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A CHARACTERIZATION OF SOME SIMPLE UNITARY GROUPS VIA ORDER AND DEGREE PATTERN OF SOLVABLE GRAPH

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Abstract. The solvable graph associated with a finite group G, denoted by $\Gamma_s(G)$, is a simple graph whose vertices are the prime divisors of $|G|$ and two distinct primes p and q are joined by an edge if and only if there exists a solvable subgroup of G whose order is divisible by pq. In this paper, we give a characterization for projective special unitary groups $U_3(q)$ with some certain conditions by the solvable graph.

1. Introduction

All groups appearing here are assumed to be finite. For a finite group G, we denote by $\pi(G)$ the set of all prime divisors of $|G|$. There are a lot of ways for studying a finite group. One of the most interesting approaches is to consider some properties of the graphs associated with it. In fact, one of the graphs associated with a finite group G is the *solvable graph* which was introduced by Abe and Iiyori in [2]. This graph is a simple and undirected graph that constructs as follows. The vertex set is $\pi(G)$, and two distinct primes p and q are adjacent

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(we write $p \approx q$) if and only if G has a solvable subgroup whose order is divisible by pq. We denote this graph by $\Gamma_{\mathrm{s}}(G)$.

The set of orders of all elements in a finite group G is denoted by $\mathrm{Spec}(G)$ and called the spectrum of G. This set is closed and partially ordered by the divisibility relation; therefore, it is determined uniquely from the subset $\mu(G)$ of all maximal elements of $\mathrm{Spec}(G)$ with respect to divisibility. According to the definition of solvable graph, for the element a in $\mu(G)$, any two primes p and q such that $pq \mid a$, are joined by an edge.

Let $p_1 < p_2 < \cdots < p_k$ be all primes in $\pi(G)$. We define

$$\mathrm{D_s}(G) = (d_{\mathrm{s}}(p_1), d_{\mathrm{s}}(p_2), \ldots, d_{\mathrm{s}}(p_k)),$$

as the *degree pattern of the solvable graph of* G, where $d_{\mathrm{s}}(p)$ signifies the degree of the vertex p in $\Gamma_{\mathrm{s}}(G)$.

We will give more attention to a prime in $\Gamma_{\mathrm{s}}(G)$ which is joined to any other primes in the graph that is called a *regular prime*. In other words, a prime p is a regular prime if $d_{\mathrm{s}}(p) = k-1$ where $|\pi(G)| = k$. We denote the set of regular primes by $\mathrm{Reg}(G)$.

Given a finite group G, denote by $h_{\mathrm{OD_s}}(G)$ the number of isomorphism classes of finite groups H such that $|H| = |G|$ and $\mathrm{D_s}(H) = \mathrm{D_s}(G)$. In terms of the function $h_{\mathrm{OD_s}}(\cdot)$, we have the following definition.

Definition 1.1. A finite group G is said to be n-fold $\mathrm{OD_s}$-characterizable if $h_{\mathrm{OD_s}}(G) = n$. The group G is $\mathrm{OD_s}$-characterizable if $h_{\mathrm{OD_s}}(G) = 1$. Moreover, we will simply say that H is $\mathrm{OD_s}$-characterizable if $H \cong G$ for every finite group G such that $|G| = |H|$ and $\mathrm{D_s}(G) = \mathrm{D_s}(H)$.

In this paper, we are interested in characterizing groups by order and degree pattern of solvable graph. In [3, 4], some simple groups were considered and shown that the following groups are $\mathrm{OD_s}$-characterizable.

- All sporadic simple groups;
- Projective special linear groups $L_2(q)$ with the following conditions:
  - (a) $p = 2$, $|\pi(q+1)| = 1$ or $|\pi(q-1)| = 1$,
  - (b) $q \equiv 1 \pmod 4$, $|\pi(q+1)| = 2$ or $|\pi(q-1)| \leqslant 2$,
  - (c) $q \equiv -1 \pmod 4$.
- Projective special linear groups $\mathrm{L}_3(\mathrm{q})$ with the following conditions:
  - (a) q is odd and $9 \nmid q-1$;
  - (b) q is even and $3 \parallel q-1$;
  - (c) $9 \mid q-1$ and $|\pi(\frac{q^2+q+1}{3})| = 1$;
  - (d) q is even, $3 \mid q+1$ and $|\pi(q^2+q+1)| = 1$.
- A finite group H such that $|\pi(H)| \geqslant 3$, $H \notin \{B_n(q), C_n(q)\}$ ($n \geqslant 3$ and q is odd), and $\mathrm{Reg}(H) = \emptyset$.

