$. It follows from Theorem 2.6 that $\mathrm{cd}(\Phi, M/K) = \mathrm{cd}(\Phi, R/\mathfrak{q}) = d$. By the above argument, we see that

$$\mathrm{Att}_R H^d_\Phi(M/K) \subseteq \{\mathfrak{p} \in \mathrm{Ass}_R\, M/K \mid \mathrm{cd}(\Phi, R/\mathfrak{p}) = d\} = \{\mathfrak{q}\}.$$

Now the short exact sequence

$$0 \to K \to M \to M/K \to 0$$

yields the following exact sequence

$$H^d_\Phi(M) \to H^d_\Phi(M/K) \to 0.$$

Consequently, we have by Lemma 4.2 that $\mathrm{Att}_R H^d_\Phi(M/K) \subseteq \mathrm{Att}_R H^d_\Phi(M)$ and then $\mathfrak{q} \in \mathrm{Att}_R H^d_\Phi(M)$, this complete the proof. $\square$

We denote by $\mathrm{mAss}_R M$ the set of minimal elements in $\mathrm{Ass}_R M$.

Corollary 4.6. *Let M be a non-zero finitely generated R-module of finite dimension d. Then*

$$\mathrm{Att}_R(H^d_\Phi(M)) = \{\mathfrak{p} \in \mathrm{mAss}_R M \mid \dim R/\mathfrak{p} = d\}.$$

Proof. Let $\mathfrak{p} \in \mathrm{Ass}_R M$ such that $\mathrm{cd}(\Phi, R/\mathfrak{p}) = \dim M$. It is clear that $\dim R/\mathfrak{p} = \dim M$ and then $\mathfrak{p} \in \mathrm{mAss}_R M$.

Now, let $\mathfrak{p} \in \mathrm{mAss}_R M$ such that $\dim R/\mathfrak{p} = d$. It follows from Theorem 4.5 that

$$\mathrm{Att}_R H^d_\Phi(M/\mathfrak{p}M) = \{\mathfrak{q} \in \mathrm{Ass}_R M/\mathfrak{p}M \mid \mathrm{cd}(\Phi, R/\mathfrak{q}) = d\} = \{\mathfrak{p}\},$$

where the second equality is get by $\mathfrak{p} \in \mathrm{mAss} M$. The short exact sequence

$$0 \to \mathfrak{p}M \to M \to M/\mathfrak{p}M \to 0$$

induces the following exact sequence

$$H^d_\Phi(M) \to H^d_\Phi(M/\mathfrak{p}M) \to 0.$$

By Lemma 4.2, we obtain that $\mathrm{Att}_R H^d_\Phi(M/\mathfrak{p}M) \subseteq \mathrm{Att}_R H^d_\Phi(M)$ and then $\mathfrak{p} \in \mathrm{Att}_R H^d_\Phi(M)$. $\square$

Corollary 4.7. *Let M be a non-zero finitely generated R-module of dimension d. Assume that $H^d_\Phi(M) \neq 0$. Then*

(i) *There exists a submodule T of M such that* $\dim M/T = d$,

(ii) $\mathrm{Ass}_R(M/T) = \{\mathfrak{p} \in \mathrm{Ass}_R M \mid \mathrm{cd}(\Phi, R/\mathfrak{p}) = d\}$,

(iii) $\mathrm{Att}_R H^d_\Phi(M) = \mathrm{Ass}_R(M/T)$.

