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2-ABSORBING I-PRIME AND 2-ABSORBING I-SECOND SUBMODULES

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Abstract. Let R be a commutative ring and let I be an ideal of R. In this paper, we will introduce the notions of 2-absorbing I-prime and 2-absorbing I-second submodules of an R-module M as a generalization of 2-absorbing and strongly 2-absorbing second submodules of M and explore some basic properties of these classes of modules.

1. Introduction

Throughout this paper, R will denote a commutative ring with identity, I an ideal of R, and $\mathbb{Z}$ will denote the ring of integers.

Let M be an R-module. A proper submodule P of M is said to be *prime* if for any $r \in R$ and $m \in M$ with $rm \in P$, we have $m \in P$ or $r \in (P :_R M)$ [6]. A non-zero submodule S of M is said to be *second* if for each $a \in R$, the homomorphism $S \xrightarrow{a} S$ is either surjective or zero [10].

The concept of 2-absorbing ideals was introduced in [4]. A proper ideal I of R is called a *2-absorbing ideal* of R if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. The notion of 2-absorbing ideals was extended to 2-absorbing submodules in [5] and [9]. A proper submodule N of an R-module M is called a *2-absorbing submodule* of M if whenever

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$abm \in N$ for some $a, b \in R$ and $m \in M$, then $am \in N$ or $bm \in N$ or $ab \in (N :_R M)$. A proper submodule N of an R-module M is said to be a *weakly 2-absorbing submodule* of M if whenever $a, b \in R$ and $m \in M$ with $0 \neq abm \in N$, then $ab \in (N :_R M)$ or $am \in N$ or $bm \in N$ [5].

In [2], the authors introduced the dual notion of 2-absorbing submodules (that is, *strongly 2-absorbing second submodules*) of an R-module M and investigated some properties of this class of modules. A non-zero submodule N of an R-module M is said to be a *strongly 2-absorbing second submodule of* M if whenever $a, b \in R$, K is a submodule of M, and $abN \subseteq K$, then $aN \subseteq K$ or $bN \subseteq K$ or $ab \in Ann_R(N)$. A non-zero submodule N of an R-module M is called a *weakly strongly 2-absorbing second submodule of* M if whenever $a, b \in R$, K is a submodule of M, $abM \not\subseteq K$, and $abN \subseteq K$, then $aN \subseteq K$ or $bN \subseteq K$ or $ab \in Ann_R(N)$ [3].

The purpose of this article is to introduce the notion of 2-absorbing I-prime and 2-absorbing I-second submodules of an R-module M as generalizations of 2-absorbing and strongly 2-absorbing second submodules and provide some useful information concerning these classes of modules. Moreover, we obtain some results analogous to those for weakly 2-absorbing submodules considered in [5] and [8].

2. 2-absorbing I-prime submodules

Definition 2.1. Let I be an ideal of R. We say that a proper submodule N of an R-module M is a *2-absorbing I-prime submodule* of M if whenever $a, b \in R$ and $m \in M$ with $abm \in N \setminus IN$, then $abM \subseteq N$ or $am \in N$ or $bm \in N$.

Example 2.2. Consider the submodule $G_n = \langle 1/p^n + \mathbb{Z} \rangle$ of the $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$ and $I = k\mathbb{Z}$, where n, k are positive integers. Then we have the following.

(a) If k divided by p, then G_n is not a 2-absorbing I-prime submodule of $\mathbb{Z}_{p^\infty}$.

(b) If k is not divided by p, then G_n is a 2-absorbing I-prime submodule of $\mathbb{Z}_{p^\infty}$.

Remark 2.3. Let I be an ideal of R. Clearly, every 2-absorbing submodule of an R-module M is a 2-absorbing I-prime submodule of M. But Examples 2.4 and 3.6 show that the converse is not true in general.

Example 2.4. The zero submodule of the $\mathbb{Z}$-module $\mathbb{Z}_{12}$ is a 2-absorbing I-prime submodule which is not 2-absorbing submodule.

