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LOCAL COHOMOLOGY MODULES AND COUSIN COMPLEXES

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Abstract. Let R be a commutative Noetherian ring with non-zero identity, $\mathfrak{a}$ an ideal of R, X an arbitrary R–module, $\mathcal{F}$ a filtration of $\mathrm{Spec}(R)$ which admits X, and s, s', t, t' non-negative integers such that $s + t = s' + t'$. In this paper, we study the membership of R–modules $\mathrm{H}^s_{\mathfrak{a}}(\mathrm{H}^{t-1}(\mathrm{C}_R(\mathcal{F}, X)))$ and $\mathrm{H}^{s'-1}(\mathrm{H}^{t'}_{\mathfrak{a}}(\mathrm{C}_R(\mathcal{F}, X)))$ in Serre subcategories of the category of R–modules and find some sufficient conditions which ensure the existence of an isomorphism between them, where $\mathrm{C}_R(\mathcal{F}, X)$ is the Cousin complex for X with respect to $\mathcal{F}$. As applications, we give some new facts and represent some older facts about the local cohomology modules and the Cousin complexes.

1. Introduction

Throughout this paper, R is a commutative Noetherian ring with non-zero identity, $\mathfrak{a}$ an ideal of R, X an arbitrary R–module, and s, t non-negative integers. We write $\mathrm{H}^s_{\mathfrak{a}}(X)$ as the sth local cohomology module of X with respect to $\mathfrak{a}$. For basic results, notations, and terminology not given in this paper, readers are referred to [4, 5, 10].

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The notion of Cousin complex was introduced in [6] and it has a commutative algebra analogue given by Sharp in [11]. In [14], Sharp generalized this concept to the Cousin complex for X with respect to a filtration $\mathcal{F} = (F_i)_{i\geq 0}$ of $\mathrm{Spec}(R)$ and denoted this complex by $\mathrm{C}_R(\mathcal{F}, X)$. He approved it as a powerful tool, for example, by characterizing Gorenstein rings, Cohen-Macaulay modules, and balanced big Cohen-Macaulay modules in terms of Cousin complexes (see [11, Theorem 5.4], [12, Theorem 2.4], and [14, Corollary 3.7]). In this paper, we study the relations between the local cohomology modules and the Cousin complexes.

In Section 2, we first introduce the Cousin spectral sequences with respect to a filtration of $\mathrm{Spec}(R)$ and an ideal of R in Theorem 2.1. Then we use this result to present some conditions for validity of an isomorphism between R–modules $\mathrm{H}^s_{\mathfrak{a}}(\mathrm{H}^{t-1}(\mathrm{C}_R(\mathcal{F}, X)))$ and $\mathrm{H}^{s'-1}(\mathrm{H}^{t'}_{\mathfrak{a}}(\mathrm{C}_R(\mathcal{F}, X)))$ in Theorem 2.2, where s', t' are non-negative integers such that $s + t = s' + t'$. Finally, in Theorems 2.3 and 2.4, we find that when $\mathrm{H}^{s-1}(\mathrm{H}^t_{\mathfrak{a}}(\mathrm{C}_R(\mathcal{F}, X)))$ and $\mathrm{H}^s_{\mathfrak{a}}(\mathrm{H}^{t-1}(\mathrm{C}_R(\mathcal{F}, X)))$ belong to a Serre subcategory of the category of R–modules (i.e. the class of R–modules which is closed under taking submodules, quotients, and extensions).

Section 3 consists of applications. Let n be a positive integer. Apply the functor $\Gamma_{\mathfrak{a}}(-)$ to the Cousin complex $\mathrm{C}_R(\mathcal{F}, X)$ and take the nth cohomology module of this complex to get $\mathrm{H}^n(\Gamma_{\mathfrak{a}}(\mathrm{C}_R(\mathcal{F}, X)))$. In Corollary 3.2, under some conditions, we observe that $\mathrm{H}^n_{\mathfrak{a}}(X)$ is in a Serre subcategory of the category of R–modules if and only if $\mathrm{H}^n(\Gamma_{\mathfrak{a}}(\mathrm{C}_R(\mathcal{F}, X)))$ is in it. We find equivalent conditions for the vanishing of Cousin cohomology modules in Corollary 3.5. In Corollary 3.9, we show that if $\mathrm{H}^i(\mathrm{C}_R(\mathcal{F}, X))$ is a finitely generated R–module for all $i \leq n-2$, then $\mathrm{H}^i_{\mathfrak{p}R_{\mathfrak{p}}}(X_{\mathfrak{p}})$ is an Artinian R–module for all $i \leq n-1$ and all $\mathfrak{p} \in F_n$, which is related to the third of Huneke's four problems in local cohomology modules [8]. We study support and annihilator of Cousin cohomology modules in Corollary 3.10. We also use these corollaries to deduce some results about the Cousin complexes with respect to the height and dimension filtrations.

2. Main results

A filtration of $\mathrm{Spec}(R)$ is a descending sequence $\mathcal{F} = (F_i)_{i\geq 0}$ of subsets of $\mathrm{Spec}(R)$ with the property that, for all $i \geq 0$, each member of $F_i \setminus F_{i+1}$ is a minimal member of F_i with respect to inclusion. We say that $\mathcal{F}$ admits an R–module X if $\mathrm{Supp}_R(X) \subseteq F_0$.

Suppose that $\mathcal{F} = (F_i)_{i\geq 0}$ is a filtration of $\mathrm{Spec}(R)$ which admits an R–module X. The Cousin complex $\mathrm{C}_R(\mathcal{F}, X)$ for X with respect to $\mathcal{F}$ is of the form

$$\mathrm{C}_R(\mathcal{F}, X) = 0 \xrightarrow{d_X^{-2}} X \xrightarrow{d_X^{-1}} X^0 \xrightarrow{d_X^0} \cdots \xrightarrow{d_X^{i-3}} X^{i-2} \xrightarrow{d_X^{i-2}} X^{i-1} \xrightarrow{d_X^{i-1}} X^i \xrightarrow{d_X^i} \cdots$$

where, for all $i \geq 0$, $X^i = \bigoplus_{\mathfrak{p}\in F_i\setminus F_{i+1}} (\mathrm{Coker}(d_X^{i-2}))_{\mathfrak{p}}$ and $d_X^{i-1}(x) = \{(x+\mathrm{Im}(d_X^{i-2}))/1\}_{\mathfrak{p}\in F_i\setminus F_{i+1}}$ for every element x of X^{i-1}; and satisfies

$$\mathrm{Supp}_R(X^i) \cup \mathrm{Supp}_R(\mathrm{Coker}(d_X^{i-2})) \cup \mathrm{Supp}_R(\mathrm{H}^{i-2}(\mathrm{C}_R(\mathcal{F}, X))) \subseteq \mathrm{Supp}_R(X) \cap F_i$$

(see [14, Definitions 1.1, Definition 1.3, Proposition 1.4, and Corollary 1.5]). We adopt the convention that $X^{-1} = X$ and $\mathrm{C}_R(\mathcal{F}, X)^i = X^i$ for all i.

Let $\mathcal{C}_{\mathcal{F}}(R)$ be the category of R–modules which are admitted by $\mathcal{F}$ and let $\mathrm{Comp}(R)$ be the category of complexes of R–modules. The Cousin functor $\mathrm{C}_R(\mathcal{F}, -)$ with respect to $\mathcal{F}$ is an R–linear and covariant functor from $\mathcal{C}_{\mathcal{F}}(R)$ to $\mathrm{Comp}(R)$. Thus if $(A^\bullet, b^\bullet)$ is a complex in $\mathcal{C}_{\mathcal{F}}(R)$, then $(\mathrm{C}_R(\mathcal{F}, A^\bullet), \mathrm{C}_R(\mathcal{F}, b^\bullet))$ is a complex in $\mathrm{Comp}(R)$ (see [3, Theorem 2.2 and Corollary 2.3]).