Proof. According to Theorem 4.5, we have $\mathrm{Att}_R H^d_\Phi(M) \subseteq \mathrm{Ass}_R M$. It follows from [11, p. 263, Proposition 4] that there is a submodule T of M such that $\mathrm{Ass}_R M/T = \mathrm{Att}_R H^d_\Phi(M)$ and $\mathrm{Ass}_R T = \mathrm{Ass}_R M \setminus \mathrm{Att}_R H^d_\Phi(M)$. It is clear that $\dim M/T = d$, and the proof is complete. $\square$

Corollary 4.8. [13, Theorem A] *Let M be a non-zero finitely generated R-module of finite dimension d and $\mathfrak{a}$ an ideal of R. Then*

$$\mathrm{Att}_R(H^d_{\mathfrak{a}}(M)) = \{\mathfrak{p} \in \mathrm{Ass}_R M \mid \mathrm{cd}(\mathfrak{a}, R/\mathfrak{p}) = d\}.$$

Before considering the case where $\mathrm{cd}(\Phi, R) = \dim R - 1$, we need the following result concerning on the annihilator of $H_\Phi^{\dim R-1}(R)$.

Theorem 4.9. *Let R be a Noetherian domain of finite dimension d and $\mathrm{cd}(\Phi, R) = d-1$. Then $\mathrm{Ann}_R(H_\Phi^{d-1}(R)) = 0$.*

Proof. Let $x \in \mathrm{Ann}_R(H_\Phi^{d-1}(R))$ and suppose that $x \neq 0$. Since R is a domain, there is a short exact sequence

$$0 \to R \xrightarrow{x^n} R \to R/x^nR \to 0$$

for all $n \geq 1$. Applying the functor Γ_Φ to the above short exact sequence, we get a following exact sequence

$$H_\Phi^{d-1}(R) \xrightarrow{x^n} H_\Phi^{d-1}(R) \to H_\Phi^{d-1}(R/x^nR) \to 0.$$

Since $x^n \in \mathrm{Ann}_R H_\Phi^{d-1}(R)$, there is an isomorphism

$$H_\Phi^{d-1}(R) \cong H_\Phi^{d-1}(R/x^nR)$$

for all $n \geq 1$. Note that $\dim R/x^nR = \dim R - 1$ for all $n \geq 1$. By Corollary 4.6 there is a prime ideal $\mathfrak{p} \in \mathrm{mAss}_R(R/x^nR)$ such that $\dim R/\mathfrak{p} = d-1$. Let $\mathfrak{q}_n$ is the $\mathfrak{p}$-primary component of x^nR. It follows from Corollary 3.4 that

$$x \in \mathrm{Ann}_R H_\Phi^{d-1}(R) \subseteq \mathfrak{q}_n$$

for all $n \geq 1$. Since $\mathfrak{p} \in \mathrm{mAss}_R(R/x^nR)$, it follows that

$$x^nR_\mathfrak{p} = \mathfrak{q}_nR_\mathfrak{p}$$

for all $n \geq 1$. This implies that

$$xR_\mathfrak{p} \subseteq \bigcap_{n\geq 1} x^nR_\mathfrak{p}.$$

By the Krulls Intersection Theorem, we can conclude that $xR_\mathfrak{p} = 0$. Since R is a domain, we have $x = 0$ which is a contradition. □

Theorem 4.10. *Let R be a ring of finite cohomological dimension $\mathrm{cd}(\Phi, R) = \dim R - 1$. Then*

$$\mathrm{Att}_R(H_\Phi^{\dim R-1}(R)) = \{\mathfrak{p} \in \mathrm{Spec}\, R \mid \mathrm{cd}(\Phi, R/\mathfrak{p}) = \dim R - 1\}.$$

Proof. By Lemma 4.3, we have

$$\mathrm{Att}_R(H_\Phi^{\dim R-1}(R)) \subseteq \{\mathfrak{p} \in \mathrm{Spec}\, R \mid \mathrm{cd}(\Phi, R/\mathfrak{p}) = \dim R - 1\}.$$

