

Research Paper

GENERALIZED STONE RESIDUATED LATTICES

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ABSTRACT. This paper introduces and investigates the notion of a generalized Stone residuated lattice. It is observed that a residuated lattice is generalized Stone if and only if it is quasicomplemented and normal. Also, it is proved that a finite residuated lattice is generalized Stone if and only if it is normal. A characterization for generalized Stone residuated lattices is given by means of the new notion of α -filters. Finally, it is shown that each non-unit element of a directly indecomposable generalized Stone residuated lattice is a dense element.

1. INTRODUCTION

Distributive pseudocomplemented lattices form an important class of distributive lattices. [1, Problem 70] asked a question inspired by M. H. Stone: “What is the most general pseudocomplemented distributive lattice in which $x^* \vee x^{**} = 1$ identically?” The first solution to this problem belongs to [11] who gave the name “*Stone lattices*” to this class of lattices. They

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characterized stone lattices as distributive pseudocomplemented lattices in which any pair of incomparable minimal prime ideals is comaximal or equivalently each prime ideal contains a unique minimal prime ideal. According to [10, Lemma III.6.3], a distributive pseudocomplemented lattice is a Stone lattice if and only if its skeleton is a subalgebra of it. As a generalization of distributive pseudocomplemented lattices [24] studied lattices which are just pseudocomplemented. It is easy to see that a distributive lattice is pseudocomplemented if and only if each its annulet is a principal ideal. By this motivation [25] and [22, 23] generalized pseudocomplemented distributive lattices, the so-called quasicomplemented lattices. Also, [12] introduced the notion of generalized Stone lattices. Motivated by characterization of Stone lattices, [5] studied distributive lattices with 0 in which each prime ideal contains a unique minimal prime ideal under the name “*normal lattices*”. He characterized generalized Stone lattices [5, Theorem 5.6] by means of normal and quasicomplemented distributive lattices with 0. The notion of α -ideals is introduced in [6] and proved that a distributive lattice is a generalized Stone lattice if and only if each prime ideal contains a unique prime α -ideal. This work is greatly motivated by the above works and a desire to extend these investigations to residuated lattices. The notion of generalized Stone residuated lattices is introduced and investigated. Our findings show that the results obtained by [5, 6] can also be reproduced via generalized Stone residuated lattices.

The paper is organized as follows. In Sec. 2, some definitions and facts about quasicomplemented and normal residuated lattices that we use in the sequel are recalled. The results in the this section are original, excepting those that we cite from the other papers. In Sec. 3, the notion of generalized Stone residuated lattices are introduced and characterized. It is observed that a residuated lattice is generalized Stone if and only if it is quasicomplemented and normal (Corollary 3.10). Also, it is proved that a finite residuated lattice is generalized Stone if and only if it is normal (Corollary 3.11). It is shown that a residuated lattice is generalized Stone if and only if any prime filter of it contains a unique prime α -filter (Theorem 3.14). This section ends with an explanation, elucidating that each non-unit element of a directly indecomposable generalized Stone residuated lattice is a dense element (Corollary 3.17).

2. RESIDUATED LATTICES

In this section, we recall some definitions, properties and results relative to residuated lattices which will be used in the following. The results in the this section are original, excepting those that we cite from the other papers.

An algebra $\mathfrak{A} = (A; \vee, \wedge, \odot, \rightarrow, 0, 1)$ is called a *residuated lattice* if $\ell(\mathfrak{A}) = (A; \vee, \wedge, 0, 1)$ is a bounded lattice, $(A; \odot, 1)$ is a commutative monoid and (\odot, \rightarrow) is an adjoint pair. A residuated lattice \mathfrak{A} is called *non-degenerate* if $0 \neq 1$. For a residuated lattice \mathfrak{A} and $a \in A$ we

put $\neg a := a \rightarrow 0$ and $a^n := a \odot \cdots \odot a$ (n times), for any integer n . An element $a \in A$ is called *idempotent* if $a^2 = a$. The set of idempotent elements of \mathfrak{A} is denoted by $\text{id}(\mathfrak{A})$. The class of residuated lattices is equational and so it forms a variety. For a survey of residuated lattices we refer to [8].