Example 2.5. Let p and q be two prime numbers. Consider the submodule $\overline{p^3}\mathbb{Z}_{p^3q}$ of the $\mathbb{Z}$-module $\mathbb{Z}_{p^3q}$ and let $I = p^3\mathbb{Z}$. Then $\overline{p^3}\mathbb{Z}_{p^3q} \setminus I\overline{p^3}\mathbb{Z}_{p^3q} = \emptyset$ and so $\overline{p^3}\mathbb{Z}_{p^3q}$ is a 2-absorbing I-prime submodule of $\mathbb{Z}_{p^3q}$. But $(p)(\bar{p})(\bar{p}) \in \overline{p^3}\mathbb{Z}_{p^3q}$, $p(\bar{p}) \notin \overline{p^3}\mathbb{Z}_{p^3q}$, and $p \notin (\overline{p^3}\mathbb{Z}_{p^3q} :_{\mathbb{Z}} \mathbb{Z}_{p^3q}) = p^3\mathbb{Z}$, implies that $\overline{p^3}\mathbb{Z}_{p^3q}$ is not a 2-absorbing submodule of $\mathbb{Z}_{p^3q}$.

Definition 2.6. Let I be an ideal of R. We say that a proper submodule N of an R-module M is a *strongly 2-absorbing I-prime submodule* of M if whenever $a, b \in R$ and K a submodule of M with $abK \subseteq N \setminus IN$, then $abM \subseteq N$ or $aK \subseteq N$ or $bK \subseteq N$.

Remark 2.7. Let I be an ideal of R and M be an R-module. Then we have the following.

(a) If $I = 0$, then the notion of 2-absorbing I-prime submodule is exactly the notion of weakly 2-absorbing submodule.

(b) If $I = R$, then every submodule is a 2-absorbing I-prime submodule. So in the rest of this paper we can assume $I \neq R$.

(c) A submodule N of M is 2-absorbing I-prime if and only if N/IN is a weakly 2-absorbing R-submodule of M/IN.

Lemma 2.8. *Let M be an R-module and I, J be ideals of R such that $I \subseteq J$. If N is a 2-absorbing I-prime submodule of M, then N is a 2-absorbing J-prime submodule of M. In particular, every weakly 2-absorbing submodule is a 2-absorbing I-prime submodule for each ideal I of R.*

Proof. The result follows from the fact that $I \subseteq J$ implies that $N \setminus JN \subseteq N \setminus IN$. □

Theorem 2.9. *Let I be an ideal of R, M be an R-module, and let N be a 2-absorbing I-prime submodule of M. If N is not 2-absorbing, then $(N :_R M)^2 N \subseteq IN$.*

Proof. Assume that $(N :_R M)^2 N \not\subseteq IN$. We show that N is 2-absorbing. Let $a, b \in R$ and $m \in M$ be such that $abm \in N$. If $abm \notin IN$, then $ab \in (N :_R M)$ or $am \in N$ or $bm \in N$ since N is 2-absorbing I-prime submodule. So assume that $abm \in IN$. Suppose first that $abN \not\subseteq IN$, say $abn_0 \notin IN$ for some $n_0 \in N$. Then $ab(m + n_0) \in N \setminus IN$. Since N is 2-absorbing I-prime submodule, we get $ab \in (N :_R M)$ or $a(m+n_0) \in N$ or $b(m+n_0) \in N$. Hence, $ab \in (N :_R M)$ or $am \in N$ or $bm \in N$. Hence we may assume that $abN \subseteq IN$. If $am(N :_R M) \not\subseteq IN$, then there exists $r_0 \in (N :_R M)$ such that $ar_0m \notin IN$. Thus $a(b + r_0)m \in N \setminus IN$. Since N is 2-absorbing I-prime submodule, we have either $a(b + r_0) \in (N :_R M)$ or $am \in N$ or $(b + r_0)m \in N$. Therefore, either $ab \in (N :_R M)$ or $am \in N$ or $bm \in N$. So we can assume without loss of generality that $am(N :_R M) \subseteq IN$. Likewise we can assume without loss of generality that $bm(N :_R M) \subseteq IN$. Since $(N :_R M)^2 N \not\subseteq IN$, there exist $a_0, b_0 \in (N :_R M)$ and $x_0 \in N$ with $a_0b_0x_0 \notin IN$. If $ab_0x_0 \notin IN$, then $a(b + b_0)(m + x_0) \in N \setminus IN$ implies that either $a(b+b_0) \in (N :_R M)$ or $a(m+x_0) \in N$ or $(b+b_0)(m+x_0) \in N$. Hence $ab \in (N :_R M)$ or $am \in N$ or $bm \in N$. So we can assume without loss of generality that $ab_0x_0 \in IN$. Likewise we can assume without loss of generality that $a_0b_0m \in IN$ and $a_0bx_0 \in IN$. Then from $(a+a_0)(b+b_0)(m+x_0) \in N \setminus IN$, we get $(a+a_0)(b+b_0) \in (N :_R M)$ or $(a+a_0)(m+x_0) \in N$

