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CHARACTERIZING SOME GROUPS WITH NILPOTENT DERIVED SUBGROUP

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Abstract. In this paper, groups with trivial intersection between Frattini and derived subgroups are considered. First, some structural properties of these groups are given in an important special case. Then, some family invariants of each n-isoclinism family of such groups are stated. In particular, an explicit bound for the order of each center factor group in terms of the order of its derived subgroup is also provided.

1. Introduction and Preliminaries

The classification of all finite groups having cyclic Sylow subgroups is done by Holder, Burnside and Zassenhaus. They have proved the following theorem.

Theorem 1.1. [8, 10.1.10] If G is a finite group all of whose Sylow subgroups are cyclic, then G has a presentation

$$G = \langle a, b | a^m = 1 = b^n, b^{-1}ab = a^r \rangle, \tag{1.1}$$

where $r^n \equiv 1 (mod\ m),\ m$ is odd, $0 \leq r < m$, and m and $n(r-1)$ are coprime.

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Conversely in a group with such a presentation all Sylow subgroups are cyclic.

This means that a finite group whose Sylow subgroups are cyclic is an extension of one cyclic group by another.

After a while, some authors focussed on finite soluble groups whose Sylow subgroups are abelian. These groups are called A-groups and were first studied by P. Hall [4]. Interest in A-groups also broadened due to an important relationship to varieties of groups discussed in [7]. Some results were obtained on this subject is as follows. Every subgroup, quotient group, and direct product of A-groups are A-groups. A finite nilpotent group is an A-group if and only if it is abelian. Moreover, Walter [10] proved that a non-abelian finite simple group is an A-group if and only if it is isomorphic to the first Janko group or to $PSL(2,q)$ where $q > 3$ and either $q = 2^n$ or $q \equiv 3,\ 5\ mod\ 8$. The important theorem on A-groups, which was stated without proof by P. Hall[4], is proved by Taunt[9].

Theorem 1.1. *Let G be an A-group. Then the meet of the center and the derived subgroup of G is the identity.*

Although Taunt [9] showed that $G' \cap Z(G) = 1$, for each A-group G, but the converse of Taunt's statement is not true in general. A counterexample is the simple group $PSL(2,17)$. Note that, the Frattini subgroup of $PSL(2,17)$ is trivial. Therefore, the meet of the center and the derived subgroup of $PSL(2,17)$ is trivial whereas it is not an A-group [10]. Hence, one of the problems that the paper follows is finding a set of groups satisfying the converse of Taunt's statement. More precisely, since, $\varphi(G) \cap G' = 1$ implies that $G' \cap Z(G) = 1$, we will focus on the family of groups in which $\varphi(G) \cap G' = 1$, for each group G, and then try to obtain some structural properties of A-groups in this family of groups. In particular, an explicit bound for the order of each center factor group in terms of the order of its derived subgroup is also provided.

We begin by introducing a slight generalization of the concept of upper central series of a group and some other concepts which are needed later.

Let G be any group and α be an ordinal number. The terms $\zeta_\alpha G$ of the upper central series of G are defined by the usual rules

$$\zeta_0(G) = 1 \quad \text{and} \quad \zeta_{\alpha+1}G/\zeta_\alpha G = \zeta(G/\zeta_\alpha G)$$

together with the completeness condition

$$\zeta_\lambda G = \cup_{\alpha<\lambda}\zeta_\alpha G$$

where λ is a limit ordinal. Since the cardinality of G cannot be exceeded, there is an ordinal β such that $\zeta_\beta G = \zeta_{\beta+1}G = \dots$, a terminal subgroup called the hypercenter of G. It is sometimes convenient to call $\zeta_\alpha G$ the α-center of G.

Now, we recall that the notion of n-isoclinism and some related results [5].

The equivalence relation n-isoclinism partitions the class of all groups into families. According to this notion, two groups G and H are n-isoclinic if there exist isomorphisms

$$\alpha : \frac{G}{Z_n(G)} \to \frac{H}{Z_n(H)} \quad \text{and} \quad \beta : \gamma_{n+1}(G) \to \gamma_{n+1}(H),$$

such that $\beta([g_1, g_2, \ldots, g_{n+1}]) = [h_1, h_2, \ldots, h_{n+1}]$, where $g_i \in G$, $h_i Z_n(H) = \alpha(g_i Z_n(G))$, for each $1 \leq i \leq n+1$. In this case, we write $G \sim_n H$. 1-isoclinic groups G and H are briefly called isoclinic and shown by $G \sim H$.

Any quantity depending on a variable group and which is the same for any two groups of the same family is called family invariant. Thus the derived subgroup, the central quotient group and also the intersection of derived subgroup and center subgroup are some family invariants in each isoclinism family. A group S in which $Z(S) \subseteq \gamma_{n+1}(S)$ is called an n-stem group and 1-stem group is briefly called stem group.