In the first theorem, we introduce the Cousin spectral sequences with respect to a filtration $\mathcal{F}$ and an ideal $\mathfrak{a} = \langle a_1, \ldots, a_t\rangle$ of R that are crucial in this paper. In its proof,

$$\mathrm{C}_R(X)^\bullet = 0 \longrightarrow \mathrm{C}_R(X)^0 \longrightarrow \cdots \longrightarrow \mathrm{C}_R(X)^i \longrightarrow \cdots \longrightarrow \mathrm{C}_R(X)^t \longrightarrow 0$$

denotes the Čech complex of X with respect to $a_1, \ldots, a_t$. Note that, for all $1 \leq i \leq t$, $\mathrm{C}_R(X)^i = \bigoplus_{j\in\varphi(i,t)} X_{a_{j(1)}\cdots a_{j(i)}}$ where $\varphi(i,t) = \{(j(1), \ldots, j(i)) \in \mathbb{N}^i : 1 \leq j(1) < \cdots < j(i) \leq t\}$. It is well known that the ith cohomology module of $\mathrm{C}_R(X)^\bullet$ is isomorphic to the ith local cohomology module $\mathrm{H}^i_{\mathfrak{a}}(X)$ (see [4, Theorem 5.1.19]).

Theorem 2.1. *Let $\mathcal{F} = (F_i)_{i\geq 0}$ be a filtration of* $\mathrm{Spec}(R)$ *which admits an R–module X and let $\mathfrak{a}$ be an ideal of R. Then we have third quadrant spectral sequences*

(i) ${}^{I}\mathrm{E}_2^{p,q} := \mathrm{H}^p_{\mathfrak{a}}(\mathrm{H}^{q-1}(\mathrm{C}_R(\mathcal{F}, X))) \underset{p}{\Longrightarrow} \mathrm{H}^{p+q}$ *and*
(ii) ${}^{II}\mathrm{E}_2^{p,q} := \mathrm{H}^{p-1}(\mathrm{H}^q_{\mathfrak{a}}(\mathrm{C}_R(\mathcal{F}, X))) \underset{p}{\Longrightarrow} \mathrm{H}^{p+q}$.

Proof. Let $\mathfrak{a} = \langle a_1, \ldots, a_t\rangle$ and let $\mathrm{C}_R(X)^\bullet$ be the Čech complex of X with respect to $a_1, \ldots, a_t$. Since $\mathrm{Supp}_R(\mathrm{C}_R(X)^p) \subseteq \mathrm{Supp}_R(X)$ for all $0 \leq p \leq t$, $\mathrm{C}_R(X)^\bullet$ is a complex in $\mathrm{C}_{\mathcal{F}}(R)$, and so $\mathrm{C}_R(\mathcal{F}, \mathrm{C}_R(X)^\bullet)$ is a complex in $\mathrm{Comp}(R)$. Therefore $\mathcal{T} = \{\mathrm{C}_R(\mathcal{F}, \mathrm{C}_R(X)^p)^{q-1}\}$ is a third quadrant bicomplex. We denote the total complex of $\mathcal{T}$ by $\mathrm{Tot}(\mathcal{T})$.

(i) The first filtration has ${}^{I}\mathrm{E}_2$ term the iterated homology $\mathrm{H}'^p\,\mathrm{H}''^{p,q}(\mathcal{T})$. We have

$$\begin{aligned}
\mathrm{H}''^{p,q}(\mathcal{T}) &\cong \mathrm{H}^{q-1}(\mathrm{C}_R(\mathcal{F}, \mathrm{C}_R(X)^p)) \\
&\cong \mathrm{H}^{q-1}(\mathrm{C}_R(\mathcal{F}, \textstyle\bigoplus_{j\in\varphi(p,t)} X_{a_{j(1)}\cdots a_{j(p)}})) \\
&\cong \mathrm{H}^{q-1}(\textstyle\bigoplus_{j\in\varphi(p,t)} \mathrm{C}_R(\mathcal{F}, X_{a_{j(1)}\cdots a_{j(p)}})) \\
&\cong \textstyle\bigoplus_{j\in\varphi(p,t)} \mathrm{H}^{q-1}(\mathrm{C}_R(\mathcal{F}, X_{a_{j(1)}\cdots a_{j(p)}})) \\
&\cong \textstyle\bigoplus_{j\in\varphi(p,t)} \mathrm{H}^{q-1}(\mathrm{C}_R(\mathcal{F}, X)_{a_{j(1)}\cdots a_{j(p)}}) \\
&\cong \textstyle\bigoplus_{j\in\varphi(p,t)} (\mathrm{H}^{q-1}(\mathrm{C}_R(\mathcal{F}, X)))_{a_{j(1)}\cdots a_{j(p)}} \\
&\cong \mathrm{C}_R(\mathrm{H}^{q-1}(\mathrm{C}_R(\mathcal{F}, X)))^p
\end{aligned}$$

in which the fifth isomorphism is given by [9, Lemma 2.9]. Hence

$$\begin{aligned} {}^{I}\,\mathrm{E}_2^{p,q} &\cong \mathrm{H}'^{p}\,\mathrm{H}''^{p,q}(\mathcal{T}) \\ &\cong \mathrm{H}^{p}(\mathrm{C}_R(\mathrm{H}^{q-1}(\mathrm{C}_R(\mathcal{F},X)))^{\bullet}) \\ &\cong \mathrm{H}^{p}_{\mathfrak{a}}(\mathrm{H}^{q-1}(\mathrm{C}_R(\mathcal{F},X))) \end{aligned}$$

that yields the third quadrant spectral sequence

$${}^{I}\,\mathrm{E}_2^{p,q} := \mathrm{H}^{p}_{\mathfrak{a}}(\mathrm{H}^{q-1}(\mathrm{C}_R(\mathcal{F},X))) \underset{p}{\Longrightarrow} \mathrm{H}^{p+q}(\mathrm{Tot}(\mathcal{T})).$$

(ii) The second filtration has ${}^{II}\,\mathrm{E}_2$ term the iterated homology $\mathrm{H}''^{p}\,\mathrm{H}'^{q,p}(\mathcal{T})$. We have

$$\begin{aligned} \mathrm{H}'^{q,p}(\mathcal{T}) &\cong \mathrm{H}^{q}(\mathrm{C}_R(\mathcal{F},\mathrm{C}_R(X)^{\bullet})^{p-1}) \\ &\cong \mathrm{H}^{q}(\mathrm{C}_R(\mathrm{C}_R(\mathcal{F},X)^{p-1})^{\bullet}) \\ &\cong \mathrm{H}^{q}_{\mathfrak{a}}(\mathrm{C}_R(\mathcal{F},X)^{p-1}). \end{aligned}$$

Thus

$$\begin{aligned} {}^{II}\,\mathrm{E}_2^{p,q} &\cong \mathrm{H}''^{p}\,\mathrm{H}'^{q,p}(\mathcal{T}) \\ &\cong \mathrm{H}^{p}(\mathrm{H}^{q}_{\mathfrak{a}}(\mathrm{C}_R(\mathcal{F},X)^{\bullet-1})) \\ &\cong \mathrm{H}^{p}(\mathrm{H}^{q}_{\mathfrak{a}}(\{-1\}\,\mathrm{C}_R(\mathcal{F},X))) \\ &\cong \mathrm{H}^{p-1}(\mathrm{H}^{q}_{\mathfrak{a}}(\mathrm{C}_R(\mathcal{F},X))) \end{aligned}$$

that gives the third quadrant spectral sequence

$${}^{II}\,\mathrm{E}_2^{p,q} := \mathrm{H}^{p-1}(\mathrm{H}^{q}_{\mathfrak{a}}(\mathrm{C}_R(\mathcal{F},X))) \underset{p}{\Longrightarrow} \mathrm{H}^{p+q}(\mathrm{Tot}(\mathcal{T})).$$

The proof is completed. □

In the next theorem, we find some conditions for validity of an isomorphism between $\mathrm{H}^{s}_{\mathfrak{a}}(\mathrm{H}^{t-1}(\mathrm{C}_R(\mathcal{F},X)))$ and $\mathrm{H}^{s'-1}(\mathrm{H}^{t'}_{\mathfrak{a}}(\mathrm{C}_R(\mathcal{F},X)))$ where $s+t=s'+t'$.