Now, let $d = \dim R$ and $\mathfrak{p} \in \mathrm{Spec}\, R$ such that $\mathrm{cd}(\Phi, R/\mathfrak{p}) = d-1$. Since $H_\Phi^{d-1}(R/\mathfrak{p}) \neq 0$, it follows that $\dim R/\mathfrak{p} \geq d-1$. Let $\overline{R} = R/\mathfrak{p}$, we have by [8, 2.5] an isomorphism

$$H_\Phi^{d-1}(R/\mathfrak{p}) \cong H_{\Phi\overline{R}}^{d-1}(\overline{R}).$$

Note that $\dim \overline{R} = \dim R/\mathfrak{p} \geq d-1$, it follows from Corollary 3.6 and Theorem 4.9 that $\mathrm{Ann}_{\overline{R}} H^{d-1}_{\Phi\overline{R}}(\overline{R}) = 0$. Thus, we can claim that $\mathrm{Ann}_R H^{d-1}_{\Phi}(R/\mathfrak{p}) = \mathfrak{p}$ and then $\mathfrak{p} \in \mathrm{Att}_R H^{d-1}_{\Phi}(R/\mathfrak{p})$ by Definition 4.1. On the other hand, the short exact sequence

$$0 \to \mathfrak{p} \to R \to R/\mathfrak{p} \to 0,$$

in conjunction with $\mathrm{cd}(\Phi, \mathfrak{p}) \leq d-1$, gives rise to a long exact sequence

$$H^{d-1}_{\Phi}(\mathfrak{p}) \to H^{d-1}_{\Phi}(R) \to H^{d-1}_{\Phi}(R/\mathfrak{p}) \to 0.$$

This implies by Lemma 4.2 that $\mathrm{Att}_R H^{d-1}_{\Phi}(R/\mathfrak{p}) \subseteq \mathrm{Att}_R H^{d-1}_{\Phi}(R)$. Consequently, $\mathfrak{p} \in \mathrm{Att}_R H^{d-1}_{\Phi}(R)$, and the proof is complete. □

The following theorem is another main result of this section which extends the main results of [4, Theorem 3.7], [16, Theorem 2.3(i)] and [5, Theorem 3.3].

Theorem 4.11. *Let M be a non-zero finitely generated R-module of finite cohomological dimension* $\mathrm{cd}(\Phi, M) = \dim M - 1$. *Then*

$$\mathrm{Att}_R(H^{\dim M-1}_{\Phi}(M)) = \{\mathfrak{p} \in \mathrm{Supp}_R M \mid \mathrm{cd}(\Phi, R/\mathfrak{p}) = \dim M - 1\}.$$

Proof. Let $\overline{R} = R/\mathrm{Ann}_R M$ and $d = \dim M$. It is clear that $\dim \overline{R} = \dim M$ and by Corollary 2.4 we have $\mathrm{cd}(\Phi, \overline{R}) = \dim M - 1$. By [8, 2.5], there is an isomorphism

$$H^{d-1}_{\Phi}(M) \cong H^{d-1}_{\Phi\overline{R}}(M).$$

By the same method in the proof of Theorem 4.4 we have

$$\mathrm{Att}_{\overline{R}}(H^{d-1}_{\Phi\overline{R}}(M)) = \mathrm{Att}_{\overline{R}}(H^{d-1}_{\Phi\overline{R}}(\overline{R})) \cap \mathrm{Supp}_{\overline{R}} M = \mathrm{Att}_{\overline{R}}(H^{d-1}_{\Phi\overline{R}}(\overline{R})).$$

and by Theorem 4.10

$$\mathrm{Att}_{\overline{R}}(H^{d-1}_{\Phi\overline{R}}(\overline{R})) = \{\mathfrak{p} \in \mathrm{Spec}\,\overline{R} \mid \mathrm{cd}(\Phi\overline{R}, \overline{R}/\mathfrak{p}) = d-1\}.$$

This implies that

$$\mathrm{Att}_R(H^{\dim M-1}_{\Phi}(M)) = \{\mathfrak{p} \in \mathrm{Supp}_R M \mid \mathrm{cd}(\Phi, R/\mathfrak{p}) = \dim M - 1\},$$

which completes the proof. □

Corollary 4.12. [5, Theorem 3.3] *Let M be a non-zero finitely generated R-module of finite cohomological dimension* $\mathrm{cd}(\Phi, M) = \dim M - 1$ *and* $\mathfrak{a}$ *an ideal of* R. *Then*

$$\mathrm{Att}_R(H^{\dim M-1}_{\mathfrak{a}}(M)) = \{\mathfrak{p} \in \mathrm{Supp}_R M \mid \mathrm{cd}(\mathfrak{a}, R/\mathfrak{p}) = \dim M - 1\}.$$

