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Research Paper

BOOLEAN EXPRESSION BASED ON HYPERGRAPHS WITH ALGORITHM

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Abstract. This paper introduces a novel concept of Boolean function–based hypergraph with respect to any given T.B.T(total binary truth table). This study defines a notation of kernel set on switching functions and proves that every T.B.T corresponds to a Minimum Boolean expression via kernel set and presents some conditions on T.B.T to obtain a Minimum irreducible Boolean expression from switching functions. Finally, we present an algorithm and so Python programming(with complete and original codes) such that for any given T.B.T, introduces a Minimum irreducible switching expression.

1. Introduction

The concept of hypergraph has been introduced by Berge as a generalization of graph around 1960 and one of the initial concerns was to extend some classical results of graph theory and the notion of hypergraph has been considered as a useful tool to analyze the structure of a

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system. Graphs and hypergraphs can be used to describe the network systems. Today, some features of hypergraphs are used in computer science, notably in machine learning, and there has been a lot of research about using hypergraphs in relational databases, which might be viewed as a sort of data mining. The reason why hypergraphs seem apt to depict relations in information systems, social networks, document centered information processing, web information systems and computer science, are the relationships among services within a service oriented architecture[4, 5, 9]. With respect to the classical hypergraph, Smarandache (2019) added the supervertices (a group of vertices put all together form a supervertex), in order to form a superhypergraph (SHG). Therefore, each SHG-vertex and each SHG-edge belong to $P(V)$, where V is the set of vertices, and $P(V)$ means the power set of V. Further on, since in our world we encounter complex and sophisticated groups of individuals and complex and sophisticated connections between them, Smarandache extended the superhypergraph to n-superhypergraph, by extending $P(V)$ to $P^n(V)$ that is the n-power set of the set V (i.e. power-set of the power-set of... the powerset of V, n times)[10, 11]. Further materials regarding graphs and hypergraphs are available in the literature too [1, 4, 5, 6, 7, 8]. Two major new works on logic were published by prominent British mathematicians, formal logic by Augustus De Morgan (1806–1871) and the mathematical analysis of Logic by George Boole (1815–1864). Both authors sought to stretch the boundaries of traditional logic by developing a general method for representing and manipulating logically valid inferences or to develop mechanical modes of making transitions. Historically, propositional logic and electrical engineering have been the main nurturing fields for the development of research on Boolean functions. However, because they are such fundamental mathematical objects, Boolean functions have also been used to model a large number of applications in a variety of areas. In most applications, however, more information is available regarding the process that generates the function of interest asillus trated by Electrical engineering, Artificial neural networks, Reliability theory, Game theory, Integer programming, Distributed computing systems, etc. In other view, Boolean expressions have been widely used to represent a decision/predicate and a number of branch testing and techniques have been reported in the literature[2, 3].

Regarding these points, this paper considers the concepts of, Boolean expression, hypergraphs and with respect to combination of these concepts, applies it in computer science. In this paper we introduce the notation of switching functions and investigates the relation between hypergraphs and switching functions. This study, for any Boolean function constructs a hyperdiagram which is called a hyperdiagram based on a given Boolean function and investigate some condition it be a hypergraph based on a given Boolean function. Also for all arbitrary hypergraph, is extracted a Boolean function titled Boolean functionable hypergraph. There is a natural question that, do it correspond a switching expression from any given T.B.T. The main our motivation from this paper is extraction an irreducible switching expression from any T.B.T. So we define the concept of Boolean function–based hypergraph and the notation of unitors set of Boolean functions. In final, we apply these concepts and prove that every T.B.T corresponds to a Minimum Boolean expression via kernels set and presents some conditions on T.B.T to obtain a Minimum irreducible Boolean expression from switching functions.

2. **Preliminaries**

In this section, we recall some definitions and results, which we need in what follows.

Let X be an arbitrary set. Then we denote $P^*(X) = P(X) \setminus \emptyset$, where $P(X)$ is the power set of X.

Definition 2.1. [1] Let $G = \{x_1, x_2, \ldots, x_n\}$ be a finite set. A *hypergraph* on G is a pair $H = (G, \{E_i\}_{i=1}^m)$ such that for all $1 \leq i \leq m, \emptyset \neq E_i \subseteq G$ and $\bigcup\limits_{i=1}^{m} E_i = G$. The elements $x_1, x_2, \ldots, x_n$ of G are called *vertices*, and the sets $E_1, E_2, \ldots, E_m$ are called the *edges* (*hyperedges*) of the hypergraph H. For each $1 \leq k \leq m$ if $|E_k| \geq 2$, then E_k is represented by a continuous curve joining its vertices, if $|E_k| = 1$ by a cycle on the element (*loop*). If for all $1 \leq k \leq m$ $|E_k| = 2$, the hypergraph becomes an ordinary (undirected) graph.

Definition 2.2. [4] Let $G = \{x_1, x_2, \ldots, x_n\}$ be a finite set. A *hyperdiagram* on G is a pair $H = (G, \{E_k\}_{k=1}^m)$ such that for all $1 \leq k \leq m, E_k \subseteq G$ and $|E_k| \geq 1$. Clearly every hypergraph is a hyperdiagram, while the converse is not necessarily true.

We say that two hyperdiagrams $H = (G, \{E_k\}_{k=1}^m)$ and $H' = (G', \{E'_k\}_{k=1}^{m'})$ are isomorphic if $m = m'$ and there exists a bijection $\varphi : G \to G'$ and a permutation $\tau : \{1, 2, \ldots, m\} \to \{1, 2, \ldots, m'\}$ such that for all $x, y \in G$, if for some $1 \leq i \leq m, x, y \in E_i$, then $\varphi(x), \varphi(y) \in E_{\tau(i)}$, if for all $1 \leq i \leq m, x, y \notin E_i$, then $\varphi(x), \varphi(y) \notin E_{\tau(i)}$ and if for some $1 \leq i \leq m, x \in E_i$, for all $1 \leq j \leq m, y \notin E_j$, then $\varphi(x) \in E_{\tau(i)}$ and $\varphi(y) \notin E_j$. Since every hypergraph is a hyperdiagram, define an isomorphic hypergraphs in a similar a way.

