



ON THE CONVERSE OF BAER'S THEOREM FOR GENERALIZATIONS OF GROUPS WITH TRIVIAL FRATTINI SUBGROUPS

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ABSTRACT. In 2012, Guo and Gong proved that if G is a finite nonabelian group with $\Phi(G) = 1$, then $|G : Z(G)| < |G'| |U(G)|$, in which $U(G)$ is the nilpotent residual of G . We show that the assumption of finiteness of the group can be omitted. Moreover, we investigate converse of Schur and Baer's theorems for groups that can be seen as generalizations of groups with trivial Frattini subgroups and state some properties of n -isoclinism families of these groups.

1. INTRODUCTION AND PRELIMINARIES

For an arbitrary group G , Schur's classical theorem states that if $G/Z(G)$ is finite, then G' is finite. A generalization of Schur's theorem is proved by Baer. Baer showed that if $Z_n(G)$ has finite index, then $\gamma_{n+1}(G)$ is also finite in which $Z_n(G)$ denotes the n th term of the upper central series of the group G and $\gamma_{n+1}(G)$ denotes the $(n + 1)$ th term of the lower central series. Extra-special p -groups are examples that show that not only the converses of Schur's theorem is not true (for example, an infinite case) but also there is no general upper bound for the index of the center of a finite group in terms of the order of its derived subgroup. The

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groups constructed by Hall [4] show that the converse of Baer's theorem is not true. But, he proved a partial converse of Baer's theorem. Hall observed that if $\gamma_{n+1}(G)$ is finite, then $G/Z_{2n}(G)$ is bounded above in terms of the order of $\gamma_{n+1}(G)$. Podoski and Szegedy proved a stronger version of this theorem for case $n = 1$, as follows.

Theorem 1.1. [11, Theorem 1] *If G is a group and $|G' : G' \cap Z(G)| = n$, then*

$$|G : Z_2(G)| \leq n^{2 \log_2 n}.$$

Recall that the Frattini subgroup of an arbitrary group G , denoted by $\Phi(G)$, is defined to be the intersection of all the maximal subgroups, with the stipulation that it shall equal G if G should prove to have no maximal subgroups. The subgroup generated by all the normal nilpotent subgroups of a group G , denoted by $F(G)$, is called the Fitting subgroup of G .

The following result, proved in [3], shows that any group with the trivial Frattini subgroup satisfies the converse of Schur's theorem.

Theorem 1.2. [3, Theorem 1.1] *Let G be a nonabelian group such that G' be finite and $\Phi(G) = 1$. Then $|G : Z(G)| < |G'|^2$.*

Finite case of Theorem 1.2 was also independently proved in [9, Theorem A]. Recall that the nilpotent residual of G , denoted by $U(G)$, is the intersection of all normal subgroups of G such that quotients by them are nilpotent. In general $G/U(G)$ need not be a nilpotent group, since if G is a free group, then the intersection of all the terms of the lower central series of G is trivial [12, 6.1.10] and so $U(G) = 1$. Hence $G \cong G/U(G)$ is not nilpotent.

As $U(G) \leq G'$ for a group G , the following results have been achieved for a finite group G .

Theorem 1.3. [1, Theorems 1 and 2] *Let G be a finite nonabelian group such that $\Phi(G) = 1$. Then*

- *if G is of odd order, then $|U(G)| > |G : Z(G)|^{1/2}$.*
- *if G is of even order and $\pi(G) \cap (\mathcal{M} \cup \mathcal{F}) = \emptyset$, then $|U(G)| > |G : Z(G)|^{1/2}$, where $\pi(G)$, \mathcal{M} , and \mathcal{F} denote the sets of primes dividing $|G|$, Mersenne primes, and Fermat primes, respectively.*

Theorem 1.4. [2, Theorem 0.4] *Let G be a finite nonabelian group such that $\Phi(G) = 1$. Then*

$$|G : Z(G)| < |G'| |U(G)|$$

In this present work, it is first shown that the assumption of finiteness of the group stated in Theorem 1.4 can be omitted. Moreover, we give a smaller bound for the index $Z_2(G)$ of a group G with respect to the bound stated in Theorem 1.1 with an additional condition on the group G and also extend Theorem 1.4 to some groups with nontrivial Frattini subgroups.