or $(b+b_0)(m+x_0) \in N$. Therefore, either $ab \in (N :_R M)$ or $am \in N$ or $bm \in N$, and so N is 2-absorbing. □

Lemma 2.10. *Let I be an ideal of R, N be a 2-absorbing I-prime submodule of an R-module M, and $a, b \in R$. If for some submodule K of M, $abK \subseteq N$ and $2abK \nsubseteq IN$, then either $ab \in (N :_R M)$ or $aK \subseteq N$ or $bK \subseteq N$.*

Proof. Suppose $ab \notin (N :_R M)$. Then one can see that it is enough to show that $K \subseteq (N :_M a) \cup (N :_M b)$. Let z an arbitrary element of K. If $abz \notin IN$, then as N is 2-absorbing I-prime and $ab \notin (N :_R M)$, either $az \in N$ or $bz \in N$ and so $z \in (N :_M a) \cup (N :_M b)$. Now let $abz \in IN$. Since $2abK \notin IN$, for some $x \in K$, we have $2abx \notin IN$ and so $abx \in N \setminus IN$. As N is 2-absorbing I-prime and $ab \notin (N :_R M)$, either $ax \in N$ or $bx \in N$. Now we have $ab(x+z) \in N \setminus IN$ and since $ab \notin (N :_R M)$, either $a(x+z) \in N$ or $b(x+z) \in N$. We consider three cases.

Case(I). $ax \in N$ and $bx \in N$. Note that $a(x+z) \in N$ or $b(x+z) \in N$, so either $az \in N$ or $bz \in N$.

Case(II). $ax \in N$ and $bx \notin N$. On the contrary let $az \notin N$. Then $a(x+z) \notin N$ and so $b(x+z) \in N$. So $a(x+z+x) \notin N$ and $b(x+z+x) \notin N$. Now as N is 2-absorbing I-prime and $ab \notin (N :_R M)$, then $ab(x+z+x) \in IN$ and so $2abx \in IN$, which is a contradiction. Therefore, $az \in N$.

Case(III). $ax \notin N$ and $bx \in N$. Then proof is similar to that of Case(II). □

Corollary 2.11. *Let I be an ideal of R, N be a 2-absorbing I-prime submodule of an R-module M such that for each $a, b \in R$ and submodule K of M, $2abK \nsubseteq IN$, then N is a strongly 2-absorbing I-prime submodule of M.*

Proof. This follows from Lemma 2.10. □

Lemma 2.12. *Let I and B be ideals of R and K, N two submodules of an R-module M, such that $aBK \subseteq N$, where $a \in R$. If N is 2-absorbing I-prime and $4aBK \nsubseteq IN$, then either $aB \subseteq (N :_R M)$ or $aK \subseteq N$ or $BK \subseteq N$.*