5. Attached primes of local cohomology modules with respect to a pair of ideals

In [25], Takahashi, Yoshino and Yoshizawa introduced an extension of local cohomology modules which is called the local cohomology modules with respect to a pair ideals. Let I, J be two ideals of R and

$$W(I,J) = \{\mathfrak{p} \in \operatorname{Spec} R \mid I^n \subseteq \mathfrak{p} + J \text{ for some integer } n\}.$$

The functor $\Gamma_{I,J}$ from the category of R-modules to itself is defined by

$$\Gamma_{I,J}(M) = \{x \in M \mid \operatorname{Supp}_R(Rx) \subseteq W(I,J)\},$$

where M is an R-module. The functor $\Gamma_{I,J}$ is R-linear and left exact. For an integer i, the i-th right derived functor of $\Gamma_{I,J}$ is called the i-th local cohomology functor $H^i_{I,J}$. Let M be an R-module, we call $H^i_{I,J}(M)$ to be the i-th local cohomology modules of M with respect to (I,J). Let

$$\widetilde{W}(I,J) = \{\mathfrak{a} \text{ is an ideal of } R \mid I^n \subseteq \mathfrak{a} + J \text{ for some integer } n\}$$

and we define a partial order on $\widetilde{W}(I,J)$ by letting $\mathfrak{a} \leq \mathfrak{b}$ if $\mathfrak{a} \supseteq \mathfrak{b}$ for $\mathfrak{a}, \mathfrak{b} \in \widetilde{W}(I,J)$. It follows from [25, Theorem 3.2] that

$$H^i_{I,J}(M) \cong \varinjlim_{\mathfrak{a} \in \widetilde{W}(I,J)} H^i_{\mathfrak{a}}(M)$$

for all $i \geq 0$ and for any R-module M. It is clear that $\widetilde{W}(I,J)$ is a system of ideals of R. We denote by

$$\operatorname{cd}(I,J,M) := \sup\{i \mid H^i_{I,J}(M) \neq 0\}$$

and call the cohomological dimension of M with respect to (I,J). It is clear that the local cohomology modules with respect to a pair of ideals are special cases of local cohomology modules with respect to a system of ideals. Therefore, we can implies the following results.

Proposition 5.1. *Let M be a non-zero finitely generated R-module with cohomological dimension $c = \operatorname{cd}(I,J,M)$. Then*

$$\operatorname{Ann}_R H^c_{I,J}(M) = \operatorname{Ann}_R(M/T_R(I,J,M)),$$

where $T_R(I,J,M)$ is the largest submodule of M such that $\operatorname{cd}(I,J,T_R(I,J,M)) < c$.

Proof. It follows from Theorem 3.3. □

Proposition 5.2. *Let M be a non-zero finitely generated R-module of finite dimension d. Then*

$$\operatorname{Att}_R(H^d_{I,J}(M)) = \{\mathfrak{p} \in \operatorname{Ass}_R M \mid \operatorname{cd}(I,J,R/\mathfrak{p}) = d\}.$$

Proof. It follows from Theorem 4.5. □

Proposition 5.3. *Let M be a finitely generated R-module of finite cohomological dimension* $\mathrm{cd}(I, J, M) = \dim M - 1$. *Then*

$$\mathrm{Att}_R(H^{\dim M-1}_{I,J}(M)) = \{\mathfrak{p} \in \mathrm{Supp}_R M \mid \mathrm{cd}(I, J, R/\mathfrak{p}) = \dim M - 1\}.$$

Proof. It follows from Theorem 4.11. □

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