Hall in 1939 introduced the concept of isoclinism. This concept is an equivalence relation on the class of all groups, which is weaker than isomorphism and such that all abelian groups fall into one equivalence class.

Later Hall generalized the notion of isoclinism to that of isologism, which is in fact isoclinism and n -isoclinism with respect to variety of all abelian groups and nilpotent groups of class at most n , respectively. Given a variety v , the groups $V(G)$ and $V^*(G)$ are the corresponding verbal and marginal subgroups of G , respectively. It is easy to see that if v is the variety of all nilpotent groups of class at most n , then $V(G) = \gamma_{n+1}(G)$ and $V^*(G) = Z_n(G)$ for a group G .

Definition 1.5. [6] Let $n \geq 0$ and let G and H be two groups. An n -isoclinism between G and H is a pair of isomorphisms (α, β) with $\alpha : G/Z_n(G) \rightarrow H/Z_n(H)$ and $\beta : \gamma_{n+1}(G) \rightarrow \gamma_{n+1}(H)$ such that

$$\beta([g_1, g_2, \dots, g_{n+1}]) = [h_1, h_2, \dots, h_{n+1}],$$

where $g_i \in G$ and $h_i Z_n(H) = \alpha(g_i Z_n(G))$ for each $1 \leq i \leq n + 1$.

Whenever the groups G and H are n -isoclinic, we write $G \sim_n H$.

It is clear that the concepts of isoclinism and 1-isoclinism coincide. The notion of n -isoclinism yields an equivalence relation on the class of all groups that generalizes isoclinism; each of these equivalence classes is called an n -isoclinism family. It is not difficult to deduce from Definition 1.5 that an n -isoclinism induces an $(n + 1)$ -isoclinism.

Following Hall, we call any quantity depending on a variable group and which is the same for any two groups of the same n -isoclinism class a family invariant.

Theorem 1.6. [6, Theorem 3.12] *Let G and H be groups, and suppose that there exists an n -isoclinism (α, β) between G and H . Then*

- (1) $\alpha(\gamma_{i+1}(G)Z_n(G)/Z_n(G)) = \gamma_{i+1}(H)Z_n(H)/Z_n(H)$ for all $i \geq 0$.
- (2) $\beta(\gamma_{n+1}(G) \cap Z_i(G)) = \gamma_{n+1}(H) \cap Z_i(H)$ for all $i \geq 0$.

Thus $\gamma_{n+1}(G) \cap Z_n(G)$ is a family invariant for n -isoclinism classes, and it is well known that $\gamma_{n+1}(G) \cap Z_n(G) \subseteq \Phi(G)$ for each group G .

Given a variety v and a group G , Hekster in [7, Section 8] asked does there exist a group S such that $S \sim_v G$ and $V^*(S) \subseteq V(S)$? In general, examples show that the answer to this question is not positive (see [7, Example 8.1]).

In what follows, we state some properties of groups with the property $\gamma_{n+1}(G) \cap Z_n(G) = 1$ that can be seen as a generalization of groups with trivial Frattini subgroups and give a positive answer to Hekster's question for these groups. Using mentioned results, we can obtain a bound for the converse of Baer's theorem for a group G such that $\Phi(G) \cap \gamma_{n+1}(G) = 1$ for some natural number n .

2. ON THE HALL'S THEOREM

In this section, it is shown that the assumption of finiteness of the group stated in Theorem 1.4 can be omitted. Moreover, we give an upper bound for the index $Z_2(G)$ of some groups. These groups can consider as generalizations of groups with trivial Frattini subgroups. It is necessary to note that if $\Phi(G) = 1$, then $Z(G) = Z_2(G)$ and $\gamma_2(G/Z(G)) \cong \gamma_2(G)$ for a group G . Now for the reminder, we need the following lemmas.