Proof. Let $aB \nsubseteq (N :_R M)$. Then $aj \notin (N :_R M)$ for some $j \in B$. First we claim that there exists $b \in B$ such that $4abK \nsubseteq IN$, and $ab \notin (N :_R M)$. Since $4aBK \nsubseteq IN$, for some $j_0 \in B$, $4aj_0K \nsubseteq IN$. If $aj_0 \notin (N :_R M)$ or $4ajK \nsubseteq IN$, then by putting $b = j_0$ or $b = j$, we get the result. So let $aj_0 \in (N :_R M)$ and $4ajK \subseteq IN$. Hence $4a(j+j_0)K \subseteq N \setminus IN$ and $a(j+j_0) \notin (N :_R M)$. Consequently, we find $b = j + j_0 \in B$, such that $4abK \nsubseteq IN$, and $ab \notin (N :_R M)$. So $2abK \nsubseteq IN$ and by Lemma 2.10, $K \subseteq (N :_M a) \cup (N :_M b)$. If $aK \subseteq N$, there is nothing to prove. So assume that $aK \nsubseteq N$. Thus $bK \subseteq N$. Now we show that $B \subseteq ((N :_R M) :_R a) \cup (N :_R K)$. Let $c \in B$. If $2acK \nsubseteq IN$, then by Lemma 2.10, either $ac \in (N :_R M)$ or $aK \subseteq N$ or $cK \subseteq N$. But as we assumed $aK \nsubseteq N$, $c \in ((N :_R M) :_R a) \cup (N :_R K)$. Next assume $2acK \subseteq IN$. Then $2a(b+c)K \subseteq N \setminus IN$ and Lemma 2.10, implies that either $a(b+c) \in (N :_R M)$ or $aK \subseteq N$ or $(b+c)K \subseteq N$. Then as $aK \nsubseteq N$, we have $b+c \in ((N :_R M) :_R a) \cup (N :_R K)$. If $b+c \in (N :_R K)$, then $c \in (N :_R K)$, since $b \in (N :_R K)$. So let $b+c \in ((N :_R M) :_R a) \setminus (N :_R K)$. Consider $2a(b+c+b)K = 4abK + 2acK \nsubseteq IN$ and $2a(b+c+b)K \subseteq N$. Since $ab \notin (N :_R M)$ and $a(b+c) \in (N :_R M)$, we have $a(b+c+b) \notin (N :_R M)$. By Lemma 2.10, $K \subseteq (N :_M a) \cup (N :_M b+c+b)$. But since $b+c \notin (N :_R K)$ and $b \in (N :_R K)$, we have $b+c+b \notin (N :_R K)$, and so $K \subseteq (N :_M a)$, which is impossible. Therefore $b+c \in (N :_R K)$ and since $b \in (N :_R K)$, $c \in (N :_R K)$. Consequently $B \subseteq ((N :_R M) :_R a) \cup (N :_R K)$. Therefore, $BK \subseteq N$ since $aB \nsubseteq (N :_R M)$. □

Theorem 2.13. *Let I, A, and B be ideals of R and N, K be two submodules of an R-module M. If N is a 2-absorbing I-prime submodule of M, $ABK \subseteq N \setminus IN$, and $8(AB+(A+B)(N :_R M)(K+N) \nsubseteq IN$, then either $AB \subseteq (N :_R M)$ or $AK \subseteq N$ or $BK \subseteq N$. In particular, this holds if the group $(M/IN, +)$ has no elements of order 2.*

Proof. Note that

$$8(AB + (A+B)(N :_R M))(K+N) =$$
$$8ABK + 8ABN + 8A(N :_R M)K + 8B(N :_R M)K +$$
$$8A(N :_R M)N + 8B(N :_R M)N \nsubseteq IN.$$

Therefore one of the following different types is satisfied.