3. **Switching expression based on hypergraph**

In this section, we apply the notation of total binary truth table(T.B.T) on Boolean variables and introduce the concept of hypergraphable Boolean functions, Boolean functionable hypergraphs and investigate some of their properties. We consider every (*switching*)*Boolean function* $f : B_n \to B = \{0, 1\}$ by $f(x_1, x_2, \ldots, x_n) = \prod\limits_{j=1}^{m} \sum\limits_{i=1}^{k_j} \overline{x_i}$, where for all $1 \leq i \leq n, \overline{x_i}$ is a *literal* (Boolean variable or the complement of a Boolean variable) and $m, j, k_j \in \mathbb{N}$. Let $n \in \mathbb{N}, m \in$

TABLE 1. T. B. T with n variables $\mathcal{T}(f^{(0)}, f^{(1)}, \ldots, f^{(m)}, x_1, x_2, \ldots, x_n)$

x_1	x_2	...	x_n	$f^{(0)}(x_1, \ldots, x_n)$	$f^{(1)}(x_1, \ldots, x_n)$	...	$f^{(m)}(x_1, \ldots, x_n)$
0	0	...	0	$f_1^{(0)}(x_1, \ldots, x_n)$	$f_1^{(1)}(x_1, \ldots, x_n)$	...	$f_1^{(m)}(x_1, \ldots, x_n)$
0	0	...	1	$f_2^{(0)}(x_1, \ldots, x_n)$	$f_2^{(1)}(x_1, \ldots, x_n)$	...	$f_2^{(m)}(x_1, \ldots, x_n)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
0	0	...	1	$f_{2^n-1}^{(0)}(x_1, \ldots, x_n)$	$f_{2^n-1}^{(1)}(x_1, \ldots, x_n)$	...	$f_{2^n-1}^{(m)}(x_1, \ldots, x_n)$
1	1	...	1	$f_{2^n}^{(0)}(x_1, \ldots, x_n)$	$f_{2^n}^{(1)}(x_1, \ldots, x_n)$	...	$f_{2^n}^{(m)}(x_1, \ldots, x_n)$

TABLE 2. T.B.T $\mathcal{T}(f, g, x_1, x_2, x_3)$

x_1	x_2	x_3	f	g	$f+g$	$f.g$	$c(f)$
0	0	0	1	0	1	0	0
0	0	1	0	0	0	0	1
0	1	0	1	1	1	1	0
0	1	1	0	1	1	0	1
1	0	0	0	1	1	0	1
1	0	1	1	1	1	1	0
1	1	0	0	0	0	0	1
1	1	1	1	1	1	1	0

$\mathbb{N}^*$, $x_1, x_2, \ldots, x_n$ be arbitrary Boolean variables and for all $0 \leq j \leq m$, $f^{(m)}(x_1, x_2, \ldots, x_n)$ be Boolean functions. We will denote a total binary truth table(T.B.T) on Boolean variables $x_1, x_2, \ldots, x_n$ by a set $\mathcal{T}(f^{(0)}, f^{(1)}, \ldots, f^{(m)}, \overline{x_1}, \ldots, \overline{x_n}) = \{f^{(0)}, f^{(1)}, \ldots, f^{(m)}, (x_1, \ldots, x_n)\}$, where for all $0 \leq j \leq m$, $f^{(m)}(x_1, x_2, \ldots, x_n)$, are Boolean functions(see a Table 1) and for $m = 0$, we will denote it by $\mathcal{T}(f, x_1, x_2, \ldots, x_n)$.

$c(f(x_1, \ldots, x_n)) = 1 - f(x_1, \ldots, x_n)$.

Example 3.1. Consider a T.B.T $\mathcal{T}(f, g, x_1, x_2, x_3)$ in Table 2. The binary operations together with a unary operation is computed in this Table.

Theorem 3.2. $(\mathcal{T}(f, f', x_1, \ldots, x_n), +, ., c)$ *is a Boolean algebra.*

Definition 3.3. Let $n \in \mathbb{N}$ and $\mathcal{T}(f, x_1, x_2, \ldots, x_n)$ be a T.B.T. For all $1 \leq j \leq 2^n$ define $Kernel(f_j) = \{(x_1, x_2, \ldots, x_n) \mid f_j(x_1, x_2, \ldots, x_n) = 0\}$ and will denote by $Ker(f_j)$, in a similar a way $Kernel(f)$ is defined and it is denoted by $Ker(f)$.

Theorem 3.4. *Let $n \in \mathbb{N}, 1 \leq j \leq 2^n$ and $\mathcal{T}(f, f', x_1, x_2, \ldots, x_n)$ be a T.B.T. Then*

(i) $Ker(f) = \bigcup_{j=1}^{2^n} Ker(f_j)$;

(ii) $Ker(f') = Ker(f)$ *if and only if* $f \sim f'$;

(iii) *If* $Ker(f) \subseteq Ker(f')$, *then* $(f + f') \sim f$;

(iv) *If* $Ker(f') \subseteq Ker(f)$, *then* $(f.f') \sim f$;

(v) $Ker(f + f') = Ker(f) \cap Ker(f')$;

(vi) $Ker(f.f') = Ker(f) \cup Ker(f')$;

(vii) $Ker(c(f)) = B_n \setminus Ker(f)$.

Proof. Let $n \in \mathbb{N}$ and $(x_1, \ldots, x_n) \in B_n$.

(i) Since $Ker(f) = \{(x_1, \ldots, x_n) \mid \prod_{j=1}^{2^n} f_j(x_1, \ldots, x_n) = 0\}$, we get

$$(x_1, \ldots, x_n) \in Ker(f) \Leftrightarrow \exists\ 1 \leq j \leq 2^n \text{ such that } \ f_j(x_1, \ldots, x_n) = 0$$

$$\Leftrightarrow \quad \text{for some } \ 1 \leq j \leq 2^n, (x_1, \ldots, x_n) \in Ker(f_j) \Leftrightarrow (x_1, \ldots, x_n) \in \bigcup_{j=1}^{2^n} Ker(f_j).$$

(ii) Let $(x_1, x_2, \ldots, x_n) \in Ker(f)$. Then $f(x_1, x_2, \ldots, x_n) = 0$ and because $f \sim f'$, we get that $f'(x_1, x_2, \ldots, x_n) = 0$. It follows that $(x_1, x_2, \ldots, x_n) \in Ker(f')$ and $Ker(f) \subseteq Ker(f')$ and in a similar a way $Ker(f') \subseteq Ker(f)$. Let $Ker(f') = Ker(f)$. Then $f(x_1, x_2, \ldots, x_n) = 0$ implies that $f'(x_1, x_2, \ldots, x_n) = 0$ and $f(x_1, x_2, \ldots, x_n) = 1$ implies that $f'(x_1, x_2, \ldots, x_n) = 1$. So $f \sim f'$.

(iii) If $(x_1, x_2, \ldots, x_n) \in Ker(f)$, then $(f + f')(x_1, x_2, \ldots, x_n) = f(x_1, x_2, \ldots, x_n) + f'(x_1, x_2, \ldots, x_n) = 0 = f(x_1, x_2, \ldots, x_n)$. If $(x_1, x_2, \ldots, x_n) \notin Ker(f)$, then $(f + f')(x_1, x_2, \ldots, x_n) = f(x_1, x_2, \ldots, x_n) + f'(x_1, x_2, \ldots, x_n) = 1 = f(x_1, x_2, \ldots, x_n)$.

(iv) If $(x_1, x_2, \ldots, x_n) \in Ker(f)$, then $(f.f')(x_1, x_2, \ldots, x_n) = f(x_1, x_2, \ldots, x_n).f'(x_1, x_2, \ldots, x_n) = 0 = f(x_1, x_2, \ldots, x_n)$. If $(x_1, x_2, \ldots, x_n) \notin Ker(f)$, because $Ker(f') \subseteq Ker(f)$, we get that $(x_1, \ldots, x_n)$ $\notin Ker(f')$ and so $(f.f')(x_1, x_2, \ldots, x_n) = f(x_1, x_2, \ldots, x_n).f'(x_1, x_2, \ldots, x_n) = 1 = f(x_1, x_2, \ldots, x_n)$.

