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REGULAR AND STRONGLY SOFT Γ- RELATIONS ON FUZZY SOFT Γ-HYPERRINGS

S. OSTADHADI-DEHKORDI∗ AND B. SHEIKH HOSEINI

Abstract. The concept of fuzzy soft Γ-hyperrings introduced by J. Zhan et al. as a generalization of the soft rings. In this paper, we prove the equivalence relation μ^* defined by J. Zhan et al. is a strongly soft Γ-regular relation and hyperoperations defined on quotient fuzzy soft Γ-hyperrings are just operations. Also, we define the equivalence relation μ_I^* as a generalization the relation μ^* and consider quotient fuzzy soft Γ-hyperrings and isomorphism theorems by this regular relation.

1. Introduction

The mathematical theories such as probability theory, fuzzy set theory [18], rough set theory [15, 16], vague set theory [4] and the interval mathematics [5] were established by researchers to modeling uncertainties appearing in the above fields. Fuzzy set were proposed by Zadeh in 1965 [18], order to manage data with non-statical uncertainly. They allow us to describe partial membership of objects in a set by ill-defined boundaries. The membership is actually a generalization of a set-characteristic function. Namely, in a classical set theory a subset

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∗Corresponding author

A of a set X is induced by its characteristic function χ_A mappings the elements of X with the elements of the set$\{0, 1\}$. Fuzzy set theory defines a fuzzy subset A of a set X by its membership functions μ_A as a mapping from the elements of X to the unity interval $[0, 1]$. Soft set theory, introduced by Molodtsov[10], has been considered as an effective mathematical tool for modeling uncertainties. The soft set theory is a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches.

Algebraic hyperstructures represent a natural generalization of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the result of this composition is a set. Hyperstructure theory was born in 1934, when Marty [9], a French mathematician, at the 8^{th} Congress of Scandinavian Mathematicians gave the definition of hypergroup and illustrated some of their applications, with utility in the study of groups, algebraic functions and rational fractions. The more general structure that satisfies the ring-like axioms is the hyperring in the general sense: $(R, +, \cdot)$ is a hyperring if $+$ and" $\cdot$ " are two hyperoperations such that $(R, +)$ is a hypergroup and " $\cdot$ " is an associative hyperoperation, which is distributive with respect to $+$. There are different types of hyperrings. If only the addition $+$ is a hyperoperation and the multiplication" $\cdot$ " is a usual operation, then we say that R is an additive hyperring. A special case of this type is the hyperring introduced by Krasner [6]. In [11], Nobusawa introduced the notion of Γ-ring, as more general than ring. Barnes [1] weakened slightly the conditions in the definition of the Γ-ring in the sense of Nobusawa. After these two papers are published, many mathematicians made good works on Γ-ring in the sense of Barnes and Nobusawa. The concept of Γ-semihyperrings is a generalization of semirings, a generalization of semihyperrings and a generalization of Γ-semirings [12, 13, 14].

By using a certain type of equivalence relations, we can connect soft Γ-hyperrings to soft Γ-rings. These equivalence relations are called strongly soft Γ-regular relations and regular soft Γ-relations, respectively. More exactly, starting with a soft Γ- hyperrings and using a strong soft Γ-regular relation, we can construct a soft Γ-rings structure on the quotient set. In this paper, we consider the relation μ^* defined by Zhan et al. [19] and we prove this relation is strongly regular and all hyperadditions defined on quotient soft Γ-hyperrings are just additions. Moreover, we define the relation μ_I^* as a generalization μ^* where I is a Γ-hyperideal.

2. **Preliminaries**

Definition 2.1. [17] A hypergroupoid $(H, \circ)$ is called a *hypergroup* when for all $x, y, z \in H$, we have $x \circ (y \circ z) = (x \circ y) \circ z$ and $x \circ H = H \circ x = H$.

Example 2.2. Let $\mathbb{R}$ be the real numbers. Then, $\mathbb{R}$ is a hypergroup with respect the following hyperoperations:

$$x \circ y = \{z \in \mathbb{R} : n \leq z < n + 1\},$$

where $x, y \in \mathbb{R}$ and $n = max\{[x], [y]\}$.

Example 2.3. Let $\mathbb{R}$ be real numbers and $x, y \in \mathbb{R}$. Then, we define $x \circ y$ all different real elements in the open interval between x and y and $x \circ x = x$. Then, $(\mathbb{R}, \circ)$ is a hypergroup.

Definition 2.4. [2] A *hyperring* is an algebraic structure $(R, +, \cdot)$ which satisfies the following axioms:

(1) $(R, +)$ is a canonical hypergroup, i.e., for every $x, y, z \in R$,

(a) $(x + y) + z = x + (y + z)$,

(b) $x + y = y + x$,

(c) there exists $0 \in R$ such that $0 + x = x$, for all $x \in R$,

(d) for every $x \in R$ there exists a unique element $x^{'} \in R$ such that $0 \in x + x^{'}$ (we shall write $-x$ for $x^{'}$ and we call it the opposite of x),

(e) $z \in x + y$ implies $y \in -x + z$ and $x \in z - y$,

(2) Relating to the multiplication, $(R, \cdot)$ is a semigroup having zero as a bilaterally absorbing element, i.e., $0 \cdot x = x \cdot 0 = 0$ for all $x \in R$,

(3) The multiplication is distributive with respect to the hyperoperation $+$, i.e., for every $x, y, z \in R$, we have $x \cdot (y + z) = x \cdot y + x \cdot z$.