(i) $8ABK \nsubseteq IN$. Then for some $b \in B$, we have $8bAK \nsubseteq IN$. So $4bAK \nsubseteq IN$ and by Lemma 2.12, either $bA \subseteq (N :_R M)$ or $bK \subseteq N$ or $AK \subseteq N$. If $AK \subseteq N$, then we have the result. So we suppose that $AK \nsubseteq N$. So $b \in ((N :_R M) :_R A) \cup (N :_R K)$. Now we show that $B \subseteq ((N :_R M) :_R A) \cup (N :_R K)$. To see this let $c \in B$. If $4cAK \nsubseteq IN$, then according to Lemma 2.12, since $AK \nsubseteq N$, $c \in ((N :_R M) :_R A) \cup (N :_R K)$. Now let $4cAK \subseteq IN$. So $4(b+c)AK \subseteq N \setminus IN$. So by Lemma 2.12, since $AK \nsubseteq N$, $b+c \in ((N :_R M) :_R A) \cup (N :_R K)$. We consider the following four cases.

Case(1). $b+c \in ((N :_R M) :_R A)$ and $b \in ((N :_R M) :_R A)$. Then $c \in ((N :_R M) :_R A)$.

Case(2). $b+c \in (N :_R K)$ and $b \in (N :_R K)$. Hence $c \in (N :_R K)$.

Case(3). $b \in ((N :_R M) :_R A) \setminus (N :_R K)$ and $b+c \in (N :_R K) \setminus ((N :_R M) :_R A)$. Therefore $b+c+b \notin ((N :_R M) :_R A)$ and $b+c+b \notin (N :_R K)$ and so $b+c+b \notin (L :_R$

$A) \cup (N :_R K)$. We consider $4(b+c+b)AK = 8bAK + 4cAK \not\subseteq IN$. Hence by Lemma 2.12, as $AK \not\subseteq N$, $b+c+b \in ((N :_R M) :_R A) \cup (N :_R K)$, which is impossible. Hence as $b \in ((N :_R M) :_R A) \cup (N :_R K)$ and $b+c \in ((N :_R M) :_R A) \cup (N :_R K)$, one of the followings holds.

(a) $b \in (N :_R K)$ and $b+c \in (N :_R K) \setminus (L :_R A)$. So $c \in (N :_R K)$.

(b) $b \in ((N :_R M) :_R A) \setminus (N :_R K)$ and $b+c \in ((N :_R M) :_R A)$. Hence $c \in ((N :_R M) :_R A)$.

Case(4). $a+c \in ((N :_R M) :_R A) \setminus (N :_R K)$ and $a \in (N :_R K) \setminus ((N :_R M) :_R A)$. Similar to Case (3), we get $c \in ((N :_R M) :_R A) \cup (N :_R K)$ Consequently $B \subseteq ((N :_R M) :_R A) \cup (N :_R K)$.

(ii) If $8ABN \not\subseteq IN$ and $8ABK \subseteq IN$, then $8AB(K+N) \subseteq N \setminus IN$, and then by part (i), either $BA \subseteq (N :_R M)$ or $B(K+N) \subseteq N$ or $A(K+N) \subseteq N$ and so either $BA \subseteq (N :_R M)$ or $BK \subseteq N$or $AK \subseteq N$.

(iii) Let $8B(N :_R M)K \not\subseteq IN$ and $8ABK \subseteq IN$. Then $8B(A+(N :_R M))K \not\subseteq IN$ and so according to part (i), either $B(A+(N :_R M)) \subseteq (N :_R M)$ or $BK \subseteq N$ or $(A+(N :_R M))K \subseteq N$ and so either $BA \subseteq (N :_R M)$ or $BK \subseteq N$ or $AK \subseteq N$. Similarly if $8A(N :_R M)K \notin IN$, we get the result.

(iv) Let $8B(N :_R M)N \not\subseteq IN$ and $8ABK \subseteq IN$, $8ABN \subseteq IN$, $8B(N :_R M)K \subseteq IN$, $8A(N :_R M)K \subseteq IN$. Then $8B(A+(N :_R M))(K+N) \not\subseteq IN$, and so part (i) implies that either $B(A+(N :_R M)) \subseteq (N :_R M)$ or $B(K+N) \subseteq N$ or $(A+(N :_R M))(K+N) \subseteq N$. Hence either $BA \subseteq (N :_R M)$ or $BK \subseteq N$ or $AK \subseteq N$. Clearly if $8A(N :_R M)N \not\subseteq IN$, we have the result.