Lemma 2.1. [3, Lemma 2.1] *Let G be a group such that $\Phi(G) = 1$. Then*

$$Z(G/Z(G)) = 1 \text{ and } \Phi(G/Z(G)) = 1.$$

Lemma 2.2. [6, Lemma 3.5] *Let G be a group, $N \trianglelefteq G$, and $n \geq 0$. Then*

$$G/N \sim_n G/(N \cap \gamma_{n+1}(G)).$$

Lemma 2.3. *Let G be a group such that $\gamma_{n+1}(G)$ be finite for some natural number n and let N be a normal subgroup of G . Then $U(G)$ is the smallest term in the lower central series of G and so $G/U(G)$ is nilpotent. Moreover, $U(G/N) = U(G)N/N$.*

Proof. Let G be a group such that $\gamma_{n+1}(G)$ be finite for some natural number n . Then the lower central series of G terminates after a finite number of steps. Denote the smallest term in the lower central series of G by $\gamma_k(G)$. Let M be an arbitrary normal subgroup of G such that G/M is nilpotent. Thus $\gamma_i(G) \subseteq M$ for some natural number i and so $\gamma_k(G) \subseteq \gamma_i(G) \subseteq M$. It implies that $\gamma_k(G) \subseteq U(G)$. In addition, $G/\gamma_k(G)$ is nilpotent and hence $U(G) \subseteq \gamma_k(G)$. It follows that $U(G) = \gamma_k(G)$ is the smallest term in the lower central series of G . Now, $\gamma_{n+1}(G/N)$ is finite, since $\gamma_{n+1}(G/N) \cong \gamma_{n+1}(G)/(\gamma_{n+1}(G) \cap N)$ and $\gamma_{n+1}(G)$ is finite. Similarly, $U(G/N)$ is the smallest term in the lower central series of G/N and the result follows. \square

Lemma 2.4. *Let G and H be two groups such that $G \sim_n H$ and $\gamma_{n+1}(G)$ be finite for some natural number n . Then $|U(G)| = |U(H)|$.*

Proof. It follows from Lemma 2.3 and this fact that an n -isoclinism induces an $(n+1)$ -isoclinism. \square

Theorem 2.5. *Let G be a nonabelian group such that $G' \cap \Phi(G) = G' \cap Z(G)$ and $\gamma_2(G/Z(G))$ is finite. Then*

$$|G : Z_2(G)| < |\gamma_2(G/Z(G))||U(G/Z(G))|.$$

Proof. Let G be a group such that $G' \cap \Phi(G) = G' \cap Z(G)$. Then $G/\Phi(G) \sim G/Z(G)$, by Lemma 2.2. Put $\bar{G} = G/\Phi(G)$. It implies that $\gamma_2(\bar{G}) \cong \gamma_2(G/Z(G))$ and $\bar{G}/Z(\bar{G}) \cong G/Z(G)/Z(G/Z(G)) \cong G/Z_2(G)$. Hence $\gamma_2(\bar{G})$ is also finite and so $\bar{G}/Z_2(\bar{G})$ is finite, by [4, Theorem 2]. Since $\Phi(\bar{G}) = 1$, we have $\bar{G}' \cap Z(\bar{G}) = 1$. It implies that $Z(\bar{G}/Z(\bar{G})) = 1$ and

that $Z(\bar{G}) = Z_2(\bar{G})$. Lemma 2.1 yields $\Phi(\bar{G}/Z(\bar{G})) = 1$. It is sufficient that we get the order of $\bar{G}/Z(\bar{G})$. Applying Theorem 1.4 for the finite group $\bar{G}/Z(\bar{G})$, we obtain

$$|\bar{G}/Z(\bar{G})| < |\gamma_2(\bar{G}/Z(\bar{G}))||U(\bar{G}/Z(\bar{G}))| \leq |\gamma_2(\bar{G})||U(\bar{G})|,$$

because $Z(\bar{G}/Z(\bar{G})) = 1$, by Lemma 2.1, and $|U(\bar{G}/Z(\bar{G}))| = |U(\bar{G})/(U(\bar{G}) \cap Z(\bar{G}))| \leq |U(\bar{G})|$, by Lemma 2.3. Now, the isoclinism between \bar{G} and $G/Z(G)$ implies that for each natural number k , we have $\gamma_{k+1}(G/Z(G)) \cong \gamma_{k+1}(\bar{G})$ since $\bar{G} \sim_k G/Z(G)$. Hence, $|\gamma_2(\bar{G})| = |\gamma_2(G/Z(G))|$ and $|U(\bar{G})| = |U(G/Z(G))|$, by Lemma 2.3. The result follows. \square

As a corollary of Theorem 2.5, we show that Theorem 1.4 holds for any nonabelian group (not necessarily finite).