Remark 2.5. The following elementary facts follow easily from the axioms: $-(-x) = x$ and $-(x + y) = -x - y$ for all $x, y \in R$.

Definition 2.6. Let $(R, +, \cdot)$ be a hyperring and I be a nonempty subset of R. Then, I is said to be *subhyperring* of R, when $(I, +, \cdot)$ is itself a hyperring. A subhyperring I of R is a *left(right) hyperideal* of R when $r \cdot x \in I$ $(x \cdot r \in I)$ for all $r \in R$ and $x \in I$. Also, I is called a *hyperideal* if I is both a left and right hyperideal.

Example 2.7. Let $(\mathbb{Z}, +, \cdot)$ be integer numbers ring. Then, $(\mathbb{Z}, \oplus, \cdot)$ is a hyperring by following hyperoperations:

$$n \oplus n = \{t \in \mathbb{Z} : t \leq n\}, \ n \oplus m = max\{n, m\},$$

where $n, m \in \mathbb{Z}$ and $n \neq m$.

Definition 2.8. [19] Let $(R, \oplus)$ and $(\Gamma, +)$ be two canonical hypergroups. Then, R is called a Γ-*hyperring*, if the following conditions are satisfied for all $x, y, z \in R$ and for all $\alpha, \beta \in \Gamma$,

(1) $x\alpha y \in R$,

(2) $(x \oplus y)\alpha z = x\alpha z \oplus x\alpha z$,
$x\alpha(y \oplus z) = x\alpha y \oplus x\alpha z$,
$x(\alpha + \beta)y = x\alpha y \oplus x\beta z$,

(3) $x\alpha(y\beta z) = (x\alpha y)\beta z$.

Definition 2.9. Let R be a Γ-hyperring and I be a nonempty subset of R. Then, I is called Γ-*subhyperring* when $(I, \oplus)$ is a Γ-hyperring. A Γ-subhyperring I is called Γ-*hyperideal* when $x\alpha r \in I,\ r\alpha x \in I$, for every $x \in I$, $r \in R$ and $\alpha \in \Gamma$.

Example 2.10. Let R be a hyperring and let $M_{m,n}(R)$ be the set of all matrices by the size $m \times n$ with entries of R. We define $\circ : M_{m,n}(R) \times M_{n,m}(R) \times M_{m,n}(R) \longrightarrow M_{m,n}(R)$ by

$$A \circ B \circ C = \{Z \in M_{m,n}(R) | Z \in ABC \ \ A, C \in M_{m,n}(R) \ \ B \in M_{n,m}(R)\}.$$

Then, $M_{m,n}(R)$ is an $M_{n,m}(R)$-hyperring.

Example 2.11. Let $(M, +, \cdot)$ be a hyperring and Γ be an hyperideal of M. Define $\circ$: $M \times \Gamma \times M \longrightarrow M$ by $a \circ \gamma \circ b = \{z \in M | z \in a \cdot \gamma \cdot b\}$. Then, M is a Γ-hyperring.

Definition 2.12. [7] A fuzzy subset μ of a Γ-hyperring R is called a *fuzzy* Γ-*hyperideal* of R, when for every $x, y \in R$ and $\alpha, \beta \in \Gamma$, the following conditions hold:

(1) $min\{\mu(x), \mu(y)\} \leq inf_{z \in x+y}\mu(z)$,

(2) $\mu(x) \leq \mu(-x)$,

(3) $max\{\mu(x), \mu(y)\} \leq \mu(x\alpha y)$,

(4) $\mu(x) \leq inf_{z \in -y+x+y}\mu(z)$.

Example 2.13. In Example 2.11, let I and J be Γ-hyperideals of Γ-hyperring $(M, +, \cdot)$ and $J \subseteq \Gamma$, then I is a J-hyperideal of Γ-hyperring M, since $IJM \subseteq I$ and $MJI \subseteq I$. Now define μ and ν on I and J, respectively as follows:

$$\mu(x) = \begin{cases} 0.6 & if\ x \in I \\ 0 & otherwise \end{cases} \qquad \nu(\delta) = \begin{cases} 0.2 & if\ \delta \in J \\ 0 & otherwise \end{cases}$$

Then, μ is fuzzy subhyperring of M and ν is fuzzy subhypergroup of Γ. It is easy to see that μ is a fuzzy ν-hyperideal of Γ-hyperring of M.

Remark 2.14. Let μ be a fuzzy Γ-hyperideal of hyperring R. Then, it is clear that

$$\mu(-x) = \mu(x),\ min\{\mu(x), \mu(y)\} \leq inf\mu(z)_{z \in x-y},\ \mu(x) \leq \mu(0).$$

Definition 2.15. Let R be a Γ-hyperring and A be a non-empty set. A *set valued function* $F : A \longrightarrow P(R)$ can be defined as $F(x) = \{y \in R : (x, y) \in \rho\}$ for all $x \in A$, where ρ is an arbitrary binary relation between an element of A and element of R. Then, the pair (F, A) is a soft set over R.

Definition 2.16. Let (F, N) be a soft set over Γ-hyperring A. Then, (F, N) is called a *soft Γ-hyperring over A*, when $F(a)$ is a Γ-subhyperring of A, for all $a \in N \subseteq A$.