For the particular case suppose the group $(M/IN, +)$ has no subgroups of order 2. Then we show that $8ABK \not\subseteq IN$, and so by part (i), the result is given. If $8ABK \subseteq IN$, then consider $l \in ABK \setminus IN$. As $8l \in IN$, so the group $(M/IN, +)$ has a subgroup of order 2, 4 or 8, which implies that it has an element of order 2, a contradiction □

3. 2-absorbing I-second submodules

Let M be an R-module. A proper submodule N of M is said to be *completely irreducible* if $N = \bigcap_{i \in I} N_i$, where $\{N_i\}_{i \in I}$ is a family of submodules of M, implies that $N = N_i$ for some $i \in I$. It is easy to see that every submodule of M is an intersection of completely irreducible submodules of M [7].

We frequently use the following basic fact without further comment.

Remark 3.1. Let N and K be two submodules of an R-module M. To prove $N \subseteq K$, it is enough to show that if L is a completely irreducible submodule of M such that $K \subseteq L$, then $N \subseteq L$.

Theorem 3.2. *Let I be an ideal of R. For a non-zero submodule S of an R-module M the following statements are equivalent:*

(a) *For each $a, b \in R$, a submodule K of M, $ab \in (K :_R S) \setminus (K :_R (S :_M I))$ implies that either $aS \subseteq K$ or $bS \subseteq K$ or $abS = 0$;*

(b) *For each $ab \notin (abS :_R (S :_M I))$, we have either $abS = aS$ or $abS = bS$ or $abS = 0$.*

Proof. $(a) \Rightarrow (b)$ Let $ab \notin (abS :_R (S :_M I))$. Then as $abS \subseteq abS$, we have either $aS \subseteq abS$ or $bS \subseteq abS$ or $abS = 0$ by part (a). Thus $abS = aS$ or $abS = bS$ or $abS = 0$.

$(b) \Rightarrow (a)$ Let $a, b \in R$ and K be a submodule of M such that $ab \in (K :_R S) \setminus (K :_R (S :_M I))$. Then if $ab \in (abS :_R (S :_M I))$, then $ab \in (K :_R (S :_M I))$, which is a contradiction. Thus $ab \notin (abS :_R (S :_M I))$. Now by part (b), either $abS = aS$ or $abS = bS$ or $abS = 0$. So either $aS \subseteq K$ or $bS \subseteq K$ or $abS = 0$, as needed. □

Definition 3.3. Let I be an ideal of R. We say that a non-zero submodule S of an R-module M is a *2-absorbing I-second submodule* of M if satisfies the equivalent conditions of Theorem 3.2. This can be regarded as a dual notion of 2-absorbing I-prime submodules.

In Definition 3.3, when $I = 0$, then S is a weakly strongly 2-absorbing second submodule of M.

Remark 3.4. Let I be an ideal of R. Clearly, every strongly 2-absorbing second submodule is a 2-absorbing I-second submodule. But the Examples 3.5 and 3.6 show that converse is not true in general.

Example 3.5. If $I = 0$, then every R-module is a 2-absorbing I-second submodule of itself but every R-module is not a strongly 2-absorbing second R-module. For example, the $\mathbb{Z}$-module $\mathbb{Z}$ is weak 2-absorbing second which is not strongly 2-absorbing second.

Example 3.6. Let p and q be two prime numbers. Consider the submodule $\bar{q}\mathbb{Z}_{p^3q}$ of the $\mathbb{Z}$-module $\mathbb{Z}_{p^3q}$ and let $I = p^3\mathbb{Z}$. Then for each submodule K of $\mathbb{Z}_{p^3q}$,

$$(K :_{\mathbb{Z}} \bar{q}\mathbb{Z}_{p^3q}) \setminus (K :_{\mathbb{Z}} (\bar{q}\mathbb{Z}_{p^3q} :_{\mathbb{Z}_{p^3q}} I)) = \emptyset$$

and so $\bar{q}\mathbb{Z}_{p^3q}$ is a 2-absorbing I-second submodule of $\mathbb{Z}_{p^3q}$. But $(p)(p)(\bar{q}\mathbb{Z}_{p^3q}) \neq (p)(\bar{q}\mathbb{Z}_{p^3q})$ and $(p)(p)(\bar{q}\mathbb{Z}_{p^3q}) \neq \bar{0}$ implies that $\bar{q}\mathbb{Z}_{p^3q}$ is not a strongly 2-absorbing second submodule of $\mathbb{Z}_{p^3q}$ by [2, Theorem 3.3 (c)].