We will show that the projective special unitary groups $U_3(q)$ defined over a field of characteristic 3 with certain conditions, are $\mathrm{OD_s}$-characterizable. In fact, we prove the following theorem.

Theorem. *The simple groups $U_3(q)$ where $q = 3^f$ $(f \geqslant 2)$, such that $|\pi(q^2 - q + 1)| = 1$ are $\mathrm{OD_s}$-characterizable.*

2. Preliminaries

We begin this section with some obtained results on solvable graphs of finite groups which will be used for further studies.

Lemma 2.1. ([2, Lemma 3]) *The solvable graph of a finite group is always a connected graph.*

Lemma 2.2. ([2]) *Let G be a non-abelian simple group. Then $\Gamma_{\mathrm{s}}(G)$ is an incomplete graph.*

Lemma 2.3. ([1, Lemma 3]) *Let G be a finite group such that $\mathrm{Reg}(G) = \emptyset$. Then G is a non-abelian simple group.*

Lemma 2.4. ([2, Lemma 2]) *Let G be a group, H a subgroup of G and N a normal subgroup of G. Then*

(1) *If p and q are joined in $\Gamma_{\mathrm{s}}(H)$ for $p, q \in \pi(H)$, then p and q are joined in $\Gamma_{\mathrm{s}}(G)$, that is, $\Gamma_{\mathrm{s}}(H)$ is a subgraph of $\Gamma_{\mathrm{s}}(G)$.*

(2) *If p and q are joined in $\Gamma_{\mathrm{s}}(G/N)$ for $p, q \in \pi(G/N)$, then p and q are joined in $\Gamma_{\mathrm{s}}(G)$, that is, $\Gamma_{\mathrm{s}}(G/N)$ is a subgraph of $\Gamma_{\mathrm{s}}(G)$.*

(3) *For $p \in \pi(N)$ and $q \in \pi(G) \setminus \pi(N)$, p and q are joined in $\Gamma_{\mathrm{s}}(G)$.*

Lemma 2.5. ([3, Corollary 1]) *Let N be a normal Hall subgroup of a finite group G. Then $\Gamma_{\mathrm{s}}(G)$ is complete if and only if $\Gamma_{\mathrm{s}}(N)$ and $\Gamma_{\mathrm{s}}(G/N)$ are complete, too.*

Lemma 2.6. *Let K be a finite group and G a subgroup of K which is a simple group with $|K : G| = 2$. Then we have:*

$$\Gamma_{\mathrm{s}}(K) - \{2\} = \Gamma_{\mathrm{s}}(G) - \{2\}.$$

In particular, if $r \in \pi(G) - \{2\}$, then $d_{\mathrm{s}_G}(r) \leqslant d_{\mathrm{s}_K}(r) \leqslant d_{\mathrm{s}_G}(r) + 1$, and moreover; if 2 is a regular prime in $\Gamma_{\mathrm{s}}(G)$, then $d_{\mathrm{s}_K}(r) = d_{\mathrm{s}_G}(r)$.

Proof. We first claim that every subgroup of K of odd order is a subgroup of G. Suppose that the claim is false and there exists a subgroup $H \leqslant K$ of odd order such that $H \nleqslant G$. Thus there is an element $x \in K \setminus G$ such that $o(x) = m$ where m is an odd number. Then we have $x^{-1} = x^{m-1} \in G$ since $|K : G| = 2$ and $m - 1$ is even. It follows that $x \in G$, which is a contradiction.