Definition 2.17. Let $(R, +, \cdot)$ be a Γ-hyperring, E be a parameter set, $A \subseteq E$ and F be a mapping given by $f : A \times \Gamma \times A \longrightarrow [0, 1]^R$, where $[0, 1]^R$ denote the collection of all fuzzy subsets of R. Then, $f(a, \alpha, b) = f(a\alpha b) = f_{a\alpha b}$, where $f_{a\alpha b} = f_\mu : R \longrightarrow [0, 1]$ is called a *fuzzy soft Γ-hyperring over R* when for each $a, b \in A$ and $\alpha \in \Gamma$, the corresponding fuzzy subset f_μ of f is a fuzzy soft Γ-hyperring of R, i.e. for all $x, y \in R$:

(1) $f_\mu(x + y) \geqslant f_\mu(x) \wedge f_\mu(y)$,
(2) $f_\mu(-x) \geqslant f_\mu(x)$,
(3) $f_\mu(x\alpha y) \geqslant f_\mu(x) \wedge f_\mu(y)$.

Example 2.18. Let $R = \mathbb{Z}_6 = \Gamma = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}\}$ and define $\overline{x}\alpha\overline{y} = \overline{x\alpha y}$, for all $\overline{x}, \overline{y} \in R$, $\alpha \in \Gamma$. Then, R is a Γ-hyperring and let $f : A \times \Gamma \times A \longrightarrow \mathcal{P}(R)$ be a set valued function defined by

$$f_a = \{(\overline{0}, 0.8), (\overline{1}, 0.2), (\overline{2}, 0.4), (\overline{3}, 0.2), (\overline{4}, 0.4), (\overline{5}, 0.2)\},$$
$$f_b = \{(\overline{0}, 0.9), (\overline{1}, 0.5), (\overline{2}, 0.5), (\overline{3}, 0.1), (\overline{4}, 0.5), (\overline{5}, 0.5)\},$$
$$f_c = \{(\overline{0}, 0.7), (\overline{1}, 0.3), (\overline{2}, 0.3), (\overline{3}, 0.3), (\overline{4}, 0.3), (\overline{5}, 0.3)\}.$$

Then, (f, N) is a fuzzy soft set over R. Also, we see that f_μ is a fuzzy ideal of R for all $\mu \in A \times \Gamma \times A$, thus (f, N) is a fuzzy soft Γ-hyperring over R.

Definition 2.19. Let (F, A) be a soft set over Γ- hyperring R. Then, the set $Supp(F, A) = \{x \in A : F(x) \neq \emptyset\}$ is called the *support* of the soft set (F, A). A soft set (F, A) is *non-null* if $Supp(F, A) \neq \emptyset$.

Definition 2.20. Let (F, A) be a non-null soft set over Γ-hyperring R and A, B are nonempty subsets of $F(x)$, where $x \in A$ and ρ be an equivalence relation on R. Then,

(i) $A\overline{\rho}B$ means that for every $a \in A$ there exists $b \in B$ such that $\rho(a) = \rho(b)$ and for every $b \in B$ there exists $a \in A$ such that $\rho(a) = \rho(b)$.
(ii) $A\overline{\overline{\rho}}B$ means that for every $a \in A$ and $b \in B$, $\rho(a) = \rho(b)$.

Definition 2.21. Let (F, A) be a non-null soft set over Γ-hyperring R and ρ be an equivalence relation on R. Then, we say that ρ is a *regular soft Γ-relation*, when for every $x \in A$, $a, b, c \in F(x)$ and $\alpha \in \Gamma$,

$$a\rho b \Longrightarrow (a + c)\overline{\rho}(b + c),\ (a\alpha c)\rho(b\alpha c).$$

Also, the equivalence relation ρ is called *strongly soft Γ- regular*, when

$$(a + c)\overline{\overline{\rho}}(b + c),\ (a\alpha c)\rho(b\alpha c).$$

Theorem 2.22. *Let (F,A) be a soft Γ-hyperring over R and ρ be a regular soft Γ-relation on R. Then, $(F/\rho,A)$ is a soft $\widehat{\Gamma}$- hyperring over R/ρ by $(F/\rho)(x)=F(x)/\rho$ and*

$$\rho(a)\oplus\rho(b)=\{\rho(c):c\in a+b\},\ \rho(a)\widehat{\alpha}\rho(b)=\rho(a\alpha b),$$

such that $a,b\in F(x)$ and $\widehat{\Gamma}=\{\widehat{\alpha}:\alpha\in\Gamma\}$. Also, when the hyperoperation $(F/\rho)(x)$ is well-defined, then the relation ρ is the regular soft Γ-relation.

Proof. Suppose that ρ is a regular soft Γ-relation. First, we check that the hyperoperation $\oplus$ is well-defined on F/ρ. Let $\rho(a_1)=\rho(a_2)$ and $\rho(b_1)=\rho(b_2)$. We check that $\rho(a_1)\oplus\rho(b_1)=\rho(a_2)\oplus\rho(b_2)$. We have $a_1\rho a_2$ and $b_1\rho b_2$. Since ρ is regular soft Γ-relation, it follows that $(a_1+b_1)\overline{\rho}(a_2+b_1)$ and $(a_2+b_1)\overline{\rho}(a_2+b_2)$. Hence for all $c_1\in a_1+b_1$, there is $c_2\in a_2+b_2$ such that $\rho(c_1)=\rho(c_2)$. It follows that $\rho(a_1)\oplus\rho(b_1)\subseteq\rho(a_2)\oplus\rho(b_2)$ and similarly, we obtain the converse inclusion. By a routine argument we can see another properties.