Proposition 3.7. *Let M be an R-module and I, J be ideals of R such that $I \subseteq J$. If N is a 2-absorbing I-second submodule of M, then N is a 2-absorbing J-second submodule of M. In particular, every weakly strongly 2-absorbing second submodule is a 2-absorbing I-second submodule for each ideal I of R.*

Proof. This is clear. □

Theorem 3.8. *Let I be an ideal of R and let N be a 2-absorbing I-second submodule of an R-module M which is not a strongly 2-absorbing second submodule. Then $Ann^2_R(N)(N :_M I) \subseteq N$.*

Proof. Assume on the contrary that $Ann^2_R(N)(N :_M I) \not\subseteq N$. We show that N is a strongly 2-absorbing second submodule of M. Let $a, b \in R$ and K be a submodule of M such that $abN \subseteq K$. If $ab(N :_M I) \not\subseteq K$, then we are done because N is a 2-absorbing I-second submodule of M. Thus suppose that $ab(N :_M I) \subseteq K$. If $ab(N :_M I) \not\subseteq N$, then $ab(N :_M I) \not\subseteq N \cap K$. Hence $abN \subseteq N \cap K$ implies that either $aN \subseteq N \cap K \subseteq K$ or $bN \subseteq N \cap K \subseteq K$ or $abN = 0$, as needed. So let $ab(N :_M I) \subseteq N$. If $aAnn_R(N)(N :_M I) \not\subseteq K$, then $a(b + Ann_R(N))(N :_M I) \not\subseteq K$. Thus $a(b + Ann_R(N))N \subseteq K$ implies that either $aN \subseteq K$ or $bN = (b + Ann_R(N))N \subseteq K$ or $abN = a(b+Ann_R(N))N = 0$, as required. So let $aAnn_R(N)(N :_M I) \subseteq K$. Similarly, we can assume without loss of generality that $bAnn_R(N)(N :_M I) \subseteq K$. Since $Ann_R(N)^2 \not\subseteq (N :_R (N :_M I))$, there exist $a_1, b_1 \in Ann_R(N)$ such that $a_1b_1(N :_M I) \not\subseteq N$. Thus there exists a completely irreducible submodule L of M such that $N \subseteq L$ and $a_1b_1(N :_M I) \not\subseteq L$ by Remark 3.1. If $ab_1(N :_M I) \not\subseteq L$, then $a(b + b_1)(N :_M I) \not\subseteq L \cap K$. Thus $a(b + b_1)N \subseteq L \cap K$ implies that either $aN \subseteq L \cap K \subseteq K$ or $bN = (b + b_1)N \subseteq L \cap K \subseteq K$ or $abN = a(b + b_1)N = 0$, as needed. So let $ab_1(N :_M I) \subseteq L$. Similarly, we can assume without loss of generality that $a_1b(N :_M I) \subseteq L$. Therefore, $(a+a_1)(b+b_1)(N :_M I) \not\subseteq L \cap K$. Hence, $(a+a_1)(b+b_1)N \subseteq L \cap K$ implies that either $aN = (a+a_1)N \subseteq K$ or $bN = (b+b_1)N \subseteq K$ or $abN = (a+a_1)(b+b_1)N = 0$, as desired. □

Theorem 3.9. *Let I be an ideal of R, $a \in R$, and let M be an R-module. If $Ann_R(IN)(IN :_M a) \subseteq a(IN :_M aI)Ann_R(IN)$ and $(IN :_M a)$ is is a 2-absorbing I-second submodule of M, then $Ann_R(IN)(IN :_M a)$ is a strongly 2-absorbing second submodule of M.*