Remark 2.1. [4, Proposition 2.6] Let \mathfrak{A} be a residuated lattice. The following conditions are satisfied for any $x, y, z \in A$:

$$\begin{aligned} r_1 \quad & x \odot (y \vee z) = (x \odot y) \vee (x \odot z); \\ r_2 \quad & x \vee (y \odot z) \geq (x \vee y) \odot (x \vee z). \end{aligned}$$

Example 2.2. [19, Example 2.1] Let $A_6 = \{0, a, b, c, d, 1\}$ be a lattice whose Hasse diagram is given by Figure 1. Routine calculation shows that $\mathfrak{A}_6 = (A_6; \vee, \wedge, \odot, \rightarrow, 0, 1)$ is a residuated lattice where the commutative operation “ \odot ” is given by Table 1 and the operation “ \rightarrow ” is defined by $x \rightarrow y = \bigvee \{a \in A_6 \mid x \odot a \leq y\}$, for any $x, y \in A_6$.

\odot	0	a	b	c	d	1
0	0	0	0	0	0	0
a	a	a	0	a	a	a
b	a	0	a	b	b	b
c	c	c	c	c	c	c
d	d	d	d	d	d	d
1	1	1	1	1	1	1

TABLE 1

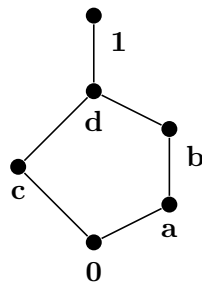


FIGURE 1

Example 2.3. [15, Example 2.2] Let $A_7 = \{0, a, b, c, d, e, 1\}$ be a lattice whose Hasse diagram is given by Figure 2. Routine calculation shows that $\mathfrak{A}_7 = (A_7; \vee, \wedge, \odot, \rightarrow, 0, 1)$ is a residuated

lattice where the commutative operation “ \odot ” is given by Table 2 and the operation “ \rightarrow ” is defined by $x \rightarrow y = \bigvee \{a \in A_7 \mid x \odot a \leq y\}$, for any $x, y \in A_7$.

\odot	0	a	b	c	d	e	1
0	0	0	0	0	0	0	0
		a	a	a	a	a	a
		b	b	a	b	a	b
		c	a	a	c	c	c
		d	b	c	d	d	d
		e	e	e	e	e	e
		1	1	1	1	1	1

TABLE 2

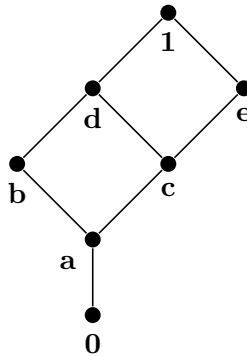


FIGURE 2

Let \mathfrak{A} be a residuated lattice. A non-void subset F of A is called a *filter* of \mathfrak{A} if $x, y \in F$ implies $x \odot y \in F$ and $x \vee y \in F$ for any $x \in F$ and $y \in A$. The set of filters of \mathfrak{A} is denoted by $\mathcal{F}(\mathfrak{A})$. A filter F of \mathfrak{A} is called *proper* if $F \neq A$. Clearly, F is a proper filter if and only if $0 \notin F$. For any subset X of A the *filter of \mathfrak{A} generated by X* is denoted by $\mathcal{F}(X)$. For each $x \in A$, the filter generated by $\{x\}$ is denoted by $\mathcal{F}(x)$ and called *principal filter*. The set of principal filters is denoted by $\mathcal{PF}(\mathfrak{A})$. For the basic facts concerning filters of a residuated lattice we refer to [20, 14, 18]. Following [10, §5.7] a complete lattice \mathfrak{A} is called a *frame* if it satisfies the join infinite distributive law (JID) i.e., for any $a \in A$ and $S \subseteq A$, $a \wedge \bigvee S = \bigvee \{a \wedge s \mid s \in S\}$. Following [4, Theorem 3.17] $(\mathcal{F}(\mathfrak{A}); \cap, \bigvee, \{1\}, A)$ is a frame in which $\bigvee \mathcal{F} = \mathcal{F}(\bigcup \mathcal{F})$, for any $\mathcal{F} \subseteq \mathcal{F}(\mathfrak{A})$.