Corollary 2.6. *Let G be an arbitrary nonabelian group such that $\Phi(G) = 1$ and G' be finite. Then $|G : Z(G)| < |G'| |U(G)|$.*

Proof. Let $\Phi(G) = 1$. Then $Z_2(G) = Z(G)$, $\gamma_2(G/Z(G)) \cong \gamma_2(G)$, $U(G/Z(G)) \cong U(G)$ and $G' \cap \Phi(G) = G' \cap Z(G) = 1$, since $U(G) \cap Z(G) \subseteq G' \cap Z(G) \subseteq \Phi(G)$. Apply Theorem 2.5 and the result follows. \square

The proof of Corollary 2.6 shows that the triviality of Frattini subgroup implies that $G' \cap Z(G) = G' \cap \Phi(G) = 1$. Nevertheless, there are groups G with nontrivial Frattini subgroups whereas $G' \cap Z(G) = G' \cap \Phi(G)$ and the recent group might be trivial or nontrivial. In other words, Theorem 2.5 includes the bigger class of groups with respect to the class of groups with trivial Frattini subgroups.

Example 2.7. Let $G = D_{24} = \langle a, b | a^{12} = b^2 = 1, b^{-1}ab = a^{-1} \rangle$, the dihedral group of order 24. Then $Z(G) = \langle a^6 \rangle \cong \mathbb{Z}_2$, $G' = \langle a^2 \rangle \cong \mathbb{Z}_6$, $\Phi(G) \cong \mathbb{Z}_2$ and so $G' \cap Z(G) = G' \cap \Phi(G) = \langle a^6 \rangle \neq 1$.

On the other hand, Theorem 2.5 gives a smaller bound with respect to the bound stated in Theorem 1.1 with an additional condition on the group.

3. SOME PROPERTIES OF GROUPS G WITH $\gamma_{n+1}(G) \cap Z_n(G) = 1$

It is well known that $\gamma_{n+1}(G) \cap Z_n(G) \subseteq \Phi(G)$ for each group G (see [6, Proposition 2.6]). In what follows, we intend to state some properties of groups G such that $\gamma_{n+1}(G) \cap Z_n(G) = 1$ in the special case $\Phi(G) = 1$. The following lemma is vital in our results of this paper.

Lemma 3.1. [6, Theorem 2.3 and Lemma 2.5] *Let G be a group and let $N \trianglelefteq G$. Then*

- (1) *if $N \cap Z(G) = 1$, then $N \cap Z_n(G) = 1$ for all $n \geq 1$.*
- (2) *if $N \cap \gamma_{n+1}(G) = 1$ for some $n \geq 1$, then $N \subseteq Z_n(G)$.*

If G is any group and α is an ordinal, then the terms $\zeta_\alpha(G)$ of the upper central series of G are defined by the usual rules

$$\zeta_0(G) = 1 \text{ and } \zeta_{\alpha+1}(G)/\zeta_\alpha(G) = \zeta(G/\zeta_\alpha(G))$$

together with the completeness condition

$$\zeta_\lambda(G) = \cup_{\alpha < \lambda} \zeta_\alpha(G)$$

where λ is a limit ordinal. Since the cardinality of G cannot be exceeded, there is an ordinal β such that $\zeta_\beta(G) = \zeta_{\beta+1}(G) = \dots$ in which a terminal subgroup is called the hypercenter of G .

Theorem 3.2. *Let G be a group such that $\gamma_{n+1}(G) \cap Z_n(G) = 1$. Then $Z_n(G)$ is the hypercenter of G .*

Proof. Let $\gamma_{n+1}(G) \cap Z_n(G) = 1$. Then $\gamma_{n+1}(G) \cap Z(G) \subseteq \gamma_{n+1}(G) \cap Z_n(G) = 1$. Since $\gamma_{n+1}(G) \cap Z(G) = 1$, we have $\gamma_{n+1}(G) \cap Z_{n+i}(G) = 1$ for each $i \geq 1$, by the part (1) of Lemma 3.1. Thus $Z_{n+i}(G) \subseteq Z_n(G)$ for each $i \geq 1$, by the part (2) of Lemma 3.1. Hence $Z_n(G)$ is the hypercenter of G . \square

In what follows, it is shown that each $(n+i)$ -isoclinic family of groups G with $\gamma_{n+1}(G) \cap Z_n(G) = 1$ is contained in an n -isoclinic family when $n, i \geq 1$.