(v) By definition, $(f + f')(x_1, x_2, \ldots, x_n) = 0$ if and only if $f(x_1, x_2, \ldots, x_n) + f'(x_1, x_2, \ldots, x_n) = 0$ if and only if $f(x_1, x_2, \ldots, x_n) = 0$ and $f'(x_1, x_2, \ldots, x_n) = 0$ if and only if $(x_1, x_2, \ldots, x_n) \in Ker(f) \cap Ker(f')$.

(vi) By definition, $(f.f')(x_1, x_2, \ldots, x_n) = 0$ if and only if $f(x_1, x_2, \ldots, x_n).f'(x_1, x_2, \ldots, x_n) = 0$ if and only if $f(x_1, x_2, \ldots, x_n) = 0$ or $f'(x_1, x_2, \ldots, x_n) = 0$ if and only if $(x_1, x_2, \ldots, x_n) \in Ker(f) \cup Ker(f')$.

(vii) $(x_1, x_2, \ldots, x_n) \in Ker(c(f))$ if and only if $c(f)(x_1, x_2, \ldots, x_n) = 0$ if and only if $1 - f(x_1, x_2, \ldots, x_n)$
$= 0$ if and only if $f(x_1, x_2, \ldots, x_n) = 1$ if and only if $(x_1, x_2, \ldots, x_n) \notin Ker(f)$. □

Example 3.5. Consider a T.B.T $\mathcal{T}(f, g, x_1, x_2, x_3)$ in Table 2. Computations show that $Ker(f + g) = \{(0, 0, 1), (1, 1, 0)\} = Ker(f) \cap Ker(g) = \{(0, 0, 1), (1, 0, 0), (0, 1, 1), (1, 1, 0)\} \cap \{(0, 0, 0), (0, 0, 1), (1, 1, 0)\}$ and $Ker(f.g) = \{(0, 0, 0), (0, 0, 1), (0, 1, 1), (1, 0, 0), (1, 1, 0)\} = Ker(f) \cup Ker(g)$.

Corollary 3.6. *Let $n \in \mathbb{N}, 1 \leq j \leq n$ and $\mathcal{T}(f, f', x_1, x_2, \ldots, x_n)$ be a T.B.T. If $(f + f') \sim f$, then $|Ker(f)| \leq |Ker(f')|$.*

Proof. Since $(f + f') \sim f$, by Theorem 3.4, $Ker(f + f') = Ker(f)$. It follows that $Ker(f) \cap Ker(f') = Ker(f + f') = Ker(f)$ and so $Ker(f) \subseteq Ker(f')$. □

Corollary 3.7. *Let $n \in \mathbb{N}, 1 \leq j \leq n$ and $\mathcal{T}(f, f', x_1, x_2, \ldots, x_n)$ be a T.B.T. If $(f.f') \sim f$, then $|Ker(f')| \leq |Ker(f)|$.*

Proof. Since $(f.f') \sim f$, by Theorem 3.4, $Ker(f.f') = Ker(f)$. It follows that $Ker(f) \cup Ker(f') = Ker(f.f') = Ker(f)$ and so $Ker(f') \subseteq Ker(f)$. □

In this section, consider any T.B.T and with respect to concept of Kernels set, try to extract associated switching expression to given T.B.T.

Theorem 3.8. *Let $n \in \mathbb{N}$. Then every $\mathcal{T}(f \not\equiv 0, x_1, x_2, \ldots, x_n)$ corresponds to a hypergraph.*

Proof. Let $x_1, x_2, \ldots, x_n$ be arbitrary Boolean variables. Consider a total binary truth table (T.B.T)$\mathcal{T}$
$(f, x_1, x_2, \ldots, x_n)$ in Table 1. Suppose that for $k \in \mathbb{N}$ and for all $i \in \{j_1, j_2, \ldots, j_k\}$, we have $Ker(f_i) \neq \emptyset$. Since $\sum_{j=1}^{n} \overline{x_j} = 0$ if and only if for all $1 \leq j \leq n, \overline{x_j} = 0$, for all $j_1 \leq i \leq j_k$ define $f_{j_i}(x_1, x_2, \ldots, x_n) = \overline{x_1} + \overline{x_2} + \ldots + \overline{x_n}$, where $\sum_{j=1}^{n} \overline{x_j} = 0$. Now, for all $j_1 \leq i \leq j_k$, set

$$E_i = \{\overline{x_1}, \overline{x_2}, \ldots, \overline{x_n} \mid \sum_{j=1}^{n} \overline{x_j} = f_j(x_1, x_2, \ldots, x_n)\}.$$

TABLE 3. T. B. T with 3 variables $\mathcal{T}(f, x_1, x_2, x_3)$

x_1	x_2	x_3	$f(x_1, x_2, x_3)$
0	0	0	$f_1(x_1, x_2, x_3) = 1$
0	0	1	$f_2(x_1, x_2, x_3) = 1$
0	1	0	$f_3(x_1, x_2, x_3) = 1$
0	1	1	$f_4(x_1, x_2, x_3) = 0$
1	0	0	$f_5(x_1, x_2, x_3) = 0$
1	0	1	$f_6(x_1, x_2, x_3) = 0$
1	1	0	$f_7(x_1, x_2, x_3) = 1$
1	1	1	$f_8(x_1, x_2, x_3) = 1$

Thus it is easy to see that $H = (G = \bigcup_{i=j_1}^{j_k} E_i, \{E_i\}_{i=j_1}^{j_k})$ is a hypergraph. □

We will call the hypergraph H in Theorem 3.8, as *Boolean function–based hypergraph* and will denote by $(H, \mathcal{T})$.

Example 3.9. Consider a T.B.T $\mathcal{T}(f, x_1, x_2, x_3)$ in Table 3. Computations show that $Ker(f_4) = \{(x_1, x_2', x_3')\}, Ker(f_5) = \{(x_1', x_2, x_3)\}, Ker(f_6) = \{(x_1', x_2, x_3')\}$. Now, it is obtained a Boolean function–based hypergraph $(H, \mathcal{T})$ in Figure 1.

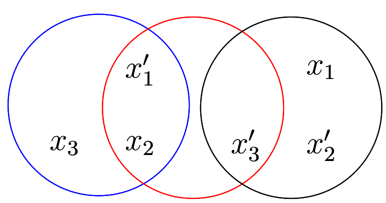

FIGURE 1. Hypergraph $(H, \mathcal{T})$

Define a relation $\sim$ on a T.B.T $\mathcal{T}(f, f', x_1, x_2, \ldots, x_n)$ by $f \sim f'$ if and only if for all $(x_1, x_2, \ldots, x_n) \in B_n$, we have $f(x_1, x_2, \ldots, x_n) = f'(x_1, x_2, \ldots, x_n)(f \equiv f')$. It is clear that $\sim$ is a congruence equivalence relation on $\mathcal{T}(f, f', x_1, x_2, \ldots, x_n)$. For $0 \leq j, j' \leq m$, we say that $\mathcal{T}(f^{(j)}, x_1, x_2, \ldots, x_n)$ and $\mathcal{T}'(f^{(j')}, x_1, x_2, \ldots, x_n)$ are equivalent, if $f^{(j)} \sim f^{(j')}$.