Note that $\pi(K) = \pi(G)$. In what follows, we will show that, if p and q are two odd primes such that $p \approx q$ in $\Gamma_{\rm s}(K)$, then $p \approx q$ in $\Gamma_{\rm s}(G)$. Assume that $p \approx q$ in $\Gamma_{\rm s}(K)$. Hence, there is a solvable subgroup $L \leqslant K$ such that $pq \mid |L|$. We consider Hall $\{p, q\}$-subgroup H of L. Now from the previous paragraph of the proof, H is a subgroup of G and so $p \approx q$ in $\Gamma_{\rm s}(G)$. □

The following lemma is a fundamental result which is gained in [2].

Lemma 2.7. ([2, Lemma 1]) *The solvable graph of a solvable group is complete.*

We are not sure if the converse of Lemma 2.7 holds. It can be shown that it is true in some special cases. So we can state the following conjecture.

Conjecture. *Let G be a finite group. Then $\Gamma_{\rm s}(G)$ is complete if and only if G is a solvable group.*

In the following lemma, we prove that if the group G has a normal Hall subgroup, the conjecture above is true.

Lemma 2.8. *Let G be a finite group possessing a normal Hall subgroup. Then $\Gamma_{\rm s}(G)$ is complete if and only if G is a solvable group.*

Proof. It is enough to consider the necessity. Suppose that G is a counterexample of minimal order to this statement which means that $\Gamma_{\rm s}(G)$ is complete while G is not solvable. Assume that N is the normal Hall subgroup of G. Now we obtain from Lemma 2.5 that $\Gamma_{\rm s}(N)$ and $\Gamma_{\rm s}(G/N)$ are complete. Thus we have by the hypothesis that N and G/N are solvable. Consequently, G is solvable which contradicts our assumption. □

Lemma 2.9. ([2, Theorem 3]) *Let G be a finite group and $\{p, q\} \subseteq \pi(G)$. Then p and q are not joined in $\Gamma_{\rm s}(G)$ if and only if there exists a series of normal subgroups of G, say*

$$1 \trianglelefteq M \triangleleft N \trianglelefteq G,$$

such that M and G/N are $\{p, q\}'$-groups and N/M is a non-abelian simple group such that p and q are not joined in $\Gamma_{\rm s}(N/M)$.

Using the notation taken from [1] and [2], such a series as in Lemma 2.9 is called a *GKS-series of G* and we will say *p and q are expressed to be disjoint by this GKS-series.*

Lemma 2.10. ([1, Lemma 4]) *Let G be a finite group with $|\pi(G)| = k$. If the number of connected components of*

$$\widetilde{\Gamma}(G) = (\Gamma_{\rm s}(G) - {\rm Reg}(G))^c$$

equals to n, then at most n GKS-series of G is necessary to express any pair of vertices of $\Gamma_{\rm s}(G)$ to be disjoint.

The following lemma is a result of Lemma 2.10.

Lemma 2.11. [3, Lemma 6] *Let G be a finite group with $|\pi(G)| = k \geqslant 4$. Moreover, let* $\mathrm{Reg}(G) \neq \emptyset$ *and*

$$\widetilde{\Gamma}(G) := (\Gamma_{\mathrm{s}}(G) - \mathrm{Reg}(G))^c.$$

If there is a prime $p \in \pi(G)$ such that $d_{\mathrm{s}}(p) = 1$ or 2, then any disjointed pair of vertices of $\Gamma_{\mathrm{s}}(G)$ *can be expressed by only one GKS-series.*

We now present a lemma which will be used in Section 3.

Lemma 2.12. [10, Lemma 2] *Let $q = p^f$, where p is a prime and f a natural number. If*

$$|\pi((q-1)/(2, q-1))| \leqslant 2 \geqslant \pi((q+1)/(2, q-1))|,$$

then $q = 4, 9, 16, 81$ or $q = p^f$, $f = 1$ or an odd prime.

In the sequel, we are going to construct the solvable graph of projective special unitary groups $U_3(q)$ where $q = 3^f$ and f is a natural number. We draw this graph in a compact form. The compact form of a graph is a graph whose vertices are displayed with disjoint subsets of the vertex set in the graph. Actually, a vertex labeled U represents the complete subgraph of the graph on U. An edge connecting U and W represents the set of edges in the graph which connect each vertex in U with each vertex in W.