Example 2.4. Consider the residuated lattice \mathfrak{A}_6 from Example 2.2 and the residuated lattice \mathfrak{A}_7 from Example 2.3. The set of their filters is presented in Table 3.

	filters
\mathfrak{A}_6	$\{1\}, \{d, 1\}, \{a, b, d, 1\}, \{c, d, 1\}, A_6$
\mathfrak{A}_7	$\{1\}, \{b, d, 1\}, \{e, 1\}, \{a, b, c, d, e, 1\}, A_7$

TABLE 3

The following proposition has a routine verification.

Proposition 2.5. *Let \mathfrak{A} be a residuated lattice and F be a filter of \mathfrak{A} . The following assertions hold for any $x, y \in A$:*

- (1) $\mathcal{F}(F, x) := F \vee \mathcal{F}(x) = \{a \in A \mid f \odot x^n \leq a, \exists f \in F \wedge \exists n \in \mathbb{N}\}$;
- (2) $x \leq y$ implies $\mathcal{F}(F, y) \subseteq \mathcal{F}(F, x)$.
- (3) $\mathcal{F}(F, x) \cap \mathcal{F}(F, y) = \mathcal{F}(F, x \vee y)$;
- (4) $\mathcal{F}(x) \vee \mathcal{F}(y) = \mathcal{F}(x \odot y)$;
- (5) $\mathcal{PF}(\mathfrak{A})$ is a sublattice of $\mathcal{F}(\mathfrak{A})$.

A proper filter of a residuated lattice \mathfrak{A} is called *maximal* if it is a maximal element in the set of all proper filters. The set of all maximal filters of \mathfrak{A} is denoted by $Max(\mathfrak{A})$. A proper filter P of \mathfrak{A} is called *prime*, if for any $x, y \in A$, $x \vee y \in P$ implies $x \in P$ or $y \in P$. The set of all prime filters of \mathfrak{A} is denoted by $Spec(\mathfrak{A})$. Since $\mathcal{F}(\mathfrak{A})$ is a distributive lattice, so $Max(\mathfrak{A}) \subseteq Spec(\mathfrak{A})$. By Zorn's lemma follows that any proper filter is contained in a maximal filter and so in a prime filter. A prime filter P is called *minimal prime* if P is a minimal element in the set of prime filters. The set of minimal prime filters of \mathfrak{A} is denoted by $Min(\mathfrak{A})$. For the basic facts concerning minimal prime filters of a residuated lattice we refer to [19, 17].

Example 2.6. Consider the residuated lattice \mathfrak{A}_6 from Example 2.2 and the residuated lattice \mathfrak{A}_7 from Example 2.3. By Example 2.4, the set of their maximal, prime and minimal prime filters is presented in Table 4.

Proposition 2.7. *In a residuated lattice any prime filter contains a minimal prime filter.*

Proof. It follows by [15, Corollary 3.25]. \square

Let \mathfrak{A} be a residuated lattice. For any subset X of A , we write $X^\perp = \{a \in A \mid a \vee x = 1, \forall x \in X\}$ and we set

prime filters			
	Maximal		Minimal
\mathfrak{A}_6	$\{a,b,d,1\}, \{c,d,1\}$	$\{d,1\}$	$\{1\}$
\mathfrak{A}_7	$\{a,b,c,d,e,1\}$		$\{b,d,1\}, \{e,1\}$

TABLE 4

- $\Gamma(\mathfrak{A}) = \{X^\perp | X \subseteq A\}$;
- $\gamma(\mathfrak{A}) = \{x^\perp | x \in A\}$;
- $\lambda(\mathfrak{A}) = \{x^{\perp\perp} | x \in A\}$.

Elements of $\Gamma(\mathfrak{A})$, $\gamma(\mathfrak{A})$ and $\lambda(\mathfrak{A})$ are called *coannihilators*, *coannulets* and *dual coannulets*, respectively. By [13, Proposition 3.13] follows that $(\Gamma(\mathfrak{A}); \cap, \vee^\Gamma, \{1\}, A)$ is a complete Boolean lattice in which for any $\mathcal{F} \subseteq \Gamma(\mathfrak{A})$ we have $\vee^\Gamma \mathcal{F} = (\cup \mathcal{F})^{\perp\perp}$ and by [16, Proposition 2.15] follows that $\gamma(\mathfrak{A})$ and $\lambda(\mathfrak{A})$ are sublattices of $\Gamma(\mathfrak{A})$. For the basic facts concerning coannihilators and coannulets of residuated lattices we refer to [13].