Theorem 3.10. *Let $0 \leq j, j' \leq m$. If $\mathcal{T}(f^{(j)}, x_1, \ldots, x_n)$ and $\mathcal{T}'(f^{(j')}, x_1, \ldots, x_n)$ are equivalent, then their Boolean function–based hypergraph are isomorphic.*

Proof. Let $0 \leq j, j' \leq m$. Since $\mathcal{T}(f^{(j)}, x_1, \ldots, x_n) \sim \mathcal{T}'(f^{(j')}, x_1, \ldots, x_n)$, for all $1 \leq i \leq 2^n$ we get that $Ker(f_i^{(j)}) = Ker(f_i^{(j')})$. Using Theorem 3.8, $\mathcal{T}(f^{(j)}, x_1, x_2, \ldots, x_n)$ corresponds to $(H, \mathcal{T})$ if and only if $\mathcal{T}(f^{(j')}, x_1, x_2, \ldots, x_n)$ corresponds to $(H', \mathcal{T}')$. Hence $(H, \mathcal{T}) \cong (H', \mathcal{T}')$. □

Definition 3.11. Let $n \in \mathbb{N}, m \in \mathbb{N}^*$, $1 \leq k \leq n$ and $\mathcal{T}(f^{(0)}, \ldots, f^{(m)}, x_1, \ldots, x_n)$ be a T.B.T, where for $0 \leq t \leq m$, $f^{(t)}(x_1, \ldots, x_n) = \prod_{i=1}^{2^n} f_i^{(t)}(x_1, x_2, \ldots, x_n)$. Then

(*i*) $Z(n, f^{(t)}, 0) = \{j \mid f_j^{(t)}(x_1, x_2, \ldots, x_n) = 0, \text{ where } 1 \leq j \leq 2^n\}$;

(*ii*) $S(k, x_1, x_2, \ldots, x_k, 0) = \{\sum_{i=1}^{n} \overline{x_i} \mid \sum_{i=1}^{k} x_i + \sum_{i=k+1}^{n} \overline{x_i} = 0\}$.

Example 3.12. Consider a T.B.T $\mathcal{T}(f, x_1, x_2, x_3)$ in Table 3. Simple computations show $Z(n, f, 0) = \{j \mid f_j(x_1, x_2, x_3) = 0, \text{ where } 1 \leq j \leq 8\} = 3, S(3, x_1, x_2, x_3, 0) = \{x_1 + x_2 + x_3\}, S(2, x_1, x_2, 0) = \{x_1 + x_2 + x_3, x_1 + x_2 + x_3'\}$, $S(1, x_1, 0) = \{x_1 + x_2 + x_3, x_1 + x_2 + x_3', x_1 + x_2' + x_3, x_1 + x_2' + x_3'\}, |S(3, x_1, x_2, x_3, 0)| = 1, |S(2, x_1, x_2, 0)| = 2$ and $|S(1, x_1, 0)| = 4$.

Theorem 3.13. *Let $n \in \mathbb{N}, 1 \leq j \leq n$ and $\mathcal{T}(f, x_1, x_2, \ldots, x_n)$ be a T.B.T. Then $|S(k = j, x_1, x_2, \ldots, x_k$
$, 0)| = 2^{n-j}$.*

Proof. Let $x_1, x_2, \ldots, x_j$ be j Boolean variables. Then $\sum_{i=1}^{j} x_i = 0$ if and only if $x_1 = x_2 = \ldots = x_j = 0$. So $\sum_{i=j+1}^{n} \overline{x_i} = 0$ or $\sum_{i=j+1}^{n} \overline{x_i} = 1$. Now, $\sum_{i=j+1}^{n} \overline{x_i} = 1$ if and only if for some $j+1 \leq t \leq n$ we have $\overline{x_t} = 1$, where have $2^{n-j} - 1$ cases. In addition, $\sum_{i=j+1}^{n} \overline{x_i} = 0$ if and only if for all $j+1 \leq t \leq n$ we have $\overline{x_t} = 0$. So $|S(j, x_1, x_2, \ldots, x_j, 0)| = 2^{n-j}$. □

Corollary 3.14. *Let $n \in \mathbb{N}, m \in \mathbb{N}^*$ and $\mathcal{T}(f, x_1, x_2, \ldots, x_n)$ be a T.B.T. Then*

(*i*) *if $k = 1$, then $|S(k, x_1, x_2, \ldots, x_k, 0)| = 2^{n-1}$;*

(*ii*) *if $k = n - 1$, then $|S(k, x_1, x_2, \ldots, x_k, 0)| = 2$;*

(*ii*) *if $k = n - i$, then $|S(k, x_1, x_2, \ldots, x_k, 0)| = 2^i$;*

(*i*) *if $k = n$, then $|S(k, x_1, x_2, \ldots, x_k, 0)| = 1$.*

Let $n \in \mathbb{N}$ and $\mathcal{T}(f^{(0)}, \ldots, f^{(m)}, x_1, \ldots, x_n)$ be a T.B.T, $1 \leq k \leq n, 0 \leq t, t' \leq m$. Define $(f^{(t)} f^{(t')})(x_1, \ldots, x_n) = \{f^{(s)}(x_1, \ldots, x_n) \mid f_j^{(s)}(x_1, \ldots, x_n) = f_j^{(t)}(x_1, \ldots, x_n) f_j^{(t')}(x_1, \ldots, x_n)$ for all $1 \leq j \leq 2^n\}$ and $(f^{(t)} + f^{(t')})(x_1, \ldots, x_n) =$

$\{f^{(s)}(x_1,\dots,x_n) \mid f_j^{(s)}(x_1,\dots,x_n) = f_j^{(t)}(x_1,\dots,x_n) + f_j^{(t')}(x_1,\dots,x_n)$ for all $1 \le j \le 2^n\}$. So we have the following Theorem.

Theorem 3.15. *Let $n \in \mathbb{N}$ and $\mathcal{T}(f^{(0)},\dots,f^{(m)},x_1,\dots,x_n)$ be a T.B.T, $1 \le k \le n, 0 \le t,t' \le m$. Then*

(i) $Z(n,\overline{x_k},0) = 2^{n-1}$;

(ii) $Z(n,0,0) = 2^n$ *and* $Z(n,1,0) = 0$;

(iii) $Z(n,\overline{x_k}.f^{(t)},0) = [Z(n,\overline{x_k},0) + Z(n,f^{(t)},0)] - Z(n,\overline{x_k}+f^{(t)},0)$.