- The set of maximal elements in the spectrum of $U_3(q)$ is as follows:

$$\mu(U_3(q)) = \{p(q+1), q^2 - 1, q^2 - q + 1\}.$$

- The maximal subgroups of $\mathrm{SU}_3(\mathrm{q})$ collected in [6] (Table 8. 5) are listed as follows.

Subgroup	Conditions	Subgroup	Conditions
${\mathrm{E}_q}^{1+2} : (q^2-1)$		$\mathrm{SU}_3(q_0).\big(\frac{q+1}{q_0+1}, 3\big)$	$q = {q_0}^r, r$ *is an odd prime*
$\mathrm{GU}_2(q)$		$d \times \mathrm{SO}_3(q)$	q *is odd*, $q \geqslant 7$
$(q+1)^2 : \mathrm{S}_3$	$q \neq 5$	${3_+}^{1+2} : Q_8.\frac{(q+1,9)}{3}$	$p = q \equiv 2 \pmod 3, q \geqslant 11$
$(q^2-q+1) : 3$	$q \neq 3, 5$		

According to the notation of [6], $d = |Z(\mathrm{SU}_3(\mathrm{q}))| = (3, \mathrm{q}+1)$. The cyclic group of order n is denoted by n. An elementary abelian group of order p^n is denoted by E_{p^n} or just by p^n. For a prime p, p_+^{1+2n} or p_-^{1+2n} is used for the particular case of an extraspecial group. For each prime number p and positive n, there are just two types of extraspecial group, which are central products of n non-abelian groups of order p^3. For an odd prime p, the subscript is $+$ or $-$ according as the group has exponent p

or p^2. For elementary abelian groups A, we write A^{m+n} to mean a group with an elementary abelian normal subgroup A^m such that the quotient is isomorphic to A^n. For two groups A and B, a split extension (resp. a non-split extension) is denoted by $A : B$ (resp. $A.B$). Moreover, $A \times B$ denotes the direct product of A and B. (See [8])

Using the information above, the compact form of $\Gamma_s(U_3(q))$ with $q = 3^f$ is found in Figure 1.

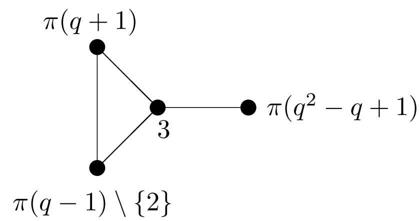

Fig. 1. $\Gamma_s(U_3(q)),\ q = 3^f$

3. Characterization of Some Projective Special Unitary Groups

In this section, we will show that the projective special unitary groups $U_3(q)$ defined over a field of characteristic 3 with some conditions, are completely characterized by their orders and degree pattern of solvable graphs.

First of all, we give a terminology which was introduced in [1]. Let m be a positive integer with the following factorization into distinct prime power factors $m = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$ for some positive integers n_i and k. Then we set

$$\mathrm{mpf}(m) := \max\{p_i^{n_i} \mid 1 \leqslant i \leqslant k\}.$$

For convenience, we bring $\mathrm{mpf}(|S|)$ in Tables 1 and 2 where S is a sporadic simple group or a simple group of Lie type.

Theorem 3.1. *All simple groups $U_3(q)$ where $q = 3^f$ $(f \geqslant 2)$, such that $|\pi(q^2 - q + 1)| = 1$ are $\mathrm{OD_s}$- characterizable.*

Proof. Let G be a finite group satisfying

$$|G| = |U_3(q)| = q^3(q^2 - 1)(q^3 + 1),$$

and $\mathrm{D_s}(G) = \mathrm{D_s}(U_3(q))$, where $q = 3^f$ and $|\pi(q^2 - q + 1)| = 1$. In what follows, we prove that $G \cong U_3(q)$.