Theorem 2.2. *Suppose that $\mathcal{F}=(F_i)_{i\geq 0}$ is a filtration of $\mathrm{Spec}(R)$ which admits an R–module X, $\mathfrak{a}$ an ideal of R, and s,s',t,t' non-negative integers such that $(n=)s+t=s'+t'$. Assume also that*

(i) $\mathrm{H}^{n-i}_{\mathfrak{a}}(\mathrm{H}^{i}(\mathrm{C}_R(\mathcal{F},X)))=0$ *for all i, $-1\leq i\leq t-2$,*
(ii) $\mathrm{H}^{n-2-i}_{\mathfrak{a}}(\mathrm{H}^{i}(\mathrm{C}_R(\mathcal{F},X)))=0$ *for all i, $t\leq i\leq n-2$,*
(iii) $\mathrm{H}^{n-1-i}_{\mathfrak{a}}(\mathrm{H}^{i}(\mathrm{C}_R(\mathcal{F},X)))=0$ *for all i, $-1\leq i\leq t-2$ or $t\leq i\leq n-1$,*
(iv) $\mathrm{H}^{n-i}(\mathrm{H}^{i}_{\mathfrak{a}}(\mathrm{C}_R(\mathcal{F},X)))=0$ *for all i, $0\leq i\leq t'-1$,*
(v) $\mathrm{H}^{n-2-i}(\mathrm{H}^{i}_{\mathfrak{a}}(\mathrm{C}_R(\mathcal{F},X)))=0$ *for all i, $t'+1\leq i\leq n-1$, and*
(vi) $\mathrm{H}^{n-1-i}(\mathrm{H}^{i}_{\mathfrak{a}}(\mathrm{C}_R(\mathcal{F},X)))=0$ *for all i, $0\leq i\leq t'-1$ or $t'+1\leq i\leq n$.*

Then

$$\mathrm{H}^{s}_{\mathfrak{a}}(\mathrm{H}^{t-1}(\mathrm{C}_R(\mathcal{F},X)))\cong \mathrm{H}^{s'-1}(\mathrm{H}^{t'}_{\mathfrak{a}}(\mathrm{C}_R(\mathcal{F},X))).$$

Proof. By Theorem 2.1(i), there is the third quadrant spectral sequence

$$^{I}\,\mathrm{E}_2^{p,q} := \mathrm{H}^p_{\mathfrak{a}}(\mathrm{H}^{q-1}(\mathrm{C}_R(\mathcal{F},X))) \underset{p}{\Longrightarrow} \mathrm{H}^{p+q}\,.$$

Let $r \geq 2$ and set $^{I}Z_r^{s,t} = \mathrm{Ker}(^{I}\,\mathrm{E}_r^{s,t} \longrightarrow {}^{I}\,\mathrm{E}_r^{s+r,t+1-r})$ and $^{I}B_r^{s,t} = \mathrm{Im}(^{I}\,\mathrm{E}_r^{s-r,t+r-1} \longrightarrow {}^{I}\,\mathrm{E}_r^{s,t})$. By assumptions (i) and (ii), we have $^{I}\,\mathrm{E}_2^{s+r,t+1-r} = 0 = {}^{I}\,\mathrm{E}_2^{s-r,t+r-1}$. Therefore $^{I}\,\mathrm{E}_r^{s+r,t+1-r} = 0 = {}^{I}\,\mathrm{E}_r^{s-r,t+r-1}$. Hence $^{I}Z_r^{s,t} = {}^{I}\,\mathrm{E}_r^{s,t}$ and $^{I}B_r^{s,t} = 0$. Thus $^{I}\,\mathrm{E}_{r+1}^{s,t} = {}^{I}\,\mathrm{E}_r^{s,t}$ and so

$$^{I}\,\mathrm{E}_\infty^{s,t} = {}^{I}\,\mathrm{E}_{n+2}^{s,t} = {}^{I}\,\mathrm{E}_{n+1}^{s,t} = \cdots = {}^{I}\,\mathrm{E}_3^{s,t} = {}^{I}\,\mathrm{E}_2^{s,t} = \mathrm{H}^s_{\mathfrak{a}}(\mathrm{H}^{t-1}(\mathrm{C}_R(\mathcal{F},X))).$$

There exists a finite filtration

$$0 = \varphi^{n+1}\,\mathrm{H}^n \subseteq \varphi^n\,\mathrm{H}^n \subseteq \cdots \subseteq \varphi^1\,\mathrm{H}^n \subseteq \varphi^0\,\mathrm{H}^n = \mathrm{H}^n$$

such that $^{I}\,\mathrm{E}_\infty^{n-r,r} \cong \varphi^{n-r}\,\mathrm{H}^n\,/\varphi^{n-r+1}\,\mathrm{H}^n$ for all $r \leq n$. Note that for each $r \neq t$, $^{I}\,\mathrm{E}_\infty^{n-r,r} = 0$ by assumption (iii). Therefore we get

$$\varphi^s\,\mathrm{H}^n = \varphi^{s-1}\,\mathrm{H}^n = \cdots = \varphi^1\,\mathrm{H}^n = \varphi^0\,\mathrm{H}^n = \mathrm{H}^n$$

and

$$0 = \varphi^{n+1}\,\mathrm{H}^n = \varphi^n\,\mathrm{H}^n = \cdots = \varphi^{s+2}\,\mathrm{H}^n = \varphi^{s+1}\,\mathrm{H}^n\,.$$

Hence $^{I}\,\mathrm{E}_\infty^{s,t} \cong \varphi^s\,\mathrm{H}^n\,/\varphi^{s+1}\,\mathrm{H}^n = \mathrm{H}^n$ and so $\mathrm{H}^s_{\mathfrak{a}}(\mathrm{H}^{t-1}(\mathrm{C}_R(\mathcal{F},X))) \cong \mathrm{H}^n$.

On the other hand, from Theorem 2.1(ii), there is the third quadrant spectral sequence

$$^{II}\,\mathrm{E}_2^{p,q} := \mathrm{H}^{p-1}(\mathrm{H}^q_{\mathfrak{a}}(\mathrm{C}_R(\mathcal{F},X))) \underset{p}{\Longrightarrow} \mathrm{H}^{p+q}\,.$$

By the same proof as above, we get $\mathrm{H}^{s'-1}(\mathrm{H}^{t'}_{\mathfrak{a}}(\mathrm{C}_R(\mathcal{F},X))) \cong \mathrm{H}^n$. Thus $\mathrm{H}^s_{\mathfrak{a}}(\mathrm{H}^{t-1}(\mathrm{C}_R(\mathcal{F},X))) \cong \mathrm{H}^{s'-1}(\mathrm{H}^{t'}_{\mathfrak{a}}(\mathrm{C}_R(\mathcal{F},X)))$ as desired. □

In the next theorems, we study the membership of R–modules $\mathrm{H}^{s-1}(\mathrm{H}^t_{\mathfrak{a}}(\mathrm{C}_R(\mathcal{F},X)))$ and $\mathrm{H}^s_{\mathfrak{a}}(\mathrm{H}^{t-1}(\mathrm{C}_R(\mathcal{F},X)))$ in Serre subcategories of the category of R–modules. Recall that a Serre subcategory $\mathcal{S}$ of the category of R–modules is a subclass of R–modules such that for any short exact sequence

$$0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0, \tag{1}$$

the module X is in $\mathcal{S}$ if and only if X' and X'' are in $\mathcal{S}$. Let $\lambda : \mathcal{S} \longrightarrow \mathcal{T}$ be a function from a Serre subcategory of the category of R–modules $\mathcal{S}$ to a partially ordered Abelian monoid $(\mathcal{T}, \circ, \preceq)$. We say that λ is a subadditive function if $\lambda(0) = 0$ and for any short exact sequence (1) in which all the terms belong to $\mathcal{S}$, $\lambda(X') \preceq \lambda(X), \lambda(X'') \preceq \lambda(X)$, and $\lambda(X) \preceq \lambda(X') \circ \lambda(X'')$ (see [15, Definition 2.3]). In this paper, $\mathcal{S}$ is a Serre subcategory of the category of R–modules, $(\mathcal{T}, \circ, \preceq)$ is a partially ordered Abelian monoid, and $\lambda : \mathcal{S} \longrightarrow \mathcal{T}$ is a subadditive function.