Proof. (i) Let $1 \le k \le n$. Then by Corollary 3.14, we obtain that $Z(n,\overline{x_k},0) = |S(1,x_1,x_2,\dots,x_k,0)| = 2^{n-1}$.

(ii) In a similar way it is obtained from Corollary 3.14.

(iii) Let $1 \le k \le n, 1 \le j \le 2^n$. Because $\overline{x_k} \sim 0$ or $\overline{x_k} \sim 1$, for all $1 \le j \le 2^n$, we get that $(\overline{x_k}.f_j^{(t)}) \sim 0$ or $(\overline{x_k}.f_j^{(t)}) \sim f_j^{(t)}$. If $(\overline{x_k}.f_j^{(t)}) \sim 0$, since for all $1 \le j \le 2^n, 0 + f_j^{(t)} = f_j^{(t)}$ by item (ii), we get that

$$\begin{aligned} Z(n,\overline{x_k}.f_j^{(t)},0) &= 2^n = 2^n + Z(n,f_j^{(t)},0) - Z(n,f_j^{(t)},0) \\ &= [Z(n,\overline{x_k},0) + Z(n,f_j^{(t)},0)] - Z(n,\overline{x_k}+f_j^{(t)},0). \end{aligned}$$

If $(\overline{x_k}.f_j^{(t)}) \sim f_j^{(t)}$, then $Z(n,\overline{x_k}.f_j^{(t)},0) = 0 + Z(n,f_j^{(t)},0)$. □

Theorem 3.16. *Let $n \in \mathbb{N}$ and $\mathcal{T}(f^{(0)},\dots,f^{(m)},x_1,\dots,x_n)$ be a T.B.T, $1 \le k \le n, 1 \le i \le 2^n, 0 \le t,t' \le m$. Then*

(i) $Z(n,f^{(t)}.f^{(t')},0) = [Z(n,f^{(t)},0) + Z(n,f^{(t')},0)] - Z(n,f^{(t)}+f^{(t')},0)$;

(ii) $Z(n,f^{(t)}+f^{(t')},0) \le \min\{Z(n,f^{(t)},0), Z(n,f^{(t')},0)\}$.

Proof. (i) Because for all $1 \le j \le 2^n$ and $1 \le t \le m$ we have $f_j^{(t)} = \sum_{i=1}^{n}\overline{x_i}$ and $\sum_{i=1}^{n} x_i \sim 0, \sum_{i=1}^{n} x_i \sim 1$ or there exists $1 \le j \le 2^n$ such that $\sum_{i=1}^{n} x_i \sim x_j$, so by Theorem 3.15$(iii)$, the proof is obtained.

(ii) By definition, for all $1 \le j,j \le 2^n$, $f_j^{(t)}(x_1,x_2,\dots,x_n) + f_j^{(t')}(x_1,x_2,\dots,x_n) = 0$ if and only if $f_j^{(t)}(x_1,x_2,\dots,x_n) = f_j^{(t')}(x_1,x_2,\dots,x_n) = 0$, so

$$\begin{aligned} &Z(n,f^{(t)}+f^{(t')},0) = \{j \mid f_j^{(t)}(x_1,x_2,\dots,x_n) = f_j^{(t')}(x_1,x_2,\dots,x_n) = 0, \text{ where } 1 \le j \le 2^n\} \\ \le\ &\{j \mid f_j^{(t)}(x_1,x_2,\dots,x_n) = 0, \text{ where } 1 \le j \le 2^n\} \end{aligned}$$

and

$$\begin{aligned} &Z(n,f^{(t)}+f^{(t')},0) = \{j \mid f_j^{(t)}(x_1,x_2,\dots,x_n) = f_j^{(t')}(x_1,x_2,\dots,x_n) = 0, \text{ where } 1 \le j \le 2^n\} \\ \le\ &\{j \mid f_j^{(t')}(x_1,x_2,\dots,x_n) = 0, \text{ where } 1 \le j \le 2^n\}. \end{aligned}$$

Hence $Z(n, f^{(t)} + f^{(t')}, 0) \leq \min\{Z(n, f^{(t)}, 0), Z(n, f^{(t')}, 0)\}$. □

Corollary 3.17. *Let $n \in \mathbb{N}$ and $\mathcal{T}(f, f', x_1, x_2, \dots, x_n)$ be a T.B.T. Then $(f.f') \sim f$ if and only if $Z(n, f.f', 0) = Z(n, f, 0)$.*

Theorem 3.18. *Let $n \in \mathbb{N}, 1 \leq j \leq n$ and $\mathcal{T}(f, f', x_1, x_2, \dots, x_n)$ be a T.B.T. If there exists some $1 \leq i_1, i_2, \dots, i_k \leq n$ such that $f' \sim \sum_{j=1}^{k} x_{i_j}$, then $Z(n, f.f', 0) > 2^{n-k}$.*

Proof. Let $1 \leq i_1, i_2, \dots, i_k \leq n$ and $f' \sim \sum_{j=1}^{k} x_{i_j}$. By Theorem 3.16,

$$\begin{aligned} Z(n, f^{(t)}.f^{(t')}, 0) &= [Z(n, f^{(t)}, 0) + Z(n, f^{(t')}, 0)] - Z(n, f^{(t)} + f^{(t')}, 0) \\ &= [Z(n, f^{(t)}, 0) + Z(n, \sum_{j=1}^{k} x_{i_j}, 0)] - Z(n, (f^{(t)}) + \sum_{j=1}^{k} x_{i_j}, 1) \\ &> Z(n, f^{(t)}, 0) + 2^{n-k} - Z(n, f^{(t)}, 0) = 2^{n-k}. \end{aligned}$$

□

Theorem 3.19. *Let $n \in \mathbb{N}, 1 \leq j \leq n$ and $\mathcal{T}(f, f', x_1, x_2, \dots, x_n)$ be a T.B.T and $f' \sim \sum_{j=1}^{k} x_{i_j}$, where $1 \leq i_1, i_2, \dots, i_k \leq n$.*

(i) If $Z(n, f, 0) < 2^{n-1}$, then $(f.f') \nsim f$;

(ii) If $Z(n, f', 0) = Z(n, f, 0)$ and $Ker(f') \subseteq Ker(f)$ imply that $f' \sim f$;

Proof. (*i*) Let $(f.f') \sim f$. Using Theorem 3.16, $Z(n, f.f', 0) = Z(n, f, 0) + Z(n, f', 0) - Z(n, f + f', 0)$. Because $Z(n, f + f', 0) \leq \min\{Z(n, f, 0), Z(n, f', 0)\}$, we get that $Z(n, f.f', 0) \geq Z(n, f, 0) + 2^{n-k} - Z(n, f, 0) = 2^{n-k}$, which is a contradiction.

