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GROUPS WHOSE SET OF VANISHING ELEMENTS IS EXACTLY A CONJUGACY CLASS

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ABSTRACT. Let G be a finite group. We say that an element g in G is a vanishing element if there exists some irreducible character χ of G such that $\chi(g) = 0$. In this paper, we classify groups whose set of vanishing elements is exactly a conjugacy class.

1. INTRODUCTION

Let G be a finite group and $\mathrm{Irr}(G)$ be the set of irreducible characters of G. We say that an element g in G is a vanishing element if there exists some $\chi \in Irr(G)$ such that $\chi(g) = 0$, otherwise, g is a non-vanishing element, in other words, g is a non-vanishing element of G if $\chi(x) \neq 0$ for all $\chi \in Irr(G)$.

Let $\mathrm{Van}(G)$ denote the set of vanishing elements of G, in other words,

$$\mathrm{Van}(G) = \{g \in G | \chi(g) = 0 \text{ for some } \chi \in \mathrm{Irr}(G)\}.$$

It is clear that $\mathrm{Van}(G)$ is the union of some conjugacy classes. In [3, Theorem 3.15], Burnside show that $\mathrm{Van}(G) = \emptyset$ if and only if G is abelian. Moreover, in [5], we using the Classification

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of the Finite Simple Groups, show that finite groups whose set of vanishing elements is the union of at most three conjugacy classes are solvable. In this paper, we classify groups whose set of vanishing elements is exactly a conjugacy class.

We summarize our notations. Let $F(G)$ denote the Fitting subgroup of G, which is the subgroup generated by all normal nilpotent subgroups of G and $cl_G(a)$ denote the conjugacy class of a in G.

2. **Main Theorem**

Before stating the main result, we mention some necessary results on vanishing and non-vanishing elements.

Theorem 2.1 ([4], Theorem D)**.** *Let G be a nilpotent group. Then each non-vanishing element of G is central.*

Lemma 2.2 ([1], Lemma 2.6)**.** *Let G ba a solvable group, and let F be the Fitting subgroup of G. If G/F is abelian, then $G \setminus F \subseteq \mathrm{Van}(G)$.*

Theorem 2.3 ([4], Theorem D)**.** *Let x be a non-vanishing element of the solvable group G. Then the image of x in $G/\mathrm{F}(G)$ has 2-power order, and in particular, if x has odd order, then $x \in \mathrm{F}(G)$. In any case, if G is not nilpotent, then x lies in the penultimate term of the ascending Fitting series.*

Lemma 2.4. *Let G be a finite group and $a, b \in G$. If $(aN)^{G/N} \cap (bN)^{bN} = \emptyset$, then $a^G \cap b^G = \emptyset$.*

Proof. Assume that $a^G \cap b^G \neq \emptyset$, then $a^G = b^G$ and $a^g = b$ for some $g \in G$. Therefore, $bN = a^g N = (aN)^{gN} \in (aN)^{G/N}$ and so $(aN)^{G/N} = (bN)^{bN}$ which is a contradiction. □

By Lemma 2.4, if G has one vanishing conjugacy class, then G/N has at most one vanishing conjugacy class. Now, we ready to prove the main theorem.

Theorem 2.5. *Let G be a finite group. Then the set of vanishing elements of G is exactly a conjugacy class of G if and only if G is a Frobenius group with the abelian kernel G' of odd order and a complement of order 2.*

Proof. Suppose that $\mathrm{Van}(G)$ is the conjugacy class C of G containing x. By Main Theorem of [5] and Theorem 3.15 of [3], G is a non-abelian solvable group and so $F(G)$ is non-trivial. Set $\bar{G} = G/F(G)$, $\bar{g} = gF(G)$ for all $g \in G$, and $\bar{C}$ be the vanishing conjugacy class of $\bar{G}$ containing $\bar{x}$. By Theorem 2.3, $\bar{y}$ has 2-power order for each non-vanishing element y in G which implies that the order of each element in $\bar{C}$ is either a prime or a 2-power. Now, we break the proof into four cases.

`Case (1):` G is nilpotent.

Since G is nilpotent, then by Theorem 2.1 $G = Z(G) \cup C$. On the other hand, if $g \in C \cap G'$, then we have

$$0 = \sum_{\chi \in Irr(G)} \chi(1)\chi(g) = \sum_{\lambda \in Lin(G)} \lambda(1)\lambda(g) = |G : G'| \tag{1}$$

which is impossible and it follows that $G' \subseteq Z(G)$. Furthermore, since $|C| \leq |G'|$, we can deduce that $|Z(G)| = |G'| = |C|$ and so $|G/Z(G)| = 2$, which is a contradiction, because G is non-abelian.

`Case (2):` $\bar{G}$ does not have any vanishing conjugacy class.

By Burnside's Theorem, $\bar{G}$ is abelian and $G' \subseteq F(G)$. By Lemma 2.2, $G \setminus F \subseteq \mathrm{Van}(G) = C$ and so $|C| = |G'| = |F(G)|$. Since every irreducible character of G vanishes on C, we can write

$$|C_G(x)| = \sum_{\chi \in Irr(G)} |\chi(x)|^2 = \sum_{\lambda \in Lin(G)} |\lambda(x)|^2 = |G : G'| = 2$$

and hence $C_G(x) = \langle x \rangle$. Therefore $G = G'\langle x \rangle$ and by Problem 7.1 of [3], we can deduce G is a Frobenius group and G' an abelian subgroup of odd order, by Theorem 13.3 of [2].

`Case (3):` The order of $\bar{x}$ is a 2-power for every $\bar{x} \in \bar{C}$.

Since $\bar{G}$ is a 2-group, then $\bar{G}$ is nilpotent with exactly a vanishing conjugacy class. This is impossible, using the proof of Case (1).

`Case (4):` The order of $\bar{x}$ is $p \neq 2$ for every $\bar{x} \in \bar{C}$.

By Equation (1), each element in $\bar{G}'$ is a non-vanishing element and so $\bar{G}'$ is a 2-group. Assume that $\bar{Q}$ is the Sylow 2-subgroup of $\bar{G}$ containing $\bar{G}'$. Since each element of $\bar{G}$ has prime power order, by Problem 7.1 of [3], we conclude that $\bar{G} = \bar{Q}\bar{P}$ is a Frobenius group with kernel $\bar{Q}$, in which $\bar{P}$ is a Sylow p-subgroup of $\bar{G}$. Consequently, by Theorem 13.3(3) of [2], since the order of each element of $\bar{P}$ is p, then $\bar{P}$ is cyclic of order p. Moreover, by Theorem 13.8 of [2], $cl_{\bar{G}}(\bar{x}) = \bar{x}\bar{Q}$ for each non-trivial element $\bar{x} \in \bar{P}$. However, $\bar{G}$ has at least two vanishing conjugacy classes whenever $p \neq 2$ by Theorem 13.8 of [2], which is a contradiction in this case also. □

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