Theorem 2.3. *Suppose that $\mathcal{F} = (F_i)_{i\geq 0}$ is a filtration of* $\operatorname{Spec}(R)$ *which admits an R–module X, $\mathfrak{a}$ an ideal of R, and s, t non-negative integers. Assume also that*

(i) $\mathrm{H}_{\mathfrak{a}}^{s+t-1-i}(\mathrm{H}^i(\mathrm{C}_R(\mathcal{F}, X))) \in \mathcal{S}$ *for all i, $-1 \leq i \leq s+t-1$,*

(ii) $\mathrm{H}^{s+t-i}(\mathrm{H}_{\mathfrak{a}}^i(\mathrm{C}_R(\mathcal{F}, X))) \in \mathcal{S}$ *for all i, $0 \leq i \leq t-1$, and*

(iii) $\mathrm{H}^{s+t-2-i}(\mathrm{H}_{\mathfrak{a}}^i(\mathrm{C}_R(\mathcal{F}, X))) \in \mathcal{S}$ *for all i, $t+1 \leq i \leq s+t-1$.*

Then $\mathrm{H}^{s-1}(\mathrm{H}_{\mathfrak{a}}^t(\mathrm{C}_R(\mathcal{F}, X))) \in \mathcal{S}$ *and*

$$\lambda(\mathrm{H}^{s-1}(\mathrm{H}_{\mathfrak{a}}^t(\mathrm{C}_R(\mathcal{F}, X)))) \preceq (\mathop{\circ}_{i=-1}^{s+t-1} \lambda(\mathrm{H}_{\mathfrak{a}}^{s+t-1-i}(\mathrm{H}^i(\mathrm{C}_R(\mathcal{F}, X))))) \circ$$
$$(\mathop{\circ}_{i=0}^{t-1} \lambda(\mathrm{H}^{s+t-i}(\mathrm{H}_{\mathfrak{a}}^i(\mathrm{C}_R(\mathcal{F}, X))))) \circ (\mathop{\circ}_{i=t+1}^{s+t-1} \lambda(\mathrm{H}^{s+t-2-i}(\mathrm{H}_{\mathfrak{a}}^i(\mathrm{C}_R(\mathcal{F}, X))))).$$

Proof. The proof is similar to that of Theorem 2.2. We bring it here for the sake of completeness. By Theorem 2.1(i), there is the third quadrant spectral sequence

$$^{I}\,\mathrm{E}_2^{p,q} := \mathrm{H}_{\mathfrak{a}}^p(\mathrm{H}^{q-1}(\mathrm{C}_R(\mathcal{F}, X))) \underset{p}{\Longrightarrow} \mathrm{H}^{p+q}.$$

For all i, $0 \leq i \leq s+t$, we have $^{I}\,\mathrm{E}_\infty^{s+t-i,i} = {}^{I}\,\mathrm{E}_{s+t+2}^{s+t-i,i}$. Thus $^{I}\,\mathrm{E}_\infty^{s+t-i,i}$ is in $\mathcal{S}$ and $\lambda(^{I}\,\mathrm{E}_\infty^{s+t-i,i}) \preceq \lambda(^{I}\,\mathrm{E}_2^{s+t-i,i})$ from the fact that $^{I}\,\mathrm{E}_{s+t+2}^{s+t-i,i}$ is a subquotient of $^{I}\,\mathrm{E}_2^{s+t-i,i}$ that is in $\mathcal{S}$ by assumption (i). There exists a finite filtration

$$0 = \varphi^{s+t+1}\,\mathrm{H}^{s+t} \subseteq \varphi^{s+t}\,\mathrm{H}^{s+t} \subseteq \cdots \subseteq \varphi^{1}\,\mathrm{H}^{s+t} \subseteq \varphi^{0}\,\mathrm{H}^{s+t} = \mathrm{H}^{s+t}$$

such that $^{I}\,\mathrm{E}_\infty^{s+t-i,i} = \varphi^{s+t-i}\,\mathrm{H}^{s+t}/\varphi^{s+t-i+1}\,\mathrm{H}^{s+t}$ for all i, $0 \leq i \leq s+t$. Now the exact sequences

$$0 \longrightarrow \varphi^{s+t-i+1}\,\mathrm{H}^{s+t} \longrightarrow \varphi^{s+t-i}\,\mathrm{H}^{s+t} \longrightarrow {}^{I}\,\mathrm{E}_\infty^{s+t-i,i} \longrightarrow 0,$$

for all i, $0 \leq i \leq s+t$, show that H^{s+t} is in $\mathcal{S}$ and

$$\lambda(\mathrm{H}^{s+t}) \preceq \mathop{\circ}_{i=0}^{s+t} \lambda(^{I}\,\mathrm{E}_\infty^{s+t-i,i}) \preceq \mathop{\circ}_{i=0}^{s+t} \lambda(^{I}\,\mathrm{E}_2^{s+t-i,i}).$$

On the other hand, there is the third quadrant spectral sequence

$$^{II}\,\mathrm{E}_2^{p,q} := \mathrm{H}^{p-1}(\mathrm{H}_{\mathfrak{a}}^q(\mathrm{C}_R(\mathcal{F}, X))) \underset{p}{\Longrightarrow} \mathrm{H}^{p+q}$$

from Theorem 2.1(ii). There exists a finite filtration

$$0 = \psi^{s+t+1}\,\mathrm{H}^{s+t} \subseteq \psi^{s+t}\,\mathrm{H}^{s+t} \subseteq \cdots \subseteq \psi^{1}\,\mathrm{H}^{s+t} \subseteq \psi^{0}\,\mathrm{H}^{s+t} = \mathrm{H}^{s+t}$$

such that $^{II}\,\mathrm{E}_\infty^{s+t-i,i} = \psi^{s+t-i}\,\mathrm{H}^{s+t}/\psi^{s+t-i+1}\,\mathrm{H}^{s+t}$ for all i, $0 \leq i \leq s+t$. Since H^{s+t} is in $\mathcal{S}$, $\psi^s\,\mathrm{H}^{s+t}$ is in $\mathcal{S}$. Hence $^{II}\,\mathrm{E}_\infty^{s,t} = \psi^s\,\mathrm{H}^{s+t}/\psi^{s+1}\,\mathrm{H}^{s+t}$ is in $\mathcal{S}$ and $\lambda(^{II}\,\mathrm{E}_\infty^{s,t}) \preceq \lambda(\psi^s\,\mathrm{H}^{s+t}) \preceq \lambda(\mathrm{H}^{s+t})$. Therefore $^{II}\,\mathrm{E}_{s+t+2}^{s,t}$ is in $\mathcal{S}$ and

$$\lambda(^{II}\,\mathrm{E}_{s+t+2}^{s,t}) \preceq \lambda(\mathrm{H}^{s+t}).$$

For all $r \geq 2$, let ${}^{II}Z_r^{s,t} = \mathrm{Ker}({}^{II}\mathrm{E}_r^{s,t} \longrightarrow {}^{II}\mathrm{E}_r^{s+r,t+1-r})$ and ${}^{II}B_r^{s,t} = \mathrm{Im}({}^{II}\mathrm{E}_r^{s-r,t+r-1} \longrightarrow {}^{II}\mathrm{E}_r^{s,t})$. We have the exact sequences

$$0 \longrightarrow {}^{II}Z_r^{s,t} \longrightarrow {}^{II}\mathrm{E}_r^{s,t} \longrightarrow {}^{II}\mathrm{E}_r^{s,t} / {}^{II}Z_r^{s,t} \longrightarrow 0$$

and

$$0 \longrightarrow {}^{II}B_r^{s,t} \longrightarrow {}^{II}Z_r^{s,t} \longrightarrow {}^{II}\mathrm{E}_{r+1}^{s,t} \longrightarrow 0.$$