Proposition 2.8. [13] *Let \mathfrak{A} be a residuated lattice. The following assertions hold for any $X, Y \subseteq A$:*

- (1) $X \subseteq Y^\perp$ implies $Y \subseteq X^\perp$;
- (2) $X \cap X^\perp \subseteq \{1\}$;
- (3) $X \subseteq X^{\perp\perp}$;
- (4) $(\mathcal{F}(X))^\perp = X^\perp$;
- (5) $x^\perp = A$ if and only if $x = 1$;
- (6) $x^\perp \vee y^\perp \subseteq x^\perp \vee^\Gamma y^\perp = (x \vee y)^\perp$.

Following [16], a residuated lattice \mathfrak{A} is called *quasicomplemented* provided that $\lambda(\mathfrak{A}) \subseteq \gamma(\mathfrak{A})$. For a residuated lattice \mathfrak{A} a subset X of A is called *dense* if $X^\perp = \{1\}$. The set of all dense elements of \mathfrak{A} is denoted by $de(\mathfrak{A})$. It is easy to check that $de(\mathfrak{A})$ is an ideal of $\ell(\mathfrak{A})$. Also, it is obvious that any non-unit element of a chain residuated lattice is dense. The following Theorem characterizes quasicomplemented residuated lattices.

Theorem 2.9. [16, Proposition 3.2] *Let \mathfrak{A} be a residuated lattice. The following assertions are equivalent:*

- (1) \mathfrak{A} is quasicomplemented;
- (2) for any $x \in A$, there exists $y \in A$ such that $x \odot y \in de(\mathfrak{A})$ and $x \vee y = 1$;
- (3) $\gamma(\mathfrak{A})$ is a Boolean lattice.

Proposition 2.10. *Let \mathfrak{A} be a residuated lattice. \mathfrak{A} is quasicomplemented provided that $\gamma(\mathfrak{A}) \subseteq \mathcal{PF}(\mathfrak{A})$.*

Proof. It follows by [16, Proposition 3.3]. \square

Corollary 2.11. *Any finite residuated lattice is quasicomplemented.*

Proof. Following Proposition 2.5(4) in a finite residuated lattice any filter is principal, so it follows by Proposition 2.10. \square

The following example produces a residuated lattice which is not quasicomplemented.

Example 2.12. Consider the residuated lattice \mathfrak{A}_6 from Example 2.2 and let X be an infinite set. Define $\overline{A_6} = \{f \in A_6^X \mid \text{Card}(\text{Cosup}(f)) < \infty\} \cup \overline{0}$, where $\text{Cosup}(f) = \{x \in X \mid f(x) \neq 1\}$ and $\overline{0}$ is the zero function. Define $\nabla, \overline{\wedge}, \overline{\odot}$ and \Rightarrow pointwise and let $\overline{1}$ be the constant function 1. One can see easily that $\overline{\mathfrak{A}_6} = (\overline{A_6}; \nabla, \overline{\wedge}, \overline{\odot}, \Rightarrow, \overline{0}, \overline{1})$ is a residuated lattice. Now, we show that $de(\overline{\mathfrak{A}_6}) = \overline{0}$. Let $\overline{0} \neq f \in \overline{A_6}$ and $x_0 \in X \setminus \text{Cosup}(f)$. Define a mapping $g : X \rightarrow A_6$ as follows:

$$g : X \longrightarrow A_6$$

$$x \longmapsto \begin{cases} 0 & \text{if } x = x_0, \\ 1 & \text{else.} \end{cases}$$

It is obvious that $g \neq \overline{1}$ and $g \in f^\perp$. This shows that the residuated lattice $\overline{\mathfrak{A}_6}$ has no non-zero dense element. Let f be an arbitrary element of $\overline{A_6}$ and there exists $g \in \overline{A_6}$ such that $f \nabla g = \overline{1}$ and $f \overline{\odot} g$ is a dense element. This implies that $f \overline{\odot} g = \overline{0}$ which is impossible. So by Theorem 2.9 follows that the residuated lattice $\overline{\mathfrak{A}_6}$ is not quasicomplemented.