Theorem 3.3. *Let G_1 and G_2 be two groups such that $\gamma_{n+1}(G_1) \cap Z_n(G_1) = \gamma_{n+1}(G_2) \cap Z_n(G_2) = 1$. Then $G_1 \sim_{n+1} G_2$ if and only if $G_1 \sim_n G_2$.*

Proof. If $G_1 \sim_{n+1} G_2$, then $G_1/Z_{n+1}(G_1) \cong G_2/Z_{n+1}(G_2)$ by an isomorphism α . Thus $G_1/Z_n(G_1) \cong G_2/Z_n(G_2)$, by Theorem 3.2. Hence α induces an isomorphism from $\gamma_{n+1}(G_1/Z_n(G_1))$ onto $\gamma_{n+1}(G_2/Z_n(G_2))$ and so $\gamma_{n+1}(G_1) \cong \gamma_{n+1}(G_2)$. The result follows. \square

Let G be a group and let $H \leq G$. Hekster proved that $H \sim_n HZ_n(G)$; see [6, Lemma 3.5]. In particular, if $G = HZ_n(G)$, then $G \sim_n H$. Conversely, if $G/Z_n(G)$ is finite and $G \sim_n H$, then $G = HZ_n(G)$.

Theorem 3.4. *Let G be a group such that $\gamma_{n+1}(G) \cap Z_n(G) = 1$, $H \leq G$ and $\gamma_{n+1}(G)$ be finite. Then $G \sim_{n+1} H$ if and only if $G \sim_n H$.*

Proof. The finiteness of $\gamma_{n+1}(G)$ implies the finiteness of $G/Z_{2n}(G)$. Since $Z_n(G)$ is the hypercenter of G , we have $Z_{2n}(G) = Z_{n+1}(G) = Z_n(G)$. Hence $G/Z_{n+1}(G)$ is finite. Let $G \sim_{n+1} H$. It implies $G = HZ_{n+1}(G) = HZ_n(G)$ and as $G \sim_n H$. The result follows. \square

Now, we intend to give a positive answer to Hekster's question in [7] for variety of all nilpotent groups of class at most n for some groups.

Theorem 3.5. [6, Lemma 6.1] *Let G be a group and let $n \geq 1$. Then there exists a group T such that $G \sim_n T$ and $Z(T) \cap \gamma_n(T) \leq \gamma_{n+1}(T)$.*

Theorem 3.6. *Let G be a group such that $\gamma_{n+1}(G) \cap Z_n(G) = 1$, then there exists a unique group S such that $S \sim_n G$ and $Z_n(S) = 1$. In particular, $Z_n(S) \subseteq \gamma_{n+1}(S)$.*

Proof. Let G be a group such that $\gamma_{n+1}(G) \cap Z_n(G) = 1$. There exists a group T_1 such that $G \sim_n T_1$ and $Z(T_1) \cap \gamma_n(T_1) \leq \gamma_{n+1}(T_1)$, by Theorem 3.5. Thus $Z(T_1) \cap \gamma_n(T_1) \leq \gamma_{n+1}(T_1) \cap Z_n(T_1)$. But, we have $\gamma_{n+1}(G) \cap Z_n(G) \cong \gamma_{n+1}(T_1) \cap Z_n(T_1)$, since $\gamma_{n+1}(G) \cap Z_n(G)$ is a family invariant for n -isoclinism classes by Theorem 1.6. Hence $Z(T_1) \cap \gamma_n(T_1) = 1$. By part (1) of Theorem 3.1, we have $Z_n(T_1) \cap \gamma_n(T_1) = 1$, and so $Z_n(T_1) \subseteq Z_{n-1}(T_1)$, by part (2) of Theorem 3.1. It implies that $Z_{n-1}(T_1) \cap \gamma_n(T_1) = 1$. Again, there exists a group T_2 such that $T_1 \sim_{n-1} T_2$. We continue this process and we conclude that there exist groups T_{n-1} and T_n such that $T_{n-1} \sim_1 T_n$ and $Z(T_n) \cap T_n \subseteq \gamma_2(T_n)$. Hence $Z(T_n) = Z(T_n) \cap \gamma_2(T_n) \cong Z(T_{n-1}) \cap \gamma_2(T_{n-1}) = 1$. Put $T_n = S$ and so we have $Z_n(S) = 1$. Now, let S_1 be another group such that $S_1 \sim_n G$ and $Z_n(S_1) \subseteq \gamma_{n+1}(S_1)$. Hence $S \sim_n S_1$ and $Z_n(S_1) \cap \gamma_{n+1}(S_1) \cong Z_n(S) \cap \gamma_{n+1}(S)$, by Theorem 1.6. Thus $Z_n(S_1) \cong Z_n(S) = 1$ and $S \cong S_1$. The result follows. \square