The solvable graph of $U_3(q)$ with $q = 3^f$ is shown in Figure 1. Considering Figure 1, we observe that

$$d_s(3) = |\pi(G)| - 1.$$

Assume that $\pi(q^2-q+1)=\{p\}$ where p is a prime. According to the degree pattern of solvable graph of G, $d_s(p)=1$. Hence it is seen from Lemma 2.11 that $\widetilde{\Gamma}(G)=(\Gamma_s(G)-\{3\})^c$ is connected and any disjoint pair of vertices of $\Gamma_s(G)$ can be expressed by only one GKS-series, say $1\trianglelefteq M\lhd N\trianglelefteq G$, such that M and G/N are 3-groups. Furthermore, using the structure of the degree pattern of the solvable graph of G, we can get that 3 is adjacent to p. It is also found from Lemma 2.9 that $p\in\pi(N/M)$. Let $|M|=3^m$ and $|G/N|=3^n$. Thus we can conclude that

$$|N/M|=3^{3f-m-n-1}(q^2-1)(q^3+1),$$

where N/M is a non-abelian simple group.

So by the classification of finite simple groups, the possibilities for N/M are as follows:

- an alternating group $\mathbb{A}_k$ on $k\geqslant 5$ letters;
- one of the 26 sporadic simple groups;
- a simple group of Lie type.

If $N/M\cong U_3(q)$, then $M=1$, $N=G$ and thus $G\cong U_3(q)$, as required. Therefore, we may suppose that N/M is isomorphic to the non-abeilan simple group $S\ncong U_3(q)$ and try to get a contradiction.

In the rest of proof we will use the following facts. The solvable graph of N/M is a subgraph of the solvable graph of G which follows that $d_s(p)=1$ in the solvable graph of N/M because $\Gamma_s(N/M)$ is connected.

Moreover, we have

$$\mathrm{mpf}(|S|)=\mathrm{mpf}(|N/M|).$$

Hence we first compute the value $\mathrm{mpf}(3^{3f-m-n-1}(3^{2f}-1)(3^{3f}+1))$.

It is good to note that $3^f-1<3^f$, $\mathrm{mpf}((3^f+1)^2)<3^{2f}$ and $3^f<3^{2f}-3^f+1=p$. So we can conclude that $\mathrm{mpf}(|N/M|)=p$ or 3^l where l is a natural number.

(1) *S is not isomorphic to an alternating group $\mathbb{A}_k$, $k\geqslant 5$.*

Suppose that S is isomorphic to an alternating group $\mathbb{A}_k$, $k\geqslant 5$. Since $d_s(p)=1$, thus it is seen from the spectrum of alternating groups obtained in [11] that $k\leqslant p+3$. On the other hand, we have

$$\frac{k!}{2}=|\mathbb{A}_k|=|S|=|N/M|=3^{3f-m-n-1}(3^{2f}-1)(3^{3f}+1)=3^{2f-m-n-1}(p-1)p(p+3^{f+1}),$$

which is a contradiction.

(2) *S is not isomorphic to one of the 26 sporadic simple groups.*

Assume that S is isomorphic to one of the 26 sporadic simple groups. As mentioned before, $\mathrm{mpf}(|S|)=\mathrm{mpf}(|N/M|)=p$ or 3^l where l is a natural number. Hence, according to Table

1, the possibilities for S are: $M^cL, J_1, J_3, Co_3, Fi_{23}, Fi'_{24}, Th$. For convenience, we collect the orders and degree pattern of solvable graphs of these groups in Table 3. (see Table 1 in [3])

We recall that the solvable graph of N/M has a vertex of degree 1. Therefore, $S \neq J_1$. Assume that $S \cong M^cL$. It is seen that $p = 11$ and so $3^f(3^f - 1) = 10$ that is impossible. Other groups may be verified similarly.

(3) *S is not isomorphic to a simple group of Lie type, except $U_3(q)$.*

Let ${}^dL_n(q)$ be a finite simple group of Lie type of rank n, defined over the finite field K of order $q = p^f$. We can observe that ${}^dL_m(q)$ contains an isomorphic copy of ${}^dL_n(q)$ where $m \geqslant n$. So we conclude that $\Gamma_s({}^dL_n(q))$ is a subgraph of $\Gamma_s({}^dL_m(q))$. It follows that in the case when $d_s(r) \geqslant 2$ in $\Gamma_s({}^dL_n(q))$ for any prime $r \in \pi({}^dL_n(q))$, then S is not isomorphic to ${}^dL_m(q)$ where $m \geqslant n$.

We only discuss on some of these cases, for example, we consider the cases $L_n(q_0)$, $C_n(q_0)$, ${}^2G_2(q_0)$. Other cases are similar, thus we omit them.