A filter F of a residuated lattice \mathfrak{A} is called an α -filter if for any $x \in F$ we have $x^{\perp\perp} \subseteq F$. The set of α -filters of \mathfrak{A} is denoted by $\alpha(\mathfrak{A})$. For any subset X of A , the α -filter generated by X is denoted by $\alpha(X)$. By [16, Proposition 5.3] follows that $(\alpha(\mathfrak{A}); \cap, \vee^\alpha, \{1\}, A)$ is a frame in which $\vee^\alpha \mathcal{F} = \alpha(\bigcup \mathcal{F})$, for any $\mathcal{F} \subseteq \alpha(\mathfrak{A})$. For the basic facts concerning α -filters and quasicomplemented residuated lattices we refer to [16].

Proposition 2.13. *Let \mathfrak{A} be residuated lattice. The following assertions hold:*

- (1) $\text{Min}(\mathfrak{A}) \subseteq \alpha(\mathfrak{A})$;
- (2) any prime filter contains a prime α -filter.

Proof.

(1): It follows by [16, Corollary 5.11].

(2): It is an immediate consequence of Proposition 2.7 and (1). \square

The next proposition characterizes quasicomplemented residuated lattice in terms of α -filters. In the sequel, for a residuated lattice \mathfrak{A} , we set $Spec_\alpha(\mathfrak{A}) = Spec(\mathfrak{A}) \cap \alpha(\mathfrak{A})$.

Proposition 2.14. [16, Corollary 5.17] *Let \mathfrak{A} be a residuated lattice. \mathfrak{A} is quasicomplemented if and only if $Spec_\alpha(\mathfrak{A}) = Min(\mathfrak{A})$.*

Let \mathfrak{A} be a residuated lattice. \mathfrak{A} is called *normal* provided that any prime filter of \mathfrak{A} contains a unique minimal prime filter. For a survey of normal residuated lattices we refer to [19].

Proposition 2.15. [19, Proposition 4.14] *Let \mathfrak{A} be a residuated lattice. The following assertions are equivalent:*

- (1) \mathfrak{A} is normal;
- (2) $\gamma(\mathfrak{A})$ is a sublattice of $\mathcal{F}(\mathfrak{A})$.

Example 2.16. Consider the residuated lattice \mathfrak{A}_6 from Example 2.2 and the residuated lattice \mathfrak{A}_7 from Example 2.3. By Example 2.6 follows that \mathfrak{A}_6 is normal and \mathfrak{A}_7 is not normal since that the prime filter $\{a, b, c, d, e, 1\}$ of \mathfrak{A}_7 contains the two minimal prime filters $\{b, d, 1\}$ and $\{e, 1\}$.

3. GENERALIZED STONE RESIDUATED LATTICE

The following proposition is an important property of principal filters of a residuated lattice, inspired by the one obtained for bounded distributive lattices by [10, Lemma 105].

Lemma 3.1. *Let F , G and H be filters of a residuated lattice \mathfrak{A} . If $G \cap H = \mathcal{F}(F, x)$ and $G \vee H = \mathcal{F}(F, y)$, then there exist $u, w \in A$ such that $G = \mathcal{F}(F, u)$ and $H = \mathcal{F}(F, w)$.*

Proof. $y \in G \vee H$ implies that $g \odot h \leq y$, for some $g \in G$ and $h \in H$. So $\mathcal{F}(F, y) \subseteq \mathcal{F}(F, g) \vee \mathcal{F}(F, h) \subseteq G \vee H = \mathcal{F}(F, y)$ and this shows that $\mathcal{F}(F, y) = \mathcal{F}(F, g) \vee \mathcal{F}(F, h)$. So, for any $z \in G$, we have $z \in \mathcal{F}(F, g) \vee (\mathcal{F}(F, h) \cap \mathcal{F}(F, z)) \subseteq \mathcal{F}(F, g \odot x)$ and this state that $G \subseteq \mathcal{F}(F, g \odot x)$. The inverse inclusion is evident and so $G = \mathcal{F}(F, g \odot x)$. By symmetry, we can obtain the other case. \square

The following example shows that the condition of the above lemma can be accrued.