This theorem asserts that the nonisomorphic structures $G/Z_n(G)$ of groups G with the above property classify the set of these groups up to n -isoclinism.

4. ON THE CONVERSE OF BAER'S THEOREM

Theorem 4.1. *Let G be a group such that $\gamma_{n+1}(G) \cap Z_n(G) = 1$ and $\gamma_{n+i}(G)$ is finite for some positive integers n and i . Then $G/Z_n(G)$ is finite.*

Proof. If $\gamma_{n+i}(G)$ is finite for some positive integers n and i , then $G/Z_{2(n+i-1)}(G)$ is finite, by [4, Theorem 2]. Theorem 3.2 implies that $Z_{2(n+i-1)}(G) = Z_n(G)$ and so the result follows. \square

Thus, the converse of Baer's theorem holds for these groups. Moreover, We can also obtain an upper bound for the order of $G/Z_n(G)$ in terms of the order of $\gamma_{n+i}(G)$, using the following theorem.

Theorem 4.2. [5, Main Theorem] *Let G be a group, let $\gamma_{n+1}(G)$ be finite, and let $G/Z_n(G)$ be finitely generated. Then*

$$\left| \frac{G}{Z_n(G)} \right| \leq |\gamma_{n+1}(G)|^{d(G/Z_n(G))^n}.$$

Corollary 4.3. *Let G be a group such that $\gamma_{n+1}(G) \cap Z_n(G) = 1$ and $\gamma_{n+i}(G)$ be finite for some positive integers n and i . Then $|G/Z_n(G)| \leq |\gamma_{n+i}(G)|^{d(G/Z_n(G))^{n+i-1}}$.*

Theorem 4.4. *Let G and H be two groups such that $G \sim_n H$, $\gamma_{n+1}(G) \cap Z_n(G) = 1$ and $\gamma_{n+i}(G)$ be finite for some positive integers i and n . Then $H/Z_n(H)$ is finite. In particular, H satisfies the converse of Baer's theorem.*

Proof. It follows from Theorems 1.6 and 4.1. □

It is obvious that $\gamma_{n+1}(G) \cap Z_n(G) \subseteq \Phi(G) \cap \gamma_{n+1}(G)$ for each group G . We will also obtain a bound smaller than the one stated in Theorem 4.2 for groups G with the property $\Phi(G) \cap \gamma_{n+1}(G) = 1$. The following lemma plays an essential role for our aim.

For the proof of the following lemma, we recall that the solvable residual of G , denoted by $S(G)$, is the intersection of all normal subgroups of G such that quotients by them are solvable.

Lemma 4.5. *Let G be a finite nonabelian group such that $\Phi(G) = 1$. Then $|G/Z(G)| < |U(G)|^3$.*

Proof. The proof can be divided into two parts according to whether $Z(G) = 1$ or not (there are certain similarities between the current proof and the proof of [2, Theorem 0.4]).