- Suppose that S is isomorphic to $L_n(q_0)$ for some integer $n \geqslant 2$ and a power q_0 of a prime p_0. Then, we have

$$|S| = |L_n(q_0)| = (n, q_0 - 1)^{-1} \cdot q_0^{n(n-1)/2} \prod_{i=2}^{n} (q_0^i - 1).$$

Using Table 2,

$$\mathrm{mpf}(|A_n(q_0)|) = q_0^{n(n-1)/2} \quad (n \geqslant 2),$$

and hence $q_0^{n(n-1)/2} = \mathrm{mpf}(|L_n(q_0)|) = \mathrm{mpf}(|S|) = \mathrm{mpf}(|N/M|)$. If $\mathrm{mpf}(|N/M|) = p$, then $q_0 = p_0 = p$ and $n(n-1)/2 = 1$, which is a contradiction. So we may assume that $\mathrm{mpf}(|N/M|) = 3^l$ for a natural number l, which implies that q_0 is a power of 3.

First consider $L_2(q_0)$. In [3], $\Gamma_s(L_2(q_0))$ for the odd number q_0, was obtained as follows:

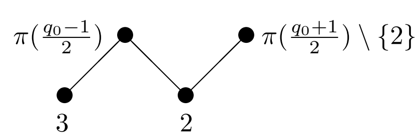

Fig. 2. $\Gamma_s(L_2(q_0)),\ 5 < q_0 \equiv -1 \pmod 4$.

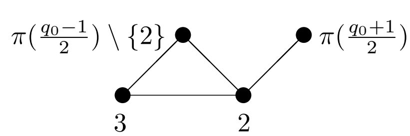

Fig. 3. $\Gamma_s(L_2(q_0)),\ 5 \leqslant q_0 \equiv 1 \pmod 4$.

Let S be isomorphic to $L_2(q_0)$. Since $\Gamma_s(S)$ has a vertex of degree 1, it forces that $|\pi(\frac{q_0+1}{2})| \leqslant 2$. Now, we can conclude from Lemma 2.12 that $q_0 = 9$ or 81. Then $p = 5$ or 41 which is a contradiction because in these cases $p \neq q^2 - q + 1$.

Suppose now that S is isomorphic to $L_3(9)$ or $L_3(81)$. Then according to the compact form of $\Gamma_s(L_3(q_0))$ found in [4], we get that $|\pi({q_0}^2 + q_0 + 1)| = 1$ that is not true.

- Assume that S is isomorphic to $C_n(q_0) = \mathrm{PSp}_{2n}(q_0)$ for some integer $n \geqslant 2$ and a power q_0 of a prime p_0. As a similar argument to above, we may suppose that $\mathrm{mpf}(|N/M|) = 3^l$ for a natural number l, which implies that q_0 is a power of 3.

  According to the spectrum of symplectic groups obtained in [7] and the maximal subgroups $\mathrm{Sp}_4(\mathrm{q_0})$ (Table 8. 12 in [6]), we find that the solvable graph of $\mathrm{PSp}_4(q_0))$ where q_0 is a power of 3 is as follows:

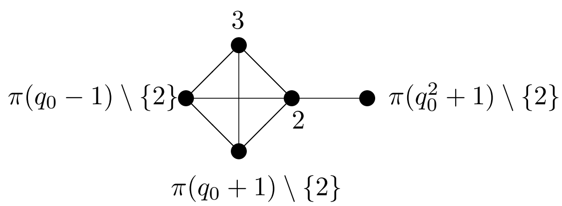

Fig. 4. $\Gamma_{\mathrm{s}}(\mathrm{PSp}_4(q_0))$, $q_0 = 3^k$.

  Since $\Gamma_{\mathrm{s}}(S)$ has a vertex of degree 1, it forces that $|\pi({q_0}^2 + 1)| = 2$. It is enough to consider ${q_0}^2$ instead of q in Lemma 2.12 and conclude that $q_0 = 9$. Then $p = 41$ which is a contradiction because in this case $p \neq q^2 - q + 1$.