(*ii*) Let $f'(x_1, \dots, x_n) = 0$. Then $Ker(f') \subseteq Ker(f)$ implies that $(x_1, \dots, x_n) \in Ker(f)$ and so $f(x_1, \dots, x_n) = 0$. Suppose that $f'(x_1, \dots, x_n) = 1$, then $Ker(f') \subseteq Ker(f)$ implies that $(x_1, \dots, x_n) \in Ker(f) \setminus Ker(f')$ or $(x_1, \dots, x_n) \notin Ker(f)$. If $(x_1, \dots, x_n) \notin Ker(f)$, then $f(x_1, \dots, x_n) = 1$ and in this case $f \sim f'$. If $(x_1, \dots, x_n) \in Ker(f)$, then $f(x_1, \dots, x_n) = 0$ and it follows that $Z(n, f', 0) < Z(n, f, 0)$, which is a contradiction. □

Theorem 3.20. *Every T.B.T corresponds to a Boolean expression.*

Proof. Let $n \in \mathbb{N}$ and $\mathcal{T}(f, x_1, x_2, \dots, x_n)$ be a T.B.T. If for all $1 \leq j \leq 2^n, f_j(x_1, x_2, \dots, x_n) = 0$, then consider $g(x_1, x_2, \dots, x_n) \equiv 0$. In a similar a way that, if for all $1 \leq j \leq 2^n, f_j(x_1, x_2, \dots, x_n) = 1$, then consider $g(x_1, x_2, \dots, x_n) \equiv 1$. Now, if there exist $k \in \mathbb{N}$

and $1 \leq j_1, j_2, \ldots, j_k \leq 2^n$ such that $f_{j_k}(x_1, x_2, \ldots, x_n) = 0$, then consider $f_j(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} \overline{x_i}$ in such a way that for all $1 \leq i \leq n, \overline{x_i} = 0$. Applying Theorem 3.8, $H = (G = \bigcup_{j=1}^{k} E_j, \{E_j\}_{j=1}^{k})$ is a hypergraph, where for all $1 \leq j \leq k$, $E_j = \{\overline{x_1}, \overline{x_2}, \ldots, \overline{x_n} \mid \sum_{j=1}^{n} \overline{x_j} = f_j(x_1, x_2, \ldots, x_n) = 0\}$ and for all $1 \leq i \neq j \leq k$ we set $F_{ij} = E_i \cap E_j$. If for all $1 \leq i \neq j \leq k, F_{ij} = \emptyset$, then consider $g(x_1, \ldots, x_n) = \prod_{1 \leq j \leq k} \sum_{\alpha \in E_j} \alpha$. In this case, since for all $1 \leq j \leq k, Ker(f_j) = Ker(\sum_{\alpha \in E_j} \alpha)$, by Theorem 3.4, $Ker(f) = Ker(g)$ and so $f(x_1, x_2, \ldots, x_n) = g(x_1, x_2, \ldots, x_n)$. If there exist $1 \leq r \leq k$ and $i \neq j \in \{i_1, i_2, \ldots, i_r\}$ such that $F_{ij} \neq \emptyset$, consider $g_{ij}(x_1, x_2, \ldots, x_n) = \sum_{\alpha \in F_{ij}} \alpha$. Clearly for all $i_1 \leq i, j \leq i_r, Ker(g_{ij}) \subseteq Ker(f)$, so if $Z(n, \prod_{\substack{i_1 \leq i \leq i_r \\ i_1 \leq j \leq i_s}} g_{ij}, 0) = Z(n, f, 0)$, thus by Theorems 3.4 and 3.19, $g \sim f$, where $g(x_1, x_2, \ldots, x_n) = \prod_{\substack{i_1 \leq i \leq i_r \\ i_1 \leq j \leq i_s}} g_{ij}(x_1, \ldots, x_n)$. But if $Z(n, \prod_{\substack{i_1 \leq i \leq i_r \\ i_1 \leq j \leq i_s}} g_{ij}, 0) < Z(n, f, 0)$, consider $1 \leq s \leq k$, $j \in \{j_1, j_2, \ldots, j_s\}$ and $f_j(x_1, x_2, \ldots, x_n) = \sum_{\beta \in E_j} \beta$ such that $Ker(\prod_{\substack{i_1 \leq i \leq i_r \\ i_1 \leq j \leq i_s}} g_{ij}) \neq Ker(f_j) \subseteq Ker(f)$ and $Z(n, (\prod_{\substack{i_1 \leq i \leq i_r \\ i_1 \leq j \leq i_s}} g_{ij}).(\prod_{j=j_1}^{j_s} f_j), 0) = Z(n, f, 0)$. Thus by Theorem 3.19, we get that $g \sim f$, where $g(x_1, x_2, \ldots, x_n) = (\prod_{j=j_1}^{j_s} f_j(x_1, \ldots, x_n)).(\prod_{\substack{i_1 \leq i \leq i_r \\ i_1 \leq j \leq i_s}} g_{ij}(x_1, \ldots, x_n))$. □

Consider a T.B.T $\mathcal{T}(f, x_1, x_2, \ldots, x_n)$ and $\mathcal{E}(f) = \{g = g_1.g_2 \ldots g_m \mid g \sim f\}$. Clearly $f \sim f$, and so $\mathcal{E}(f) \neq \emptyset$. We say that f has a Minimum Boolean expression if, there exists $g \neq f \in \mathcal{E}(f)$ such that m is the Minimum natural in such a way that $g = g_1.g_2 \ldots g_m$.

Theorem 3.21. *Every T.B.T corresponds to a Minimum Boolean expression.*

Proof. Let $n \in \mathbb{N}$ and $\mathcal{T}(f, x_1, x_2, \ldots, x_n)$ be a T.B.T. Using Theorem 3.20, $\mathcal{E}(f) \neq \emptyset$ and there exists a hypergraph $H = (G = \bigcup_{j=1}^{k} E_j, \{E_j\}_{j=1}^{k})$, where for all $1 \leq j \leq k$, $E_j = \{\overline{x_1}, \overline{x_2}, \ldots, \overline{x_n} \mid \sum_{j=1}^{n} \overline{x_j} = 0\}$. Thus there exist $m \in \mathbb{N}$ and Boolean functions $g_1, g_2, \ldots, g_m$ in such a way that $f \sim (g_1.g_2 \ldots g_m)$, where for all $1 \leq i \leq m$, there exist

$1 \leq j, j' \leq k$ in such a way that $\overline{x_1}, \overline{x_2}, \ldots, \overline{x_n} \in F_{jj'} \neq \emptyset$ and $g_i(x_1, x_2, \ldots, x_n) = \sum_{l=1}^{n} \overline{x_l}$. In addition, if for all $1 \leq j, j' \leq k, F_{jj'} = \emptyset$, we consider $k = m$ and $g_i(x_1, x_2, \ldots, x_n) = \sum_{l=1}^{n} \overline{x_l}$, where $\overline{x_1}, \overline{x_2}, \ldots, \overline{x_n} \in E_i$. If there exist $1 \leq j, j' \leq k$ in such a way that $\overline{x_1}, \overline{x_2}, \ldots, \overline{x_n} \in F_{jj'} \neq \emptyset$, we choice $1 \leq j, j' \leq k$ such that $|F_{jj'}| = n - 1$ and $\mathcal{M} = \{1 \leq j, j' \leq k \mid |F_{jj'}| = n - 1\}$. Clearly $|\mathcal{M}| \leq k$, so if $Z(n, f, 0) = Z(n, g_1.g_2 \ldots g_{|\mathcal{M}|}, 0)$, then $f \sim g$. But if $Z(n, f, 0) > Z(n, g_1.g_2 \ldots g_{|\mathcal{M}|}, 0)$, we consider the Minimum $1 \leq i \leq k$, and Boolean functions f_i such that $g' = \prod_{i} f_i, Ker(g) \neq Ker(g') \subseteq Ker(f)$ and $Z(n, f, 0) = Z(n, g'.(\prod_{j=1}^{|\mathcal{M}|} g_j), 0)$. Because $Ker(g'.(\prod_{j=1}^{|\mathcal{M}|} g_j)) \subseteq Ker(f)$, by Theorem 3.19, we get that $(g.g') \sim f$. □