Conversely, $a,b,c\in F(x)$ such that $u\in a+c$. Hence $\rho(u)\in\rho(a)\oplus\rho(c)=\rho(b)\otimes\rho(c)$. Thus, there exists $v\in b+c$ such that $\rho(u)=\rho(v)$, whence $(a\circ c)\overline{\rho}(b+c)$. In a same way, $\rho(a\alpha c)=\rho(b\alpha c)$. Therefore, the relation ρ is the regular soft Γ-relation. □

Corollary 2.23. *Let ρ be an equivalence relation on Γ-hyperring R and (F,A) be a soft Γ-hyperring over R. Then, ρ is a strongly soft Γ-relation if and only if the hyperaddition defined on F/ρ is an addition.*

3. Quotient fuzzy soft Γ-hyperrings

In this section by using a certain type of equivalence relations, we connect soft Γ-hyperrings to quotient soft Γ-hyperrings. Let (F,A) be a soft Γ-hyperring over Γ-hyperring R and μ be a fuzzy Γ- hyperideal over Γ-hyperring R. We analyze the equivalence relation μ^* used by Zhan et al. [19] and we define the relation μ^*_I as a generalization of μ^*. Also, we prove that the relation μ^* is strongly regular. Hence the hyperoperation defined by Zhan [19], on quotient soft Γ-hyperrings is just operation.

Definition 3.1. [19] Let (F,A) be a soft Γ-hyperring, μ be a fuzzy Γ- hyperideal over hyperring R and equivalence relation $\equiv$ defined as follows:

$$x\equiv y\Longleftrightarrow\ \exists\ z\in x-y:\mu(z)=\mu(0).$$

denote this relation by μ^*. The equivalence class containing the element $x\in R$ denoted by $\mu^*(x)$ and the set of all equivalence classes denoted by F/μ.

By the concept Γ-hyperideal of a Γ-hyperring, we can generalize the relation μ^* as follows:

Definition 3.2. Let (F, A) be a soft Γ-hyperring over Γ-hyperring R, I be a Γ-hyperideal of R and μ be a fuzzy Γ- hyperideal of R. Then, we define the relation μ_I^* as follows:

$$\forall a, b \in F(x),\ (a, b) \in \mu_I^* \Longleftrightarrow\ \exists\ \alpha \in I,\ a \in b + \alpha : \mu(\alpha) = \mu(0).$$

When $I = R$, the relations μ_I^* and the relation μ^* defined by Zhan et al. [19] are equals.

Proposition 3.3. *Let (F, A) be a soft Γ- hyperring over hyperring R, I be a Γ-hyperideal of R and μ be a fuzzy Γ-hyperideal of R. Then, the relation μ_I^* is an equivalence relation on R.*

Proof. Suppose that $x \in R$. Since $x = x + 0$ and $0 \in I$, we have $(x, x) \in \mu_I^*$. Then, the relation μ_I^* is reflexive. Let $(x, y) \in \mu_I^*$. Then, there exits $\alpha \in I$ such that $x \in y + \alpha$ and $\mu(\alpha) = \mu(0)$. Hence $y \in x - \alpha$ and $\mu(\alpha) = \mu(-\alpha)$. This implies that $(y, x) \in \mu_I^*$ and so μ_I^* is symmetric. Let $(x, y) \in \mu_I^*$ and $(y, z) \in \mu_I^*$. Then, there exist $\alpha_1, \alpha_2 \in I$ such that $x \in y + \alpha_1$ and $y \in z + \alpha_2$ and $\mu(\alpha_1) = \mu(\alpha_2) = \mu(0)$. So $x \in z + \alpha_1 + \alpha_2$. Thus, there exists $\alpha_3 \in \alpha_1 + \alpha_2 \subseteq I$ such that $x \in z + \alpha_3$. Also, $\alpha_3 \in \alpha_1 + \alpha_2$ implies that $min\{\mu(\alpha_1), \mu(\alpha_2)\} \leq \mu(\alpha_3)$. Hence $\mu(\alpha_3) = \mu(0)$ and $(x, z) \in \mu_I^*$. Therefore, the relation μ_I^* is an equivalence. □

By the concept of soft regular Γ- relation and Proposition 3.3, we can construct quotient soft Γ-hyperrings.