Example 3.2. Consider the residuated lattice \mathfrak{A}_6 from Example 2.2. By considering Example 2.4 we set $F = \{d, 1\}$, $G = \{a, b, d, 1\}$ and $H = \{c, d, 1\}$. It is obvious that $G \cap H = \mathcal{F}(F, e)$ and $G \vee H = \mathcal{F}(F, 0)$. Also, we have $G = \mathcal{F}(F, a)$ and $H = \mathcal{F}(F, c)$.

Corollary 3.3. *Let G and H be two filters of a residuated lattice \mathfrak{A} . The following assertions hold:*

- (1) *If $G \cap H$ and $G \vee H$ are principal, then G and H are also principal;*
- (2) *if $G \cap H = \{1\}$ and $G \vee H = A$, then there exist $g \in G$ and $h \in H$ such that $G = \mathcal{F}(g)$, $H = \mathcal{F}(h)$ and $g \odot h = 0$.*

Proof. It is a direct consequence of Lemma 3.1. \square

Let $(A; \vee, \wedge, 0, 1)$ be a bounded lattice. Recall [10, §I.6.1] that an element $x \in A$ is called *complemented* if there is an element $y \in A$ such that $x \vee y = 1$ and $x \wedge y = 0$; y is called a complement of x . Complements in a bounded lattice are generally not unique unless the lattice is distributive.

Definition 3.4. Let \mathfrak{A} be a residuated lattice. The set of complemented elements of $\mathcal{F}(\mathfrak{A})$ shall be denoted by $\beta(\mathcal{F}(\mathfrak{A}))$ and its elements are called *the direct summands* of \mathfrak{A} .

Let \mathfrak{A} be a residuated lattice. Since $\mathcal{F}(\mathfrak{A})$ is a frame, so by [10, Lemma 97] follows that $\beta(\mathcal{F}(\mathfrak{A}))$ is a bounded sublattice of $\mathcal{F}(\mathfrak{A})$ and thus a Boolean lattice.

Corollary 3.5. *Let \mathfrak{A} be a residuated lattice. Then $\beta(\mathcal{F}(\mathfrak{A})) \subseteq \mathcal{PF}(\mathfrak{A})$.*

Proof. It is evident by Corollary 3.3(2). \square

Definition 3.6. A residuated lattice \mathfrak{A} is called *generalized Stone* if for any $x \in A$ we have $x^\perp \vee x^{\perp\perp} = A$.

Proposition 3.7. *Let \mathfrak{A} be a residuated lattice. \mathfrak{A} is generalized Stone if and only if $\gamma(\mathfrak{A}) \subseteq \beta(\mathcal{F}(\mathfrak{A}))$.*

Proof. Let \mathfrak{A} be generalized Stone and $x \in A$. By Proposition 2.8(2) follows that $x^\perp \cap x^{\perp\perp} = \{1\}$ and by hypothesis we have $x^\perp \vee x^{\perp\perp} = A$. It shows that x^\perp is a direct summand of \mathfrak{A} . Conversely, consider $x \in A$. By hypothesis there exists a filter F such that $x^\perp \cap F = \{1\}$ and $x^\perp \vee F = A$. Since $x^{\perp\perp}$ is a pseudocomplement of x^\perp so we obtain that $F \subseteq x^{\perp\perp}$ and it follows that $A = x^\perp \vee F \subseteq x^\perp \vee x^{\perp\perp}$. So $x^\perp \vee x^{\perp\perp} = A$ and it shows that \mathfrak{A} is generalized Stone. \square

Proposition 3.8. *Let \mathfrak{A} be a residuated lattice. If \mathfrak{A} is generalized Stone, then \mathfrak{A} is quasi-complemented.*

Proof. It is an immediate consequence of Corollary 3.5 and Proposition 3.7. \square

Proposition 3.9. *Let \mathfrak{A} be a residuated lattice. If \mathfrak{A} is generalized Stone, then \mathfrak{A} is normal.*

Proof. Let $x, y \in A$ and $a \in (x \vee y)^\perp$. So $a \vee y \in x^\perp$ and it implies that $\mathcal{F}(a \vee y) \cap x^{\perp\perp} = \{1\}$. It means that $(\mathcal{F}(a) \cap \mathcal{F}(y)) \cap x^{\perp\perp} = \{1\}$ and it states that $\mathcal{F}(a) \cap x^{\perp\perp} \subseteq y^\perp$. Now, we have the following sequence of formulas:

$$\begin{aligned} x^\perp \vee y^\perp &\supseteq (\mathcal{F}(a) \cap x^\perp) \vee (\mathcal{F}(a) \cap x^{\perp\perp}) \\ &= \mathcal{F}(a) \cap (x^\perp \vee x^{\perp\perp}) \\ &= \mathcal{F}(a) \cap A = \mathcal{F}(a) \ni a. \end{aligned}$$

The inverse inclusion follows by Proposition 2.8(6). Thus the result holds by Proposition 2.15.