Let $n, k, \lambda \in \mathbb{N}^*$. A hypergraph $H = (G, \{E_j\}_{j=1}^{k})$ is called a λ-intersection hypergraph, if for all $1 \leq i, j \leq k$, we have $|E_i \cap E_j| = \lambda$.

Theorem 3.22. *Let $n \in \mathbb{N}$ and $\mathcal{T}(f, x_1, x_2, \ldots, x_n)$ be a T.B.T. If $(H, \mathcal{T})$ is a 0-intersection hypergraph, then the T.B.T corresponds to an irreducible Boolean expression.*

Proof. Using Theorem 3.20, $\mathcal{E}(f) \neq \emptyset$ and there exists a hypergraph $(H, \mathcal{T}) = (G = \bigcup_{j=1}^{k} E_j, \{E_j\}_{j=1}^{k})$, where for all $1 \leq j \leq k$, $E_j = \{\overline{x_1}, \overline{x_2}, \ldots, \overline{x_n} \mid \sum_{j=1}^{n} \overline{x_j} = 0\}$. Thus there exist $m \in \mathbb{N}$ and Boolean functions $g_1, g_2, \ldots, g_m$ in such a way that $f \sim (g_1.g_2 \ldots g_m)$. Since $(H, \mathcal{T})$ is an 0-intersection hypergraph, for all $1 \leq j, j' \leq m$ we get that $|F_{jj'}| = 0$. It follows that $g_j(x_1, \ldots, x_n) = \sum_{\alpha \in E_j} \alpha$ and so

$$\{\overline{x_1}, \overline{x_2}, \ldots, \overline{x_n} \mid \sum_{j=1}^{n} \overline{x_j} = g_j(x_1, x_2, \ldots, x_n) = 0\} \cap$$

$$\{\overline{x_1}, \overline{x_2}, \ldots, \overline{x_n} \mid \sum_{j=1}^{n} \overline{x_j} = g_{j'}(x_1, x_2, \ldots, x_n) = 0\} = \emptyset$$

and so $f(x_1, x_2, \ldots, x_n)$ is an irreducible Boolean expression. □

The following Example shows that the converse of Theorem 3.22, is not necessarily true.

Example 3.23. (i) Consider a T. B. T $\mathcal{T}(f, x, y, z)$ in Table 4. Clearly $(H, \mathcal{T})$ is an 2-intersection hypergraph, while the T.B.T corresponds to a reducible Boolean expression.

(ii) Consider a T. B. T $\mathcal{T}(f', x, y, z)$ in Table 4. Obviously, $(H, \mathcal{T})$ is an 1-intersection hypergraph, while the T.B.T corresponds to an reducible Boolean expression.

Table 4. T. B. T $\mathcal{T}(f,x,y,z)$

x	y	z	$f(x,y,z)$	$f'(x,y,z)$
0	0	0	0	1
0	0	1	1	1
0	1	0	1	0
0	1	1	1	1
1	0	0	0	1
1	0	1	1	1
1	1	0	1	1
1	1	1	1	0

Let $n,k,\lambda \in \mathbb{N}^*$. A hypergraph $H=(G,\{E_j\}_{j=1}^{k})$ is called a strong λ-intersection hypergraph, if for some $1\leq i,j\leq k$, we have $1\leq |E_i\cap E_j|\leq \lambda$.

The method for the construction of a Boolean expression from an T.B.T is explained in Table 5, Algorithm 1 based on Theorem 3.21.

Example 3.24. (i)Consider a T. B. T $\mathcal{T}(f,x,y,z)$ in Table 6. Let $H=\{x,y,z,x',y',z'\}$. Consider the undirected hypergraph $\mathcal{H}'=(H,E_1,E_2,E_3,E_4)$ in Figure 2, where $E_1=\{x,y,z'\}, E_2=\{x,y',z\}, E_3=\{x',y,z\}$ and $E_4=\{x',y',z'\}$.

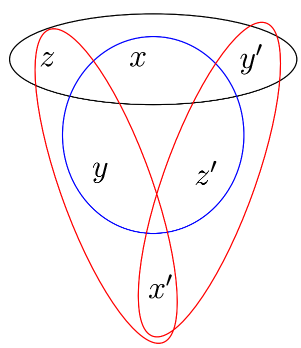

Figure 2. Undirected hypergraph $\mathcal{H}'=(H,E_1,E_2,E_3,E_4)$

Since $E_1\cap E_2=\{x\}, E_1\cap E_3=\{y\}, E_1\cap E_4=\{z'\}, E_2\cap E_3=\{z\}, E_2\cap E_4=\{y'\}$ and $E_3\cap E_4=\{x'\}$ we get that $f(x,y,z)=(x+y+z')(x+y'+z)(x'+y+z)(x'+y'+z')$.

(ii) Consider a T. B. T $\mathcal{T}(f,x,y,z)$ in Table 7. Let $H=\{x,y,z,x',y',z'\}$. Consider the undirected hypergraph $\mathcal{H}'=(H,E_1,E_2,E_3,E_4)$ in Figure 3, where $E_1=\{x,y',z\}, E_2=\{x,y',z'\}$, $E_3=\{x',y,z'\}$ and $E_4=\{x',y',z'\}$.

Table 5. Algorithm 1

Begin:

1. Input a T.B.T $\mathcal{T}(f, x_1, x_2, \ldots, x_n)$.

2. If $f \equiv 0$ or $f \equiv 1$, then consider $g \equiv 0$ or $g \equiv 1$, respectively.

3. If there exists $j \in \{1 \leq j_1, j_2, \ldots, j_s \leq 2^n\}$, such that $f_j(x_1, x_2, \ldots, x_n) = 0$, then consider $E_j = \{\overline{x_1}, \overline{x_2}, \ldots, \overline{x_n} \mid \sum_{j=1}^{n} \overline{x_j} = f_j(x_1, x_2, \ldots, x_n) = 0\}$.