Since ${}^{II}\mathrm{E}_2^{s+r,t+1-r}$ and ${}^{II}\mathrm{E}_2^{s-r,t+r-1}$ are in $\mathcal{S}$ by assumptions (ii) and (iii), ${}^{II}\mathrm{E}_r^{s+r,t+1-r}$ and ${}^{II}\mathrm{E}_r^{s-r,t+r-1}$ are also in $\mathcal{S}$, and so ${}^{II}\mathrm{E}_r^{s,t} / {}^{II}Z_r^{s,t}$ and ${}^{II}B_r^{s,t}$ are in $\mathcal{S}$. It shows that ${}^{II}\mathrm{E}_r^{s,t}$ is in $\mathcal{S}$ whenever ${}^{II}\mathrm{E}_{r+1}^{s,t}$ is in $\mathcal{S}$ and we get

$$\begin{aligned}
\lambda({}^{II}\mathrm{E}_r^{s,t}) &\preceq \lambda({}^{II}\mathrm{E}_{r+1}^{s,t}) \circ \lambda({}^{II}\mathrm{E}_r^{s,t} / {}^{II}Z_r^{s,t}) \circ \lambda({}^{II}B_r^{s,t}) \\
&\preceq \lambda({}^{II}\mathrm{E}_{r+1}^{s,t}) \circ \lambda({}^{II}\mathrm{E}_r^{s+r,t+1-r}) \circ \lambda({}^{II}\mathrm{E}_r^{s-r,t+r-1}) \\
&\preceq \lambda({}^{II}\mathrm{E}_{r+1}^{s,t}) \circ \lambda({}^{II}\mathrm{E}_2^{s+r,t+1-r}) \circ \lambda({}^{II}\mathrm{E}_2^{s-r,t+r-1}).
\end{aligned}$$

Therefore ${}^{II}\mathrm{E}_2^{s,t}$ is in $\mathcal{S}$ and we have

$$\begin{aligned}
\lambda({}^{II}\mathrm{E}_2^{s,t}) &\preceq \lambda({}^{II}\mathrm{E}_3^{s,t}) \circ \lambda({}^{II}\mathrm{E}_2^{s+2,t-1}) \circ \lambda({}^{II}\mathrm{E}_2^{s-2,t+1}) \\
&\preceq \lambda({}^{II}\mathrm{E}_4^{s,t}) \circ \lambda({}^{II}\mathrm{E}_2^{s+3,t-2}) \circ \lambda({}^{II}\mathrm{E}_2^{s+2,t-1}) \circ \lambda({}^{II}\mathrm{E}_2^{s-2,t+1}) \circ \lambda({}^{II}\mathrm{E}_2^{s-3,t+2}) \\
&\preceq \cdots \\
&\preceq \lambda({}^{II}\mathrm{E}_{s+t+2}^{s,t}) \circ (\mathop{\circ}_{i=0}^{t-1} \lambda({}^{II}\mathrm{E}_2^{s+t+1-i,i})) \circ (\mathop{\circ}_{i=t+1}^{s+t-1} \lambda({}^{II}\mathrm{E}_2^{s+t-1-i,i}))
\end{aligned}$$

which completes the proof. □

Theorem 2.4. *Suppose that $\mathcal{F} = (F_i)_{i \geq 0}$ is a filtration of $\mathrm{Spec}(R)$ which admits an R–module X, $\mathfrak{a}$ an ideal of R, and s, t non-negative integers. Assume also that*

(i) $\mathrm{H}^{s+t-1-i}(\mathrm{H}_{\mathfrak{a}}^i(\mathrm{C}_R(\mathcal{F}, X))) \in \mathcal{S}$ *for all i, $0 \leq i \leq s+t$,*

(ii) $\mathrm{H}_{\mathfrak{a}}^{s+t-i}(\mathrm{H}^i(\mathrm{C}_R(\mathcal{F}, X))) \in \mathcal{S}$ *for all i, $-1 \leq i \leq t-2$, and*

(iii) $\mathrm{H}_{\mathfrak{a}}^{s+t-2-i}(\mathrm{H}^i(\mathrm{C}_R(\mathcal{F}, X))) \in \mathcal{S}$ *for all i, $t \leq i \leq s+t-2$.*

Then $\mathrm{H}_{\mathfrak{a}}^s(\mathrm{H}^{t-1}(\mathrm{C}_R(\mathcal{F}, X))) \in \mathcal{S}$ *and*

$$\lambda(\mathrm{H}_{\mathfrak{a}}^s(\mathrm{H}^{t-1}(\mathrm{C}_R(\mathcal{F}, X)))) \preceq (\mathop{\circ}_{i=0}^{s+t} \lambda(\mathrm{H}^{s+t-1-i}(\mathrm{H}_{\mathfrak{a}}^i(\mathrm{C}_R(\mathcal{F}, X))))) \circ$$
$$(\mathop{\circ}_{i=-1}^{t-2} \lambda(\mathrm{H}_{\mathfrak{a}}^{s+t-i}(\mathrm{H}^i(\mathrm{C}_R(\mathcal{F}, X))))) \circ (\mathop{\circ}_{i=t}^{s+t-2} \lambda(\mathrm{H}_{\mathfrak{a}}^{s+t-2-i}(\mathrm{H}^i(\mathrm{C}_R(\mathcal{F}, X))))).$$

Proof. This is sufficiently similar to that of Theorem 2.3 to be omitted. We leave the proof to the reader. □

3. Applications

The following lemma is needed in this section. Here, we denote $\mathrm{V}(\mathfrak{a}) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{p} \supseteq \mathfrak{a}\}$ for an ideal $\mathfrak{a}$ of R.

Lemma 3.1. *Suppose that $\mathcal{F} = (F_i)_{i\geq 0}$ is a filtration of $\mathrm{Spec}(R)$ which admits an R–module X, $\mathfrak{a}$ an ideal of R, V a subset of $\mathrm{Spec}(R)$ such that $\mathrm{Supp}_R(X) \subseteq \mathrm{V}$, and n a positive integer such that $\mathrm{V}(\mathfrak{a}) \cap V \subseteq F_n$. Then $\mathrm{H}^j_{\mathfrak{a}}(\mathrm{C}_R(\mathcal{F}, X)^i) = 0$ for all $0 \leq i \leq n-1$ and all $j \geq 0$. In particular, $\mathrm{H}^{-1}(\mathrm{H}^j_{\mathfrak{a}}(\mathrm{C}_R(\mathcal{F}, X))) = \mathrm{H}^j_{\mathfrak{a}}(X)$ and $\mathrm{H}^i(\mathrm{H}^j_{\mathfrak{a}}(\mathrm{C}_R(\mathcal{F}, X))) = 0$ for all $0 \leq i \leq n-1$ and all $j \geq 0$.*

Proof. Let i and j be integers such that $0 \leq i \leq n-1$ and $j \geq 0$. It is enough to show that $\mathrm{H}^j_{\mathfrak{a}}((\mathrm{Coker}(d_X^{i-2}))_{\mathfrak{p}}) = 0$ for all $\mathfrak{p} \in F_i \setminus F_{i+1}$ because $\mathrm{C}_R(\mathcal{F}, X)^i = \bigoplus_{\mathfrak{p}\in F_i\setminus F_{i+1}} (\mathrm{Coker}(d_X^{i-2}))_{\mathfrak{p}}$. Let $\mathfrak{p}$ be a prime ideal in $F_i \setminus F_{i+1}$. Suppose that $\mathfrak{p} \in \mathrm{Supp}_R(\mathrm{Coker}(d_X^{i-2}))$. Then $\mathfrak{p} \in \mathrm{Supp}_R(X)$ and so $\mathfrak{p} \notin \mathrm{V}(\mathfrak{a})$ because $\mathrm{V}(\mathfrak{a}) \cap V \subseteq F_n$ by assumption. Thus there is an element $r \in \mathfrak{a} \setminus \mathfrak{p}$. Now, since $r\,\mathrm{Id}_{(\mathrm{Coker}(d_X^{i-2}))_{\mathfrak{p}}}$ is an R–isomorphism, $\mathrm{H}^j_{\mathfrak{a}}(r\,\mathrm{Id}_{(\mathrm{Coker}(d_X^{i-2}))_{\mathfrak{p}}}) = r\,\mathrm{Id}_{\mathrm{H}^j_{\mathfrak{a}}((\mathrm{Coker}(d_X^{i-2}))_{\mathfrak{p}})}$ is also an R–isomorphism and so $\mathrm{H}^j_{\mathfrak{a}}((\mathrm{Coker}(d_X^{i-2}))_{\mathfrak{p}}) = 0$. □