Proposition 3.4. *Let (F, A) be a soft Γ-hyperring over hyperring R and I be a Γ-hyperideal of R. Then, the set of all classes F/μ_I^* is a soft Γ-hyperring as follows:*

$$(F/\mu_I^*)(x) = F(x)/\mu_I^*.$$

Proof. By Proposition 3.3, the relation μ_I^* is an equivalence relation on R. Let $x_1 \mu_I^* x_2$ and $z \in F(x)$. Then, $x_1 \in x_2 + w$, where $w \in I$. Also, $x_1 + z \subseteq x_2 + z + w$ and $x_1 \alpha z \subseteq x_2 \alpha z + w \alpha z$, for every $\alpha \in \Gamma$. Hence, for every $t_1 \in x_1 + z$ there is $t_2 \in x_2 + z$ such that $t_1 \in t_2 + w$. Thus, $\mu_I^*(t_1) = \mu_I^*(t_2)$. In a same way, we can see that for every $s_1 \in x_2 + z$ there is $s_2 \in x_1 + z$ such that $\mu_I^*(s_1) = \mu_I^*(s_2)$. Since, $\mu(0) = max\{\mu(w), \mu(z)\} \leq \mu(w \alpha z)$, we have $\mu(w \alpha z) = \mu(0)$. Hence, the relation μ_I^* is a regular soft Γ-regular. Therefore, by Theorem 2.22, F/μ_I^* is a soft Γ-hyperring. □

Proposition 3.5. *Let (F, A) be a soft Γ-hyperring over Γ-hyperring R and I_1, I_2 be Γ-hyperideals of $F(x)$. Then,*

(i) $\mu^*_{I_1 \cap I_2} \subseteq \mu^*_{I_1} \cap \mu^*_{I_2}$,
(ii) $\mu^*_{I_1} \cup \mu^*_{I_2} \subseteq \mu^*_{I_1 + I_2}$.

Proof. The proof is straightforward. □

Definition 3.6. Let (F, A) be a soft Γ-hyperring over R and μ be a fuzzy Γ-hyperideal on R. Then, we say that a map $\varphi : F(x) \longrightarrow F(y)$ is a *homomorphism* if

(i) $\varphi(a + b) = \varphi(a) + \varphi(b)$,

(ii) $\varphi(a\alpha b) = \varphi(a)\alpha\varphi(b)$,

(iii) $\varphi(0) = 0$,

(iv) $\mu \circ \varphi(a) = \mu(a)$,

for every $a, b \in F(x)$ and $\alpha \in \Gamma$.

Definition 3.7. Let (F, A) be a soft Γ-hyperring over R, μ be a fuzzy Γ- hyperideal on $F(x)$ and $\varphi : F(x) \longrightarrow F(y)$ be a homomorphism. Then, we define the *kernel* of φ as follows:

$$ker\varphi = \{a \in F(x) : \varphi(a) = 0,\ \mu(a) = \mu(0)\}.$$

Proposition 3.8. *Let (F, A) be a soft Γ-hyperring and $\varphi : F(x) \longrightarrow F(y)$ be a homomorphism and μ be a fuzzy Γ-hyperiddeal on $F(x)$. Then, the kernel of φ is a Γ-subhyperiddeal of $F(x)$.*

Proof. Since $0 \in ker\varphi$, we have $ker\varphi$ is a nonempty set. Let $a, b \in ker\varphi$. Then, $\mu(a) = \mu(b) = \mu(0)$ and $\varphi(a) = \varphi(b) = 0$. This implies that $\varphi(a + b) = 0$ and $\varphi(a\alpha b) = 0$, for every $\alpha \in \Gamma$. Also, for every $c \in a + b$, $min\{\mu(a), \mu(b)\} \le \mu(c)$ and $max\{\mu(a), \mu(r)\} \le \mu(a\alpha r)$, for every $r \in F(x)$. Thus, $\mu(0) \le \mu(c)$ and $\mu(0) \le \mu(a\alpha r)$. Then, $\mu(c) = \mu(0)$ and $\mu(a\alpha r) = \mu(0)$. Therefore, $a + b \subseteq ker\varphi$ and $a\alpha r \in ker\varphi$ and $ker\varphi$ is a right Γ-hyperideal. In a same way, we can see that $ker\varphi$ is a left Γ- hyperideal. □

By Proposition 3.17, we construct quotient soft Γ-hyperring by the regular soft Γ-relation. At follows, we define fuzzy set on quotient soft Γ-hyperring.

Proposition 3.9. *Let (F, A) be a soft Γ-hyperring over hyperring R, μ be a fuzzy Γ- hyperrideal and I be a Γ-hyperideal of R. Then, there is a fuzzy set on F/μ_I^*.*

Proof. Suppose that $\widehat{\mu}$ is a fuzzy set on $F(x)/\mu_I^*$ defined by $\widehat{\mu}(\mu_I^*(a)) = \mu(a)$, where $a \in F(x)$. Let $\mu_I^*(x_1) = \mu_I^*(x_2)$, where $x_1, x_2 \in F(x)$. Then, for some $w \in I$ such that $\mu(w) = \mu(0)$, we have $x_1 \in x_2 + w$. Thus, $min\{\mu(0), \mu(x_2)\} \le \mu(x_1)$. Since $\mu(x_2) \le \mu(0)$, we have $\mu(x_2) \le \mu(x_1)$. In a same way, we can see $\mu(x_1) \le \mu(x_2)$. It is easy to see that the other properties holds. □

Next, we establish three isomorphism theorems of soft Γ-hyperrings by regular soft Γ-relation μ_I^* on soft Γ-hyperrings.