- If $Z(G) = 1$, then $F(G) \subseteq U(G)$, by [1, Lemma 5]. Now we proceed with the two cases.
 - (1) Assume that $F(G) = 1$. Since $G/U(G)$ is nilpotent for the finite group G , we have $S(G) \subseteq U(G)$, where $S(G)$ is the solvable residual of G . By [9, Proposition 2.4], $|G| < |S(G)|^2 \leq |U(G)|^2$.
 - (2) Assume that $F(G) \neq 1$. We apply induction on $|G|$. Put

$$F_0/F = \Phi(G/F), \quad F_1/F_0 = Z(G/F_0)$$

Noticing that $\Phi(G/F_0) = 1$, we have by [8, Lemma 7] that

$$Z(G/F_1) = \Phi(G/F_1) = 1.$$

Since $(F_1/F)/\Phi(G/F)$ is abelian, we see F_1 is metanilpotent by [12, Theorem 5.2.15(i)]. Suppose firstly that $F_1 = G$. Then G is metanilpotent and $|G| < |F(G)|^3$, by [8, Theorem C]. It follows from $F(G) \subseteq U(G)$ that $|G| < |U(G)|^3$. Now we assume that $F_1 \subset G$. Applying the inductive hypothesis to the factor group G/F_1 , we obtain that $|G/F_1| < |U(G/F_1)|^3 = |U(G)|^3/|U(G) \cap F_1|^3$. On the other hand, F_1 is meta-nilpotent, $\Phi(F_1) = 1$, and $F(G) = F(F_1)$, then we deduce by [8, Theorem C], that $|F_1| < |F(F_1)|^3 < |F_1 \cap U(G)|^3$. Thus we can conclude that $|G| < (|U(G)|^3/|U(G) \cap F_1|^3)|F_1| \leq (|U(G)|^3/|U(G) \cap F_1|^3)|F_1 \cap U(G)|^3$ and so $|G| < |U(G)|^3$.

- If $Z(G) \neq 1$, then by [10, Theorems 2.1 and 2.5], $G = H \times Z(G)$. Now H satisfies $\Phi(H) = Z(H) = 1$ and $U(H) = U(G)$. By the above part we see $|G/Z(G)| = |H| < |U(H)|^3 = |U(G)|^3$. The result is proved.

□

We also need the following theorem which comes from [6, Theorem 7.7]

Theorem 4.6. *Let G be a group. The following properties are equivalent.*

- (1) *The group $G/Z_n(G)$ is finite*
- (2) *The group G is n -isoclinic to a finite group.*

Theorem 4.7. *Let G be a group such that $\Phi(G) \cap \gamma_{n+1}(G) = 1$ and $\gamma_{n+1}(G)$ be finite for some natural number n . Then $|G/Z_n(G)| \leq |\gamma_{n+1}(G)|^3$.*

Proof. Let G be a group such that $\Phi(G) \cap \gamma_{n+1}(G) = 1$. Then $Z_n(G) \cap \gamma_{n+1}(G) = 1$, since $Z_n(G) \cap \gamma_{n+1}(G) \subseteq \Phi(G) \cap \gamma_{n+1}(G)$. Theorem 4.1 implies that $G/Z_n(G)$ is finite. Hence, there exists a finite group S such that $S \sim_n G$ and $Z_n(S) \cap \gamma_{n+1}(S) = 1$, by Theorems 4.6 and 1.6, respectively. Now, there is also a group S_1 such that $Z_n(S_1) = 1$ and $S_1 \sim_n S \sim_n G$, by Theorem 3.6. Hence $G/Z_n(G) \cong S_1$ and so $\Phi(G/Z_n(G)) \cong \Phi(S_1)$. We claim that $\Phi(G/Z_n(G)) = 1$. Since $\Phi(G) \cap \gamma_{n+1}(G) = 1$, we have $G \sim_n G/\Phi(G)$ by Theorem 2.2. Thus $G/Z_n(G) \cong (G/\Phi(G))/Z_n(G/\Phi(G))$. But $Z_n(G/\Phi(G)) = Z(G/\Phi(G))$ since $\Phi(G/\Phi(G)) = 1$. Hence $\Phi(G/Z_n(G)) \cong \Phi((G/\Phi(G))/Z_n(G/\Phi(G))) = 1$, by Theorem 2.1. we can deduced that $|G/Z_n(G)| = |S_1| < |U(S_1)|^3 \leq |U(G)|^3 \leq |\gamma_{n+1}(G)|^3$. The result is proved. □

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