  Let S be isomorphic to $\mathrm{PSp}_6(9)$. We found from the spectrum of symplectic groups that for every prime $r \in \pi(S) \setminus \{73\}$, $d_{\mathrm{s}}(r) \geqslant 1$. Moreover, if $73 = q^2 - q + 1$, then $q = 9$. It follows that $|\mathrm{PSp}_6(9)| \mid |U_3(9)|$ which is a contradiction.

- Suppose that S is isomorphic to Ree groups ${}^2G_2(q_0)$ where $q_0 = 3^{2k+1}$, for $k \geqslant 1$.

  According to the spectrum of ${}^2G_2(q_0)$ (see [5, Lemma 4]) and the list of maximal subgroups of ${}^2G_2(q_0)$ in [9], we can obtain that the solvable graph of ${}^2G_2(q_0))$ is as follows:

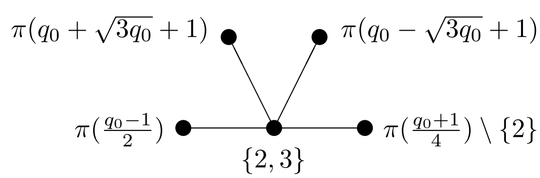

Fig. 5. $\Gamma_{\mathrm{s}}({}^2G_2(q_0)), q = 3^{2k+1} > 3$.

  Assume that $\Gamma_{\mathrm{s}}({}^2G_2(q_0))$ has a vertex of degree 1. By an easy computation, we get from Lemma 2.12 that $|\pi(\frac{q_0+1}{4})| \neq 2$ and $|\pi(\frac{q_0-1}{2})| \neq 1$. If $|\pi(q_0 + \sqrt{3q_0} + 1)| = 1$, then $q_0 + \sqrt{3q_0} = q^2 - q$. It implies that $3^{k+1}(3^k + 1) = 3^f(3^f - 1)$ that is impossible. Using a similar argument, $|\pi(q_0 - \sqrt{3q_0} + 1)| \neq 1$.

This completes the proof of theorem. □

Table 1. *The order and mpf of a sporadic simple group S.*

S	$\|S\|$	mpf($\|S\|$)	S	$\|S\|$	mpf($\|S\|$)
J_2	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$	2^7	Co_2	$2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$	2^{18}
M_{11}	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	2^4	Fi_{23}	$2^{18} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23$	3^{13}
M_{12}	$2^6 \cdot 3^3 \cdot 5 \cdot 11$	2^6	Co_1	$2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23$	2^{21}
M_{22}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	2^7	Ru	$2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29$	2^{14}
HS	$2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$	2^9	Fi'_{24}	$2^{21} \cdot 3^{16} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot$	3^{16}
M^cL	$2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$	3^6		$23 \cdot 29$	
Suz	$2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	2^{13}	$O'N$	$2^9 \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31$	2^9
Fi_{22}	$2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	2^{17}	Th	$2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31$	3^{10}
He	$2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$	2^{10}	J_4	$2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3 \cdot 23 \cdot 29 \cdot$	2^{21}
J_1	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$	19		$31 \cdot 37 \cdot 43$	
J_3	$2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19$	3^5	B	$2^{41} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot$	2^{41}
HN	$2^{14} \cdot 3^6 \cdot 5^6 \cdot 7 \cdot 11 \cdot 19$	2^{14}		$19 \cdot 23 \cdot 31 \cdot 47$	
M_{23}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	2^7	Ly	$2^8 \cdot 3^7 \cdot 5^6 \cdot 7 \cdot 11 \cdot 31 \cdot 37 \cdot 67$	5^6
M_{24}	$2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	2^{10}	M	$2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot$	2^{46}
Co_3	$2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$	3^7		$19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$	

Table 2. *The order and* mpf *of a simple group of Lie type* S.