Theorem 3.10. *Let (F, A) be a soft Γ-hyperring and $\varphi : F(x) \longrightarrow F(y)$ be a homomorphism with kernel K and μ be a fuzzy Γ-hyperideal on R. Then, $F/\mu_K^* \cong Im\varphi$.*

Proof. Suppose that $\psi : F(x)/\mu_K^* \longrightarrow Im\varphi$ defined by $\psi(\mu_K^*(x_1)) = \varphi(x_1)$, where $x_1 \in F(x)$. Let $\mu_K^*(x_1) = \mu_K^*(x_2)$. Then, for some $w \in K$, we have $x_1 \in x_2 + w$ and $\mu(w) = \mu(0),\ \varphi(w) = 0$. Consequently, $\varphi(x_1) \in \varphi(x_2 + w) = \varphi(x_2) + \varphi(w) = \varphi(x_2)$. Thus, $\varphi(x_1) = \varphi(x_2)$ and φ is well-defined. Let $\psi(\mu_K^*(x_1)) = \psi(\mu_K^*(x_2)$. Then, $0 \in \varphi(x_1 - x_2)$ and so there exists $x_3 \in x_1 - x_2$ such that $\varphi(x_3) = 0$. Hence $x_1 \in x_2 + x_3$ and $\mu \circ \varphi(x_3) = \mu(x_3)$. Thus, $\mu(x_3) = \mu(0)$ and $x_3 \in ker\varphi$. This implies that $\mu_K^*(x_1) = \mu_K^*(x_2)$. Also, by Proposition 3.9, we can define a fuzzy Γ-hyperideal $\widehat{\mu}$ on $F(x)/\mu_K^*$. Obviously, $\mu \circ \psi = \widehat{\mu}$. □

Theorem 3.11. *Let (F, A) be a soft Γ-hyperring over R, I_1 and I_2 be Γ-hyperideals of R. Then, $I_1/\mu^*_{I_1 \cap I_2} \cong (I_1 + I_2/\mu^*_{I_2})$.*

Proof. Suppose that $\varphi : I_1 \longrightarrow (I_1 + I_2)/\mu^*_{I_2}$ defined by $\varphi(a) = \mu^*_{I_2}(a)$, where $a \in F(x)$. Obviously, φ is well-defined. Let $\mu^*_{I_2}(a) \in (I_1 + I_2)/\mu^*_{I_2}$. Then, there are $a_1 \in I_1$ and $a_2 \in I_2$ such that $a \in a_1 + a_2$. Thus, $\mu^*_{I_2}(a) = \mu^*_{I_2}(a_1)$ and $\varphi(\mu^*_{I_2}(a)) = \mu^*_{I_2}(a_1)$. Hence the map φ is onto. Also,

$$\begin{aligned} ker\varphi = \{a \in I_1 : \varphi(a) &= \mu_I^*(0)\} = \{a \in I_1 : \mu_I^*(a) = \mu_I^*(0)\} \\ &= \{a \in I_1 : a \in 0 + \alpha\ \alpha \in I_2\} \\ &= I_1 \cap I_2. \end{aligned}$$

Therefore, by Theorem 3.10, $I_1/\mu^*_{I_1 \cap I_2} \cong (I_1 + I_2)/\mu^*_{I_2}$. □

Let (F, A) be a soft Γ- hyperring over R and I_1, I_2 be Γ-hyperideals of R such that $I_1 \subseteq I_2$. Then,

$$\mu^*_{I_2}/\mu^*_{I_1} = \{(\mu^*_{I_1}(a), \mu^*_{I_1}(b)) : \mu^*_{I_2}(a) = \mu^*_{I_2}(b),\ a, b \in F(x)\}.$$

Obviously, $\mu^*_{I_2}/\mu^*_{I_1}$ is a regular soft Γ-relation $F/\mu^*_{I_1}$.

Theorem 3.12. *Let (F, A) be a soft Γ- hyperring over R and I_1, I_2 be Γ- hyperideals of R such that $I_1 \subseteq I_2$. Then,*

$$(F/\mu^*_{I_1})/(\mu^*_{I_2}/\mu^*_{I_1}) \cong F/\mu^*_{I_2}.$$

Proof. Suppose that $\varphi : F(x)/\mu^*_{I_1} \longrightarrow F(x)/\mu^*_{I_2}$ defined by $\varphi(\mu^*_{I_1}(a)) = \mu^*_{I_2}(a)$, where $a \in F(x)$. Let $\mu^*_{I_1}(a_1) = \mu^*_{I_1}(a_2)$.Then, there is $b \in I_1$ such that $a_1 \in a_2 + b$. Also, $I_1 \subseteq I_2$, implies that $\mu^*_{I_2}(a_1) = \mu^*_{I_2}(a_2)$. Hence φ is well-defined. Also, $ker\varphi = \mu^*_{I_2}/\mu^*_{I_1}$. Therefore, by Theorem 3.10, the proof is complete. □

Proposition 3.13. *Let (F, A) be a soft Γ-hyperring and μ be a fuzzy Γ- hyperring of R where. Then, following sets are equals:*
$\Omega_1 = \{\mu^*(c) : c \in a + b\}$,
$\Omega_2 = \{\mu^*(c) : c \in \mu^*(a) + \mu^*(b)\}$,
$\Omega_3 = \{\mu^*(c) : \mu^*(c) \subseteq \mu^*(a) + \mu^*(b)\}$.