Proof. (a) Suppose that $(IN :_M a)$ is a 2-absorbing I-second submodule of M, $r, s \in R$, and K is a submodule of M such that $rsAnn_R(IN)(IN :_M a) \subseteq K$. Then $rs(IN :_M a) \subseteq (K :_M Ann_R(IN))$. If $rs(IN :_M aI) \not\subseteq (K :_M Ann_R(IN))$, then since $(IN :_M a)$ is 2-absorbing I-second, we have either $rAnn_R(IN)(IN :_M a) \subseteq K$ or $sAnn_R(IN)(IN :_M a) \subseteq K$ or $srAnn_R(IN)(IN :_M a) = 0$ which implies $Ann_R(IN)(IN :_M a)$ is strongly 2-absorbing second. Therefore we may assume that $rs(IN :_M aI) \subseteq (K :_M Ann_R(IN))$. Clearly, $r(s + a)(IN :_M a) \subseteq (K :_M Ann_R(IN))$. If $r(s + a)(IN :_M aI) \not\subseteq (K :_M Ann_R(IN))$, then we have either $r(IN :_M a) \subseteq (K :_M Ann_R(IN))$ or $(s + a)(IN :_M a) \subseteq (K :_M Ann_R(IN))$ or $r(s + a)Ann_R(IN)((IN :_M a)) = 0$. Therefore, $r(IN :_M a) \subseteq (K :_M Ann_R(IN))$ or $s(IN :_M a) \subseteq (K :_M Ann_R(IN))$ or $r(s + a)Ann_R(IN)((IN :_M a)) = 0$. Now suppose that $r(s + a)(IN :_M aI) \subseteq (K :_M Ann_R(IN))$. Then since $rs(IN :_M aI) \subseteq (K :_M Ann_R(IN))$, we have $ra(IN :_M aI) \subseteq (K :_M Ann_R(IN))$. Now $Ann_R(IN)(IN :_M a) \subseteq a(IN :_M aI)Ann_R(IN)$ implies that $r(IN :_M a) \subseteq (K :_M Ann_R(IN))$. Thus $rAnn_R(IN)(IN :_M a) \subseteq K$, as needed.

□

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References

[1] D. F. Anderson, A. Badawi, *On n-absorbing ideals of commutative rings*, Comm. Algebra **39** (2011), 1646-1672.

[2] H. Ansari-Toroghy, F. Farshadifar, *Some generalizations of second submodules*, Palestine Journal of Mathematics, **8** (2) (2019), 159168.

[3] H. Ansari-Toroghy, F. Farshadifar, S. Maleki-Roudposhti, *n-absorbing and strongly n-absorbing second submodules*, Bol. Soc. Paran. Mat., to appear.

[4] A. Badawi, *On 2-absorbing ideals of commutative rings*, Bull. Austral. Math. Soc. **75** (2007), 417-429.

[5] A. Y. Darani, F. Soheilnia, *2-absorbing and weakly 2-absorbing submoduels*, Thai J. Math. **9**(3) (2011), 577584.

[6] J. Dauns, *Prime submodules*, J. Reine Angew. Math. **298** (1978), 156–181.

[7] L. Fuchs, W. Heinzer, B. Olberding, *Commutative ideal theory without finiteness conditions: Irreducibility in the quotient filed*, in : Abelian Groups, Rings, Modules, and Homological Algebra, Lect. Notes Pure Appl. Math. **249** (2006), 121–145.

[8] S. Moradi, A. Azizi, *Weakly 2-absorbing submodules of modules*, Turk. J. Math. **40** (2016), 350-364.

[9] Sh. Payrovi, S. Babaei, *On 2-absorbing submodules*, Algebra Collq., **19** (2012), 913-920.

[10] S. Yassemi, *The dual notion of prime submodules*, Arch. Math. (Brno) **37** (2001), 273–278.

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