Let n be a positive integer. By definition, $\mathrm{H}^n_{\mathfrak{a}}(X) = \mathrm{H}^n(\Gamma_{\mathfrak{a}}(\mathrm{E}^\bullet))$ where $\mathrm{E}^\bullet$ is an injective resolution for X. In [11, Theorem 5.4], Sharp proved that if R is Gorenstein, then $\mathrm{C}_R(\mathcal{H}(R), R)$ is an injective resolution for R. Thus $\mathrm{H}^n_{\mathfrak{a}}(R) = \mathrm{H}^n(\Gamma_{\mathfrak{a}}(\mathrm{C}_R(\mathcal{H}(R), R)))$. Here, $\mathcal{H}(X) = (H_i(X))_{i\geq 0}$ is the X–height filtration where $H_i(X) = \{\mathfrak{p} \in \mathrm{Supp}_R(X) \mid \mathrm{ht}_X(\mathfrak{p}) \geq i\}$ for all $i \geq 0$. In the next result, under some conditions, we show that $\mathrm{H}^n_{\mathfrak{a}}(X)$ and $\mathrm{H}^n(\Gamma_{\mathfrak{a}}(\mathrm{C}_R(\mathcal{F}, X)))$ have similar properties.

In the course of the remaining parts of the paper by $\mathrm{cd}_{\mathcal{S}}(\mathfrak{a}, X)$ ($\mathcal{S}$–cohomological dimension of X with respect to $\mathfrak{a}$) we mean the largest integer i in which $\mathrm{H}^i_{\mathfrak{a}}(X)$ is not in $\mathcal{S}$ (see [2, Definition 3.4] or [1, Definition 3.5]). If $\mathcal{S} =$ the class of zero R–module, then $\mathrm{cd}_{\mathcal{S}}(\mathfrak{a}, X) = \mathrm{cd}(\mathfrak{a}, X)$ as in [7].

Corollary 3.2. *Suppose that $\mathcal{F} = (F_i)_{i\geq 0}$ is a filtration of $\mathrm{Spec}(R)$ which admits an R–module X, $\mathfrak{a}$ an ideal of R, and V a subset of $\mathrm{Spec}(R)$ such that $\mathrm{Supp}_R(X) \subseteq \mathrm{V}$. Assume also that n is a positive integer such that $\mathrm{V}(\mathfrak{a}) \cap V \subseteq F_n$ and $\mathrm{cd}_{\mathcal{S}}(\mathfrak{a}, \mathrm{H}^i(\mathrm{C}_R(\mathcal{F}, X))) \leq n - i - 2$ for all $-1 \leq i \leq n$. Then $\mathrm{H}^n_{\mathfrak{a}}(X) \in \mathcal{S}$ if and only if $\mathrm{H}^n(\Gamma_{\mathfrak{a}}(\mathrm{C}_R(\mathcal{F}, X))) \in \mathcal{S}$.*

Proof. $\Rightarrow$. By considering Lemma 3.1, apply Theorem 2.3 with $s = n+1$ and $t = 0$ to get $\mathrm{H}^n(\Gamma_{\mathfrak{a}}(\mathrm{C}_R(\mathcal{F}, X))) \in \mathcal{S}$.

$\Leftarrow$. Consider Lemma 3.1 and take $s = 0$ and $t = n$ in Theorem 2.3 to get $\mathrm{H}^{-1}(\mathrm{H}^n_{\mathfrak{a}}(\mathrm{C}_R(\mathcal{F}, X))) \in \mathcal{S}$. That is $\mathrm{H}^n_{\mathfrak{a}}(X) \in \mathcal{S}$. □

We say that the Cousin complex for an R–module X with respect to a filtration $\mathcal{F}$ of Spec(R) vanishes early if there exists an integer $i \leq \dim_R(X)$ for which $\mathrm{C}_R(\mathcal{F}, X)^i = 0$. The following result is related to the Cousin complex with respect to the X–height filtration.

Corollary 3.3. (see [13, Conjecture 1, Theorem, and its Corollary]) *Let X be an R–module, $\mathfrak{a}$ an ideal of R, and n a positive integer such that $\mathrm{ht}_X(\mathfrak{a}) \geq n$ and $\mathrm{cd}_{\mathcal{S}}(\mathfrak{a}, \mathrm{H}^i(\mathrm{C}_R(\mathcal{H}(X), X))) \leq n - i - 2$ for all $-1 \leq i \leq n$. Then $\mathrm{H}^n_{\mathfrak{a}}(X) \in \mathcal{S}$ if and only if $\mathrm{H}^n(\Gamma_{\mathfrak{a}}(\mathrm{C}_R(\mathcal{H}(X), X))) \in \mathcal{S}$. In particular, if X is a non-zero finitely generated R–module, then $\mathrm{C}_R(\mathcal{H}(X), X)$ does not vanish early.*

Proof. The first assertion follows form Corollary 3.2. For the last one, assume on the contrary that $\mathrm{C}_R(\mathcal{H}(X), X)$ vanishes early. Then there exists a positive integer $n \leq \dim_R(X)$ for which $\mathrm{C}_R(\mathcal{H}(X), X)^n = 0$. Let $\mathfrak{p} \in \mathrm{Supp}_R(X)$ such that $\mathrm{ht}_X(\mathfrak{p}) = n$ and let $\mathcal{S}$ be the class of R–modules Y where $Y_{\mathfrak{p}} = 0$. Then $\mathrm{cd}_{\mathcal{S}}(\mathfrak{p}, \mathrm{H}^i(\mathrm{C}_R(\mathcal{H}(X), X))) \leq n - i - 2$ for all $-1 \leq i \leq n$. Now, since $\mathrm{H}^n(\Gamma_{\mathfrak{p}}(\mathrm{C}_R(\mathcal{H}(X), X))) = 0$, $\mathrm{H}^n_{\mathfrak{p}}(X) \in \mathcal{S}$. That is $\mathrm{H}^n_{\mathfrak{p}R_{\mathfrak{p}}}(X_{\mathfrak{p}}) = 0$. This contradiction shows that $\mathrm{C}_R(\mathcal{H}(X), X)$ does not vanish early. □

When R is local, we have the following corollary about the Cousin complex for an R–module with respect to the dimension filtration, i.e. $\mathcal{D}(R) = (D_i(R))_{i \geq 0}$ where $D_i(R) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \dim(R) - \dim_R(R/\mathfrak{p}) \geq i\}$ for all $i \geq 0$.

Corollary 3.4. *Let R be a local ring, X an R–module, $\mathfrak{a}$ an ideal of R, and n a positive integer such that $\dim(R) - \dim_R(R/\mathfrak{a}) \geq n$ and $\mathrm{cd}_{\mathcal{S}}(\mathfrak{a}, \mathrm{H}^i(\mathrm{C}_R(\mathcal{D}(R), X))) \leq n - i - 2$ for all $-1 \leq i \leq n$. Then $\mathrm{H}^n_{\mathfrak{a}}(X) \in \mathcal{S}$ if and only if $\mathrm{H}^n(\Gamma_{\mathfrak{a}}(\mathrm{C}_R(\mathcal{D}(R), X))) \in \mathcal{S}$. In particular, if X is a non-zero finitely generated R–module such that $\dim_R(X) = \dim(R)$, then $\mathrm{C}_R(\mathcal{D}(R), X)$ does not vanish early.*

Proof. The first assertion follows form Corollary 3.2. For the last one, assume on the contrary that $\mathrm{C}_R(\mathcal{D}(R), X)$ vanishes early. Then there exists an integer $n \leq \dim_R(X)$ for which $\mathrm{C}_R(\mathcal{D}(R), X)^n = 0$. Therefore $\mathrm{C}_R(\mathcal{D}(R), X)^{\dim_R(X)} = 0$. Thus $\mathrm{H}^{\dim_R(X)}(\Gamma_{\mathfrak{m}}(\mathrm{C}_R(\mathcal{D}(R), X))) = 0$ where $\mathfrak{m}$ is the maximal ideal of R. Since $\mathrm{cd}(\mathfrak{m}, \mathrm{H}^i(\mathrm{C}_R(\mathcal{D}(R), X))) \leq \dim_R(X) - i - 2$ for all $-1 \leq i \leq \dim_R(X)$, $\mathrm{H}^{\dim_R(X)}_{\mathfrak{m}}(X) = 0$. This contradiction shows that $\mathrm{C}_R(\mathcal{D}(R), X)$ does not vanish early. □

In the next corollary, we find equivalent conditions for vanishing of Cousin cohomology modules.