□

The following corollary gives the interrelation between the subclasses of quasicomplemented, normal and generalized stone residuated lattices (See Fig. 3).

Corollary 3.10. *Let \mathfrak{A} be a residuated lattice. The following assertions are equivalent:*

- (1) \mathfrak{A} is generalized Stone;
- (2) \mathfrak{A} is quasicomplemented and normal.

Proof.

(1) \Rightarrow (2): It follows by Proposition 3.8 and 3.9

(2) \Rightarrow (1): Consider $x \in A$. So there exists $y \in A$ such that $x^{\perp\perp} = y^\perp$ and it implies that $x^\perp \vee x^{\perp\perp} = x^\perp \vee y^\perp = x^\perp \vee^\Gamma y^\perp = x^\perp \vee^\Gamma x^{\perp\perp} = A$. □

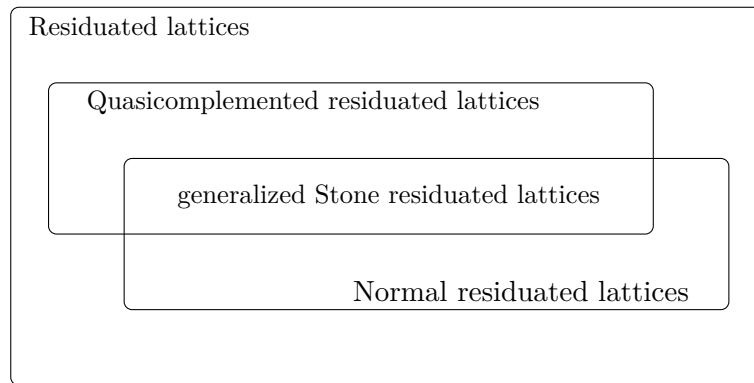


FIGURE 3

Corollary 3.11. *A finite residuated lattice is generalized Stone if and only if it is normal.*

Proof. It follows by Corollary 2.11 and 3.10. □

Example 3.12. Consider the residuated lattice \mathfrak{A}_6 from Example 2.2 and the residuated lattice \mathfrak{A}_7 from Example 2.3. By Corollary 3.11 and Example 2.16 follows that \mathfrak{A}_6 is generalized Stone and \mathfrak{A}_7 is not generalized Stone.

Following [2, p. 473], the unit real interval $\mathbb{I} = [0, 1]$ by setting $x \odot y = \max\{0, x + y - 1\}$, and $x \rightarrow y = 1 - x + y$ for all $x, y \in \mathbb{I}$ and the natural ordering of real numbers becomes an MV-algebra and so a residuated lattice.

Remark 3.13. Let X be a topological space, and \mathbb{I} endowed with the natural topology. Following [3, p.648] and [7, p. 23], the family $C(X) = \{f \in I^X \mid f \text{ is continuous}\}$ by the MV-operations induced pointwise on \mathbb{I} has a structure of MV-algebra and so a residuated lattice. A zero-set and a cozero-set of X is $Z(f) = f^{-1}(0)$ and $Coz(f) = X \setminus U(f)$ for some $f \in C(X)$, respectively. Clearly, zero-sets are closed in X and cozero-sets are open. A topological space X is called an F -space provided that disjoint cozero-sets are completely separable (i.e., they are separated by disjoint zero-sets) (see [9, 14N]). X is said to be *basically disconnected* if the closure of every cozero-set is an open set (see [9, 1H]). By [3, Theorem 4.3], a topological space X is an F -space if and only if $C(X)$ is a normal MV-algebra (Hypernormal MV-algebra in terminology of [3]). Also, by [7, Theorem 4.5], a topological space X is a basically disconnected space if and only if $C(X)$ is a generalized Stone MV-algebra (Stonian MV-algebra in terminology of [7]). Let \mathbb{R}^+ be the space of nonnegative reals with the topology induced by the usual topology of \mathbb{R} , and let $\beta(\mathbb{R}^+)$ be the Stone-Ćech compactification of \mathbb{R}^+ . The topological space $\beta(\mathbb{R}^+) \setminus \mathbb{R}^+$ is an F -space which is not basically disconnected (see [9, 14.28]). So $C(\beta(\mathbb{R}^+) \setminus \mathbb{R}^+)$ is a normal MV-algebra which is not generalized Stone. So $C(\beta(\mathbb{R}^+) \setminus \mathbb{R}^+)$ is a normal residuated lattice which is not a quasicomplemented residuated lattice.