S	Restrictions on S	$\vert S\vert$	mpf($\vert S\vert$)
$L_{n+1}(q)$	$n \geqslant 2$	$(n+1, q-1)^{-1} q^{n(n+1)/2} \prod_{i=2}^{n+1}(q^i - 1)$	$q^{n(n+1)/2}$
$L_2(q)$	$\vert\pi(q+1)\vert = 1$	$(2, q-1)^{-1} q(q-1)(q+1)$	$q+1$
$L_2(q)$	$\vert\pi(q+1)\vert \geqslant 2$	$(2, q-1)^{-1} q(q-1)(q+1)$	q
$B_n(q)$	$n \geqslant 2$	$(2, q-1)^{-1} q^{n^2} \prod_{i=1}^{n}(q^{2i} - 1)$	q^{n^2}
$C_n(q)$	$n \geqslant 2$	$(2, q-1)^{-1} q^{n^2} \prod_{i=1}^{n}(q^{2i} - 1)$	q^{n^2}
$D_n(q)$	$n \geqslant 4$	$(4, q^n - 1)^{-1} q^{n(n-1)}(q^n - 1) \prod_{i=1}^{n-1}(q^{2i} - 1)$	$q^{n(n-1)}$
$G_2(q)$		$q^6(q^6 - 1)(q^2 - 1)$	q^6
$F_4(q)$		$q^{24}(q^{12} - 1)(q^8 - 1)(q^6 - 1)(q^2 - 1)$	q^{24}
$E_6(q)$		$(3, q-1)^{-1} q^{12}(q^9 - 1)(q^5 - 1)\vert F_4(q)\vert$	q^{36}
$E_7(q)$		$(2, q-1)^{-1} q^{39}(q^{18} - 1)(q^{14} - 1)(q^{10} - 1)\vert F_4(q)\vert$	q^{63}
$E_8(q)$		$q^{96}(q^{30} - 1)(q^{12} + 1)(q^{20} - 1)(q^{18} - 1)(q^{14} - 1)$	q^{120}
		$(q^6 + 1)\vert F_4(q)\vert$	
$U_{n+1}(q)$	$(n, q) \neq (2, 3), (3, 2)$	$(n+1, q+1)^{-1} q^{n(n+1)/2} \prod_{i=2}^{n+1}(q^i - (-1)^i)$	$q^{n(n+1)/2}$
	$n \geqslant 2$		
$U_4(2)$		$2^6 \cdot 3^4 \cdot 5$	3^4
$U_3(3)$		$2^5 \cdot 3^3 \cdot 7$	2^5
${}^2B_2(q)$	$q = 2^{2m+1}$	$q^2(q^2 + 1)(q - 1)$	q^2
	$\vert\pi(q^2 + 1)\vert \geqslant 2$		
${}^2B_2(q)$	$q = 2^{2m+1}$	$q^2(q^2 + 1)(q - 1)$	$q^2 + 1$
	$\vert\pi(q^2 + 1)\vert = 1$		
${}^2D_n(q)$	$n \geqslant 4$	$(4, q^n + 1)^{-1} q^{n(n-1)}(q^n + 1) \prod_{i=1}^{n-1}(q^{2i} - 1)$	$q^{n(n-1)}$
${}^3D_4(q)$		$q^{12}(q^8 + q^4 + 1)(q^6 - 1)(q^2 - 1)$	q^{12}
${}^2G_2(q)$	$q = 3^{2m+1}$	$q^3(q^3 + 1)(q - 1)$	q^3
${}^2F_4(q)$	$q = 2^{2m+1}$	$q^{12}(q^6 + 1)(q^4 - 1)(q^3 + 1)(q - 1)$	q^{12}
${}^2E_6(q)$		$(3, q+1)^{-1} q^{12}(q^9 + 1)(q^5 + 1)\vert F_4(q)\vert$	q^{36}

Table 3.

S	$\|S\|$	$\mathrm{D_s}(S)$
M^cL	$2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$	$(3, 3, 3, 2, 1)$
J_1	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$	$(5, 4, 3, 2, 2, 2)$
J_3	$2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19$	$(3, 3, 2, 1, 1)$
Co_3	$2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$	$(4, 3, 3, 2, 3, 1)$
Fi_{23}	$2^{18} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23$	$(6, 4, 4, 3, 3, 2, 1, 1)$
Fi'_{24}	$2^{21} \cdot 3^{16} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29$	$(7, 5, 4, 4, 4, 2, 1, 1, 2)$
Th	$2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31$	$(4, 5, 3, 2, 2, 1, 1)$

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