Proof. Obviously, $\Omega_1 \subseteq \Omega_2$. Let $\mu^*(c) \in \Omega_2$. Then, $c \in \mu^*(a) + \mu^*(b)$ and for some $c_1 \in \mu^*(a)$ and $c_2 \in \mu^*(b)$ such $c \in c_1 + c_2$. This implies that there are $t_1, t_2 \in F(x)$ such that $c \in a + b - (t_1 + t_2)$, $\mu(t_1) = \mu(t_2) = \mu(0)$. Then, for some $t_3 \in t_1 + t_2$ and $d \in a + b$ we have $c \in d - t_3$. Also, $min\{\mu(t_1), \mu(t_2)\} \leq \mu(t_3)$. This implies that $\mu(t_3) = \mu(0)$ and $\mu^*(c) = \mu^*(d)$. Hence $\Omega_2 \subseteq \Omega_1$. Let $\mu^*(c) \in \Omega_3$. Then, $\mu^*(c) \subseteq \mu^*(a) + \mu^*(b)$. This, implies that for some $t_1, t_2 \in F(x)$, we have $c \in a + b - (t_1 + t_2)$, where $\mu(t_1) = \mu(t_2) = \mu(0)$. Then, for some $d \in a + b$ and $t_3 \in t_1 + t_2$ such that $c \in d - t_3$. On the other hand, $min\{\mu(t_1), \mu(t_2)\} \leq \mu(t_3)$ implies that $\mu(t_3) = \mu(0)$. Thus, $\Omega_3 \subseteq \Omega_1$. Assume that $\mu^*(c) \in \Omega_2$. Then, $c \in \mu^*(a) + \mu^*(b)$ and for some $d \in a + b$ and $t_3 \in t_1 + t_2$, we have $c \in d - t_3$, where $\mu(t_1) = \mu(t_2) = \mu(0)$. Let $w \in \mu^*(c)$. Then, for some $r \in F(x)$, $w \in c - r$ and $\mu(r) = \mu(0)$. This implies that $w \in d - t_3 - r \subseteq a + b - (t_3 + r) = (a - t_3) + (b - r)$. Hence there are $z_1 \in a - t_3$ and $z_2 \in b - r$ such that $w \in z_1 + z_2 \subseteq \mu^*(x) + \mu^*(y)$. Therefore, $\mu^*(c) \subseteq \mu^*(a) + \mu^*(b)$ and $\Omega_2 \subseteq \Omega_3 \subseteq \Omega_1$ □

In the Theorem 3.14, we prove that the relation μ^* defined by Zhan et al. [19] is an equivalence relation only according to conditions (1), (2) and (3) Definition 2.12.

Theorem 3.14. *Let (F, A) be a soft Γ-hyperring, $x \in A$ and μ be a fuzzy Γ-hyperring of $F(x)$. Then, the relation μ^* is an equivalence relation on $F(x)$.*

Proof. Suppose $a \in F(x)$. Since $F(x)$ is a hyperring $0 \in a - a$. Hence μ^* is reflexive. Let $a, b \in F(x)$ and $a\mu^*b$. Then, there exists $c \in a - b$ such that $\mu(c) = \mu(0)$. Since $-c \in b - a$ and $\mu(-c) = \mu(c) = \mu(0)$. Thus, μ^* is symmetric. Let $a\mu^*b$ and $b\mu^*c$. Then, there exist $\alpha_1 \in a - b$ and $\alpha_2 \in b - c$ such that $\mu(\alpha_1) = \mu(\alpha_2) = \mu(0)$. Hence $\alpha_1 \in (a - c) - \alpha_2$ and there exists $\beta \in a - c$ such that $\alpha_1 \in \beta - c$. Since $\beta \in \alpha_1 + c$ and $min\{\mu(\alpha_1), \mu(c\} \leq \mu(\beta)$. This implies that $\mu(\beta) = \mu(0)$. Hence $a\beta^*c$. This completes the proof. □

Definition 3.15. Let (F, A) be a soft Γ- hyperring, $x \in A$ and μ be a fuzzy set of $F(x)$. Then, $\Theta_F(B, x) = \{a \in F(x) : inf\mu(b)_{b\in B} \leq inf\mu(z)_{z\in a-a}\}$.
When $B = \{0\}$, we denote $\Theta_F(\{0\}, x)$ by $\Theta_F(0, x)$.

Theorem 3.16. *Let (F, A) be a soft Γ-hyperring, $x \in A$ and B be a nonempty subset of $F(x)$. Then, $\Theta_F(B, x)$ is a Γ-subhyperring of $F(x)$.*