Corollary 3.5. *Let $\mathcal{F} = (F_i)_{i\geq 0}$ be a filtration of* $\mathrm{Spec}(R)$ *which admits an R–module X, let* V *be a subset of* $\mathrm{Spec}(R)$ *such that* $\mathrm{Supp}_R(X) \subseteq \mathrm{V}$*, and let n be a non-negative integer. Then the following statements are equivalent:*

(i) $\mathrm{H}^i(\mathrm{C}_R(\mathcal{F}, X)) = 0$ *for all* $i \leq n-2$;

(ii) $\mathrm{H}^i_{\mathfrak{b}}(X) = 0$ *for all* $i < n-j$ *and all ideals* $\mathfrak{b}$ *of* R *with* $\mathrm{V}(\mathfrak{b}) \cap \mathrm{V} \subseteq F_{n-j}$ *where* $j = 0, 1, \ldots, n-1$;

(iii) $\mathrm{H}^i_{\mathfrak{p}}(X) = 0$ *for all* $i < n-j$ *and all* $\mathfrak{p} \in \mathrm{Supp}_R(X) \cap F_{n-j}$ *where* $j = 0, 1, \ldots, n-1$;

(iv) $\mathrm{H}^i_{\mathfrak{p}R_{\mathfrak{p}}}(X_{\mathfrak{p}}) = 0$ *for all* $i < n-j$ *and all* $\mathfrak{p} \in F_{n-j}$ *where* $j = 0, 1, \ldots, n-1$.

Proof. (i)$\Rightarrow$ (ii). Let j and i be integers such that $0 \leq j \leq n-1$ and $0 \leq i < n-j$. Let $\mathfrak{b}$ be an ideal of R such that $\mathrm{V}(\mathfrak{b}) \cap \mathrm{V} \subseteq F_{n-j}$. By considering Lemma 3.1, apply Theorem 2.3 with $\mathcal{S} =$ the class of zero R–module, $s = 0$, and $t = i$ to get $\mathrm{H}^{-1}(\mathrm{H}^i_{\mathfrak{b}}(\mathrm{C}_R(\mathcal{F}, X))) = 0$. That is $\mathrm{H}^i_{\mathfrak{b}}(X) = 0$.

(ii)$\Rightarrow$ (iii). Let j be an integer such that $0 \leq j \leq n-1$. Then $\mathrm{V}(\mathfrak{p}) \subseteq F_{n-j}$ for all $\mathfrak{p} \in \mathrm{Supp}_R(X) \cap F_{n-j}$ because $\mathrm{Supp}_R(X) \subseteq F_0$ and, for all i, each member of $F_i \setminus F_{i+1}$ is a minimal member of F_i with respect to inclusion.

(iv)$\Rightarrow$ (i). We prove by using induction on n. There is nothing to prove in the case that $n = 0$. Suppose that $n > 0$ and that $n-1$ is settled. Since we have $\mathrm{H}^i(\mathrm{C}_R(\mathcal{F}, X)) = 0$ for all $i \leq n-3$ from the induction hypothesis on $n-1$, it is enough to show that $\mathrm{H}^{n-2}(\mathrm{C}_R(\mathcal{F}, X)) = 0$. Suppose on the contrary that $\mathrm{H}^{n-2}(\mathrm{C}_R(\mathcal{F}, X)) \neq 0$. Let $\mathfrak{p} \in \mathrm{Ass}_R(\mathrm{H}^{n-2}(\mathrm{C}_R(\mathcal{F}, X)))$. Therefore $\mathfrak{p} \in \mathrm{Supp}_R(X) \cap F_n$ and $\Gamma_{\mathfrak{p}R_{\mathfrak{p}}}((\mathrm{H}^{n-2}(\mathrm{C}_R(\mathcal{F}, X)))_{\mathfrak{p}}) \neq 0$. Consider Lemma 3.1 and take $s = s' = 0$ and $t = t' = n-1$ in Theorem 2.2 to get $\mathrm{H}^{-1}(\mathrm{H}^{n-1}_{\mathfrak{p}}(\mathrm{C}_R(\mathcal{F}, X))) \cong \Gamma_{\mathfrak{p}}(\mathrm{H}^{n-2}(\mathrm{C}_R(\mathcal{F}, X)))$. That is $\mathrm{H}^{n-1}_{\mathfrak{p}}(X) \cong \Gamma_{\mathfrak{p}}(\mathrm{H}^{n-2}(\mathrm{C}_R(\mathcal{F}, X)))$. Thus $\mathrm{H}^{n-1}_{\mathfrak{p}R_{\mathfrak{p}}}(X_{\mathfrak{p}}) \cong \Gamma_{\mathfrak{p}R_{\mathfrak{p}}}((\mathrm{H}^{n-2}(\mathrm{C}_R(\mathcal{F}, X)))_{\mathfrak{p}})$ and so $\mathrm{H}^{n-1}_{\mathfrak{p}R_{\mathfrak{p}}}(X_{\mathfrak{p}}) \neq 0$. This contradiction completes the proof. □

The following three results are applications of the above corollary.

Corollary 3.6. *Let $\mathcal{F} = (F_i)_{i\geq 0}$ be a filtration of* $\mathrm{Spec}(R)$*, n a non-negative integer, and*

$$0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$$

a short exact sequence in $\mathcal{C}_{\mathcal{F}}(R)$. Then the followings hold:

(i) *If* $\mathrm{H}^i(\mathrm{C}_R(\mathcal{F}, X')) = 0 = \mathrm{H}^i(\mathrm{C}_R(\mathcal{F}, X''))$ *for all* $i \leq n-2$ *(resp.* $\mathrm{C}_R(\mathcal{F}, X')$ *and* $\mathrm{C}_R(\mathcal{F}, X'')$ *are exact), then* $\mathrm{H}^i(\mathrm{C}_R(\mathcal{F}, X)) = 0$ *for all* $i \leq n-2$ *(resp.* $\mathrm{C}_R(\mathcal{F}, X)$ *is exact).*

(ii) *If* $\mathrm{H}^i(\mathrm{C}_R(\mathcal{F}, X)) = 0 = \mathrm{H}^i(\mathrm{C}_R(\mathcal{F}, X''))$ *for all* $i \leq n-2$ *(resp.* $\mathrm{C}_R(\mathcal{F}, X)$ *and* $\mathrm{C}_R(\mathcal{F}, X'')$ *are exact), then* $\mathrm{H}^i(\mathrm{C}_R(\mathcal{F}, X')) = 0$ *for all* $i \leq n-2$ *(resp.* $\mathrm{C}_R(\mathcal{F}, X')$ *is exact).*