Theorem 3.14. *Let \mathfrak{A} be a residuated lattice. \mathfrak{A} is generalized Stone if and only if any prime filter of \mathfrak{A} contains a unique prime α -filter.*

Proof. Let \mathfrak{A} be generalized Stone and P be a prime filter of \mathfrak{A} . Following Corollary 3.10, \mathfrak{A} is quasicomplemented and normal. By Proposition 2.13(2) follows that P contains a prime α -filter. Let Q be a prime α -filter containing in P . By Proposition 2.14 follows that Q is minimal prime and so it is unique by normality of \mathfrak{A} . Conversely, let any prime filter of \mathfrak{A} contains a unique prime α -filter. By Proposition 2.13(1) follows that any prime filter contains a unique minimal prime filter and so \mathfrak{A} is normal. If P is a prime α -filter of \mathfrak{A} , then P contains a minimal prime filter \mathfrak{m} due to Proposition 2.7. This implies that $\mathfrak{m} = P$ and so \mathfrak{A} is quasicomplemented. It shows that \mathfrak{A} is generalized Stone by Corollary 3.10. \square

According to [21, Definition 7.6], a residuated lattice is called *directly indecomposable* if and only if it is not isomorphic to a direct product of two non-degenerate residuated lattices.

Proposition 3.15. *Let \mathfrak{A} , \mathfrak{A}_1 and \mathfrak{A}_2 be residuated lattices. $\mathfrak{A} \cong \mathfrak{A}_1 \times \mathfrak{A}_2$ if and only if $\mathfrak{A}_1 \cong \mathfrak{A}/F_1$ and $\mathfrak{A}_2 \cong \mathfrak{A}/F_2$, for some $F_1, F_2 \in \beta(\mathcal{F}(\mathfrak{A}))$.*

Proof. In a similar manner with [21, Theorem 7.5], we can show that $\mathfrak{A} \cong \mathfrak{A}_1 \times \mathfrak{A}_2$ if and only if there exist two filters F_1 and F_2 of \mathfrak{A} such that $\mathfrak{A}/F_1 \cong \mathfrak{A}_1$, $\mathfrak{A}/F_2 \cong \mathfrak{A}_2$, $F_1 \cap F_2 = \{1\}$ and $F_1 \vee F_2 = A$. \square

Corollary 3.16. *Let \mathfrak{A} be a residuated lattice. \mathfrak{A} is directly indecomposable if and only if $\beta(\mathcal{F}(\mathfrak{A})) = \{\{1\}, A\}$.*

Proof. It is an immediate consequence of Proposition 3.15. \square

Corollary 3.17. *Let \mathfrak{A} be a residuated lattice. If \mathfrak{A} is directly indecomposable and generalized Stone, then $de(\mathfrak{A}) = A \setminus \{1\}$.*

Proof. Let \mathfrak{A} be directly indecomposable and generalized Stone. Consider $1 \neq x \in A$. By Proposition 3.7 follows that $x^\perp \in \beta(\mathcal{F}(\mathfrak{A}))$ and by Corollary 3.16 follows that $\beta(\mathcal{F}(\mathfrak{A})) = \{\{1\}, A\}$. Following by Proposition 2.8(5) we get that $x \in de(\mathfrak{A})$. \square

Corollary 3.18. *Let \mathfrak{A} be a finite residuated lattice. If \mathfrak{A} is directly indecomposable and normal, then $de(\mathfrak{A}) = A \setminus \{1\}$.*

Proof. It is an immediate consequence of Corollary 3.11 and 3.17. \square

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