Proof. Suppose that $a = 0$. Hence $inf\mu(z)_{z\in a-a} = \mu(0)$. Since for every $b \in B$, $\mu(b) \leq \mu(0)$, we have $inf\mu(b)_{b\in B} \leq \mu(0)$. Thus, $0 \in \Theta_F(B,x)$ and $\Theta_F(B,x)$ is a nonempty set. Let $a, b \in \Theta_F(B,x)$. Then, $inf\mu(b)_{b\in B} \leq \mu(z)_{z\in a-a}$ and $inf\mu(b)_{b\in B} \leq \mu(z)_{z\in b-b}$. Then, for every $d \in a+b$ and $t \in d-d$ there exist $d_1 \in a-a$ and $d_2 \in b-b$ such that $t \in d_1 + d_2$. Also, $min\{\mu(d_1), \mu(d_2)\} \leq \mu(t)$. Thus, $inf\mu(b)_{b\in B} \leq \mu(t)$. Since t is an arbitrary element, $inf\mu(b)_{b\in B} \leq inf\mu(t)_{t\in d-d}$. This implies that $a + b \subseteq \Theta_F(B,x)$ and $\Theta_F(B,x)$ is closed with respect to the additive hyperoperation. Let $a, b \in \Theta_F(B,x)$ and $z \in a\alpha b - a\alpha b$. Then, there exists $z_1 \in b-b$ such that $z \in a\alpha z_1$. Since $z_1 \in \Theta_F(B,x)$, $inf\mu(b)_{b\in B} \leq \mu(z_1)$. Also, $max\{\mu(a), \mu(z_1)\} \leq \mu(z)$. Hence $inf\mu(b)_{b\in B} \leq \inf \mu(z)_{z\in a\alpha b - a\alpha b}$ and $\Theta_F(B,x)$ is closed with respect to the multiplication. Therefore, $\Theta_F(B,x)$ is a Γ-subhyperring of $F(x)$. □

Theorem 3.17. *Let (F,A) be a soft Γ-hyperring over Γ-hyperring R and μ be a fuzzy Γ-hyperideal of $F(x)$. Then, for every $a, b \in \Theta_F(0,x)$, $\mu(a) \oplus \mu(b)$ has exactly one element.*

Proof. Suppose that $a, b \in \Theta_F(0,x)$ and $w_1, w_2 \in a+b$. Hence $\mu(0) = inf\mu(z)_{z\in a-a}$ and $\mu(0) = inf\mu(z)_{z\in b-b}$. Let $z \in w_1 - w_2 \subseteq (a-a)+(b-b)$. Then, there exist $z_1 \in a-a$ and $z_2 \in b-b$ such that $z \in z_1 + z_2$. This implies that $min\{\mu(z_1), \mu(z_2)\} \leq \mu(z)$ and $\mu(0) \leq \mu(z)$. Also, $z \in a-b$ implies that $w_1 \in w_2 + z$ and $min\{\mu(z), \mu(w_2)\} \leq \mu(w_1)$. Since $0 \in b-b$ and $0 \in z-z$, we have $min\{\mu(b), \mu(-b)\} \leq \mu(0)$ and $min\{\mu(z), \mu(-z)\} \leq \mu(0)$. This implies that $\mu(z) \leq \mu(0)$ and $\mu(b) \leq \mu(0)$. Therefore, $\mu(w_2) \leq \mu(w_1)$. In a same way, $\mu(w_1) \leq \mu(w_2)$. Hence $\mu(w_1) = \mu(w_2)$ and the additive hyperoperation defined on F/μ is an operation. □

Theorem 3.18. *Let (F,A) be a soft Γ-hyperring over R and μ be a fuzzy Γ- hyperideal of $F(x)$, where $x \in A$. Then, following conditions are equivalent:*

(i) $\mu(b) \leq inf\mu(c)_{c\in -a+b+a}$,
(ii) $\Theta_F(0,x) = F(x)$.

Proof. Suppose that μ is a fuzzy Γ-hyperideal. Hence for every $b \in F(x)$, we have $\mu(b) \leq inf\mu(c)_{c\in a+b-a}$. Then, for $b = 0$, we have $\mu(0) \leq \inf \mu(c)_{c\in a-a}$. Since for every $c \in a-a$, $\mu(c) \leq \mu(0)$. This implies that $\mu(0) = \inf \mu(c)_{c\in a-a}$ and $\Theta_F(0,x) = F(x)$. Conversely, assume that $\Theta_F(0,x) = F(x)$. Hence for every $a \in F(x)$, $\mu(0) = \inf \mu(b)_{b\in a-a}$. this implies that

$$\inf \mu(c)_{c\in a+b-a} = \inf \mu(c)_{c\in c_1+b, c_1\in a-a} \geq min\{\mu(c_1), \mu(b)\} \geq \mu(b).$$

This completes the proof. □

Corollary 3.19. *Let (F, A) be a soft Γ-hyperring over Γ- hyperring R and μ be a fuzzy set of R. Then, by Theorem 3.18, $\Theta_F(0, x) = F(x)$ and by Theorem 3.17 the hyperaddition defined on F/μ is an operation and the relation μ^* defined in* [19] *is strongly regular.*

4. Conclusions

In this paper, we proved the relation μ^* defined by Zhan et al. [19] is strongly regular. Hence the hyperaddition defined on quotient soft Γ-hyperrings is just operation. Also, we generalize the relation μ^* defined by Zhan et al. by the equivalence relation μ_I^* where I is a Γ-hyperideal of a Γ-hyperring R.

In the future study of fuzzy structure of Γ-hyperrings, the following topics could be considered:

(i) To consider the relation μ_I^* on n-ary Γ-hyperrings,

(ii) To describe the quotient fuzzy soft Γ-hyperrings and their applications.

(iii) To define and consider a covariant functor between the category fuzzy soft Γ-hyperrings and category fuzzy soft rings.

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S. Ostadhadi-Dehkordi

Department of mathematics,

University of Hormozgan,

Hormozgan, Bandar abbas, Iran.

ostadhadi@hormozgan.ac.ir

B. Sheikh Hoseini

Department of mathematics,

Shahid Beheshti University,

Tehran, Iran.

b_sheikhhoseini@sbu.ac.ir