Proof. The proof follows from Corollary 3.5 and the long exact sequence

$$\begin{array}{lllllll} 0 & \longrightarrow & \Gamma_{\mathfrak{b}}(X') & \longrightarrow & \Gamma_{\mathfrak{b}}(X) & \longrightarrow & \Gamma_{\mathfrak{b}}(X'') \\ & \longrightarrow & \mathrm{H}^1_{\mathfrak{b}}(X') & \longrightarrow & \mathrm{H}^1_{\mathfrak{b}}(X) & \longrightarrow & \mathrm{H}^1_{\mathfrak{b}}(X'') \\ & \longrightarrow & \cdots & & & & \\ & \longrightarrow & \mathrm{H}^{i-1}_{\mathfrak{b}}(X') & \longrightarrow & \mathrm{H}^{i-1}_{\mathfrak{b}}(X) & \longrightarrow & \mathrm{H}^{i-1}_{\mathfrak{b}}(X'') \\ & \longrightarrow & \mathrm{H}^{i}_{\mathfrak{b}}(X') & \longrightarrow & \mathrm{H}^{i}_{\mathfrak{b}}(X) & \longrightarrow & \mathrm{H}^{i}_{\mathfrak{b}}(X'') \\ & \longrightarrow & \mathrm{H}^{i+1}_{\mathfrak{b}}(X') & \longrightarrow & \mathrm{H}^{i+1}_{\mathfrak{b}}(X) & \longrightarrow & \mathrm{H}^{i+1}_{\mathfrak{b}}(X'') \\ & \longrightarrow & \cdots, & & & & \end{array}$$

where $0 \leq j \leq n-1$ and $\mathfrak{b}$ is an ideal of R with $\mathrm{V}(\mathfrak{b}) \subseteq F_{n-j}$. □

Corollary 3.7. (see [12, Theorem 2.4]) *Let X be an R–module and n a non-negative integer. Then the following statements are equivalent:*

(i) $\mathrm{H}^i(\mathrm{C}_R(\mathcal{H}(X), X)) = 0$ *for all* $i \leq n-2$*;*

(ii) $\mathrm{H}^i_{\mathfrak{b}}(X) = 0$ *for all* $i < n-j$ *and all ideals* $\mathfrak{b}$ *of* R *with* $\mathrm{ht}_X(\mathfrak{b}) \geq n-j$ *where* $j = 0, 1, \ldots, n-1$*;*

(iii) $\mathrm{H}^i_{\mathfrak{p}R_{\mathfrak{p}}}(X_{\mathfrak{p}}) = 0$ *for all* $i < n-j$ *and all prime ideals* $\mathfrak{p}$ *of* R *with* $\mathrm{ht}_X(\mathfrak{p}) \geq n-j$ *where* $j = 0, 1, \ldots, n-1$.

In particular, if X is non-zero and finitely generated, then X is Cohen-Macaulay if and only if $\mathrm{C}_R(\mathcal{H}(X), X)$ is exact.

Proof. Put $\mathrm{V} = \mathrm{Supp}_R(X)$ in Corollary 3.5. □

Corollary 3.8. (see [14, Theorem 3.6]) *Let R be a local ring, X an R–module, and n a non-negative integer. Then the following statements are equivalent:*

(i) $\mathrm{H}^i(\mathrm{C}_R(\mathcal{D}(R), X)) = 0$ *for all* $i \leq n-2$*;*

(ii) *Every subset of a system of parameters for R of length n is poor regular sequences on X;*

(iii) $\mathrm{H}^i_{\mathfrak{p}R_{\mathfrak{p}}}(X_{\mathfrak{p}}) = 0$ *for all* $i < n-j$ *and all prime ideals* $\mathfrak{p}$ *of* R *with* $\dim(R) - \dim_R(R/\mathfrak{p}) \geq n-j$ *where* $j = 0, 1, \ldots, n-1$.

In particular, every system of parameters for R is poor regular sequences on X if and only if $\mathrm{C}_R(\mathcal{D}(R), X)$ is exact.

Proof. Take $\mathrm{V} = \mathrm{Spec}(R)$ in Corollary 3.5. □

An important problem in commutative algebra is to determine when $\mathrm{H}^n_{\mathfrak{a}}(X)$ is Artinian. In the next result, we show that the minimaxness of Cousin cohomology modules $\mathrm{H}^i(\mathrm{C}_R(\mathcal{F}, X))$ for all $i \leq n-2$ implies, for all $i \leq n-1$ and all $\mathfrak{p} \in F_n$, the Artinianness of $\mathrm{H}^i_{\mathfrak{p}R_{\mathfrak{p}}}(X_{\mathfrak{p}})$, which is related to the third of Huneke's four problems in local cohomology modules [8]. Recall that an R–module X is said to be minimax, if there is a finitely generated submodule X' of X such that X/X' is Artinian [16]. Thus the class of minimax modules includes all finitely generated and all Artinian modules.

Corollary 3.9. *Let $\mathcal{F} = (F_i)_{i\geq 0}$ be a filtration of* $\mathrm{Spec}(R)$ *which admits an R–module X, let* V *be a subset of* $\mathrm{Spec}(R)$ *such that* $\mathrm{Supp}_R(X) \subseteq \mathrm{V}$*, and let n be a non-negative integer such that* $\mathrm{H}^i(\mathrm{C}_R(\mathcal{F}, X))$ *is minimax for all $i \leq n-2$. Then* $\mathrm{H}^i_{\mathfrak{p}R_{\mathfrak{p}}}(X_{\mathfrak{p}})$ *is Artinian for all $i \leq n-1$ and all $\mathfrak{p} \in F_n$.*

Proof. Let $i \leq n-1$ and let $\mathfrak{p} \in F_n$. Suppose that $\mathfrak{p} \in \mathrm{Supp}_R(X)$. Then $\mathrm{V}(\mathfrak{p}) \subseteq F_n$ because $\mathrm{Supp}_R(X) \subseteq F_0$ and, for all k, each member of $F_k \setminus F_{k+1}$ is a minimal member of F_k with respect to inclusion. By considering Lemma 3.1, apply Theorem 2.3 with $\mathcal{S}$ = the class of R–modules Y where $Y_{\mathfrak{p}}$ is Artinian, $s = 0$, and $t = i$ to get $\mathrm{H}^{-1}(\mathrm{H}^i_{\mathfrak{p}}(\mathrm{C}_R(\mathcal{F}, X))) \in \mathcal{S}$. That is $\mathrm{H}^i_{\mathfrak{p}}(X) \in \mathcal{S}$. Thus $\mathrm{H}^i_{\mathfrak{p}R_{\mathfrak{p}}}(X_{\mathfrak{p}})$ is Artinian. □

As another application of Theorem 2.3, we study the relations between support (resp. annihilator) of local cohomology modules and support (resp. annihilator) of Cousin cohomology modules.

Corollary 3.10. *Let $\mathcal{F} = (F_i)_{i\geq 0}$ be a filtration of* $\mathrm{Spec}(R)$ *which admits an R–module X, $\mathfrak{a}$ an ideal of R,* V *a subset of* $\mathrm{Spec}(R)$ *such that* $\mathrm{Supp}_R(X) \subseteq \mathrm{V}$*, and n a non-negative integer such that* $\mathrm{V}(\mathfrak{a}) \cap V \subseteq F_{n+1}$*. Then the following statements hold true:*

(i) $\bigcup_{i=0}^{n} \mathrm{Supp}_R(\mathrm{H}^i_{\mathfrak{a}}(X)) \subseteq \bigcup_{i=-1}^{n-1} \mathrm{Supp}_R(\mathrm{H}^i(\mathrm{C}_R(\mathcal{F}, X)))$;

(ii) $\prod_{i=-1}^{n-1} (0 :_R \mathrm{H}^i(\mathrm{C}_R(\mathcal{F}, X))) \subseteq \bigcap_{i=0}^{n} (0 :_R \mathrm{H}^i_{\mathfrak{a}}(X))$.

Proof. (i) We have

$$\mathrm{Supp}_R(\mathrm{H}^n_{\mathfrak{a}}(X)) \subseteq \bigcup_{i=-1}^{n-1} \mathrm{Supp}_R(\mathrm{H}^{n-1-i}_{\mathfrak{a}}(\mathrm{H}^i(\mathrm{C}_R(\mathcal{F}, X))))$$

by considering Lemma 3.1 and using Theorem 2.3 with $s = 0$ and $t = n$ because $\lambda(X) = \mathrm{Supp}_R(X)$ is a subadditive function from the category of R–modules to the partially ordered Abelian monoid $(\mathrm{Spec}(R), \cup, \subseteq)$. Thus the assertion follows.

(ii) The proof is similar to the first part because $\lambda(X) = (0 :_R X)$ is a subadditive function from the category of R–modules to the partially ordered Abelian monoid $(\mathrm{Ideals}(R), ., \supseteq)$. □

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