With respect to step 3 consider the following:

4. For all $1 \leq i \neq j \leq k$ and $k \in \mathbb{N}$, set $F_{ij} = E_i \cap E_j$.

5. If for all $1 \leq i \neq j \leq k, F_{ij} = \emptyset$ or $|F_{ij}| < n-1$, then consider $g(x_1, x_2, \ldots, x_n) = \prod_{1 \leq i \leq k} \sum_{\alpha \in E_i} \alpha$.

6. If there exists $1 \leq r \leq k$, and $i \neq j \in \{i_1, i_2, \ldots, i_r\}$ such that $F_{ij} \neq \emptyset$ and $|F_{ij}| \geq n-1$, then put $g_{ij}(x_1, x_2, \ldots, x_n) = \sum_{\alpha \in F_{ij}} \alpha$.

7. If $Z(n, \prod_{\substack{i_1 \leq i \leq i_r \\ i_1 \leq j \leq i_s}} g_{ij}, 0) = Z(n, f, 0)$, then consider $g(x_1, x_2, \ldots, x_n) = \prod_{\substack{i_1 \leq i \leq i_r \\ i_1 \leq j \leq i_s}} g_{ij}(x_1, \ldots, x_n)$ that $Ker(g_{ij}) \subseteq Ker(f)$.

8. If $Z(n, \prod_{\substack{i_1 \leq i \leq i_r \\ i_1 \leq j \leq i_s}} g_{ij}, 0) < Z(n, f, 0)$, then consider $1 \leq s \leq k$, $j \in \{j_1, j_2, \ldots, j_s\}$, $f_j(x_1, x_2, \ldots, x_n) = \sum_{\beta \in E_j} \beta$ such that $Ker(\prod_{\substack{i_1 \leq i \leq i_r \\ i_1 \leq j \leq i_s}} g_{ij}) \neq Ker(f_j) \subseteq Ker(f)$ and $Z(n, (\prod_{\substack{i_1 \leq i \leq i_r \\ i_1 \leq j \leq i_s}} g_{ij}).(\prod_{j=j_1}^{j_s} f_j), 0) = Z(n, f, 0)$.

End.

Since $E_1 \cap E_2 = \{x, y'\}$, $E_1 \cap E_3 = \emptyset$, $E_1 \cap E_4 = \{y'\}$, $E_2 \cap E_3 = \{z'\}$, $E_2 \cap E_4 = \{y', z'\}$ and $E_3 \cap E_4 = \{x', z'\}$, we get that $f(x, y) = (x + y')(y' + z')(x' + z')$.

3.1. **Irreducible Switching Expression Based on a Program.**

In this subsection, we present a program(Python programming) to accesses of Irreducible Boolean expression for any given T.B.T, based on the Algorithm 1, Table 5.

```
import xlrd
import re
```

TABLE 6. T. B. T $\mathcal{T}(f,x,y)$

x	y	z	$f(x,y,z)$
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	0

TABLE 7. T. B. T $\mathcal{T}(f,x,y,z)$

x	y	z	$f(x,y,z)$
0	0	0	1
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	0

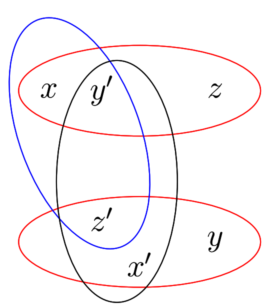

FIGURE 3. Undirected hypergraph $\mathcal{H}' = (H, E_1, E_2, E_3, E_4)$

```

sheet = xlrd.open_workbook(”input.xlsx”).sheet_by_index(0)
```

```

element_names = sheet.row_values(0)[:-1]
elements = []
for i in range(1, sheet.nrows):
    elements.append(sheet.row_values(i))

if len(elements) != pow(2, len(element_names)):
    print("Error in input file, rows count isn't equal to 2^n.")
    exit(0)

E_i = []
for i, elm in enumerate(elements):
    if(elm[-1] == 0):
        E = []
        for c in range(0, len(elm)-1):
            if(elm[c] == 1):
                E.append(element_names[c]+"'")
            else:
                E.append(element_names[c])
        E_i.append([i, E])

def print_g(g_expr):
    print("g("+",".join(element_names)+") = "+g_expr)
    input("")
    exit(0)

F_ij = []
for i in E_i:
    F = []
    for j in E_i:
        if(i != j):
            intersection = [x for x in i[1] if x in j[1]]
```

```
            if(len(intersection) >= (len(element_names)-1)):
                F.append([j[0], intersection])
    if(F):
        F_ij.append([i[0], F])

F_ij_len = True
for i in F_ij:
    if len(i) >= len(element_names)-1:
        F_ij_len = False

if(F_ij == [] or F_ij_len):
    g = []
    for i in E_i:
        g.append("("+"+".join(i[1])+")")
    print_g("".join(g))


g_ij = []
for i in F_ij:
    for j in i[1]:
        g_ij.append("("+"+".join(j[1])+")")
g_ij = list(dict.fromkeys(g_ij))
sigma_g_ij = "".join(g_ij)


def calc_mult_and(expr):
    expr = re.findall('\((.*?)\)',expr)
    for i in expr:
        v = i.split("+")
        if "1" not in v:
            return 0
    return 1
```

```
false_g_ij = []
for elm in elements:
    tmp_g = sigma_g_ij
    for c in range(0, len(elm)-1):
        if(elm[c] == 1):
            tmp_g = tmp_g.replace(element_names[c]+"'", "0")
            tmp_g = tmp_g.replace(element_names[c], "1")
        else:
            tmp_g = tmp_g.replace(element_names[c]+"'", "1")
            tmp_g = tmp_g.replace(element_names[c], "0")
    false_g_ij.append(calc_mult_and(tmp_g))

f_j = []
for i, elm in enumerate(elements):
    if(elm[-1] == 0 and false_g_ij[i] == 1):
        f_j.append("("+"+".join([x for x in E_i if x[0] == i][0][1])+")")

print_g("".join(g_ij + f_j))
```

Remark 3.25. We take n as number of variables in table, and k as number of elements of E_j. Then

(*i*) $T(E_j) = O(n \times 2^n)$, because one outer loop through rows and one inner loop through columns.

(*ii*) $T(F_{ij}) = O(k^2 \times n)$, because two nested loops through elements of E_j and one inner loop for getting intersection.

(*iii*) If for $1 \leq j \leq 2^n$, $F_j(x_1, ..., x_n) = 0$ or $F_j(x_1, ..., x_n) = 1$ complexity is $O(2^n)$ for one loop through rows.

(*iv*) Others complexity is $T(E_j) + T(F_{ij}) = O(n \times (k^2 + 2^n))$.

4. **Conclusion**

The current paper has defined and considered the notion of Boolean function–based hypergraph, also is shown that every T.B.T corresponds to a Boolean expression. It investigated to correspond every T.B.T to a Minimum and irreducible Boolean expression. For simplifying the complex computations, we introduce an Algorithm and based on this algorithm we extracted a Python programming. We hope that these results are helpful for further studies in Boolean function theory. In our future studies, we hope to obtain more results regarding irreducible Boolean function, graphs, hypergraphs, decision tree based on Boolean function–based hypergraph and their applications.

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