One of the problems that we like to follow is given an explicit bound for the order of each center factor group in terms of the order of its derived subgroup. There are numerous interesting bound which obtained by some authors. A famous Theorem of P. Hall [8, p.423] says that the factor group $G/Z_2(G)$ is finite, whenever G is an arbitrary group with finite derived subgroup. Therefore, the finiteness of G' implies that the finiteness of G, when $Z_2(G)$ is trivial. Herzog *et al.*[6] assumed a stronger condition, namely, the Frattini subgroup and the center of G is trivial and proved that not only such group G is finite but also there exists an explicit bound for the order of G in terms of the order of G'. In fact, they can prove that $|G| \leq |G'|^3$ for each group G with finite derived subgroup in which $\varphi(G) = Z(G) = 1$. Furthermore, they conjectured that $|G| \leq |G'|^2$ for such groups. Halasi and Podoski [2] proved this conjecture and extended it by showing the following result.

Theorem 1.2. Let G be a group such that G' is finite and $\varphi(G) = 1$. Then

$$|G/Z(G)| \leq |G'|^2.$$

Equality holds if and only if G is abelian.

We are now in a position to state the major results of this paper.

Theorem A. *Let G be a finite group such that G' is nilpotent. Then G is an A-group and $\varphi(G)$ is central if and only if $\varphi(G) \cap G' = 1$.*

Theorem B. *Let G be a group such that $\varphi(G) \cap G' = 1$ and G' be finite nilpotent. Then*

(i) *$Z(G)$ is the intersection of all non-normal maximal subgroups of G,*

(ii) *$Z(G)$ is the hypercenter of G and $G/Z(G)$ is finite and also $\varphi(G/Z(G)) = 1$,*

(iii) *$G = HG'$, where G' is elementary abelian, H is abelian, $H \cap G' = 1$, and $Z(G) \subseteq H$,*

(iv) *every stem group that is isoclinic to G is an A-group.*

The next result generalizes Theorem 1.2 and the main result of [1].

Theorem C. *Let G be a non abelian group such that G' is finite and $\varphi(G) \cap G' = 1$. If $G^{\mathcal{N}}$ is the smallest normal subgroup of G such that $G/G^{\mathcal{N}}$ is nilpotent, then*

$$|G'| \leq |G/Z(G)| < |G'||G^{\mathcal{N}}|.$$

2. Main results

In 1904, Schur proved that the finiteness of $G/Z(G)$ for each group G implies that the finiteness of G'. The converse of this statement is not true in general. Infinite extra-special p-groups are desirable counterexamples. Now, we give a set of groups satisfying the converse of Schur's theorem.

Lemma 2.1. Let G be a group such that $\varphi(G) \cap G' = 1$. Then $Z(G)$ is the hypercenter of G. Moreover, if G' is finite, then $G/Z(G)$ is finite.

Proof. It is easy to see that $Z(G) \cap G'$ is trivial, because of $\varphi(G) \cap G' = 1$. Let $x \in Z_2(G)$. Then for each $g \in G$ we will have $[x, g] \in Z(G) \cap G'$, and so $Z_2(G) = Z(G)$. Now, one can obtain the result by P. Hall's Theorem [8, p.423]. □

Following the method used in the process of the proof of [2, Lemma 2.1], one can obtain the following result.

Proposition 2.2. Let G be a group such that $\varphi(G) \cap G' = 1$. Then $Z(G)$ is the intersection of all non-normal maximal subgroups of G and hence $\varphi(G) \subseteq Z(G)$. Moreover, $Z(G/Z(G))$ is trivial.

Proof. Let D be the intersection of all non-normal maximal subgroups of G. It is clear that D is a normal subgroup and every non-normal maximal subgroup contains $Z(G)$. Furthermore, $D \cap G' \subseteq \varphi(G) \cap G'$ because of $G' \subseteq M$ for each normal maximal subgroup M. Hence $D = Z(G)$. By Lemma 2.1, we conclude that $Z(G/Z(G)) = 1$, and the proof is complete. □

Notice that the condition $\varphi(G) \cap G' = 1$ for a group G is equivalent to $D \cap G' = 1$, where D is the intersection of all non-normal maximal subgroups of G.

The following examples give a group G such that $\varphi(G) \cap G' = 1$ whereas $\varphi(G) \neq 1$.

Example 2.3. Let p be a prime number, S_n the symmetric group of degree n and $G = S_n \times \mathbb{Z}_{p^t}$ such that $n \geq 3$ and $t \geqslant 2$. It is easy to see that $\varphi(G) = 1 \times \mathbb{Z}_{p^{t-1}}$ and $G' = A_n \times 1$.

Other example is provided by using Theorem 1.1.

Example 2.4. Let $G = \langle a, b | a^5 = 1 = b^{32}, b^{-1}ab = a^2 \rangle$. It is easy to see that $\langle b \rangle$ is a maximal subgroup of G. Since $G' = \langle a \rangle$ and $\varphi(G) \subseteq \langle b \rangle$, we have $\varphi(G) \cap G' = 1$. Now, let $K = \langle a \rangle \times \langle b^4 \rangle$. Since K is a normal subgroup of G and $\varphi(K) \cong \mathbb{Z}_4$, so $\varphi(G) \neq 1$.

Here, some structural properties of the derived subgroup of a group G such that $\varphi(G) \cap G' = 1$ are given. First, recall that a finite group G is said to be elementary if and only if each subgroup of G has trivial Frattini subgroup. Moreover, a group satisfies the normalizer condition if each proper subgroup is smaller than its normalizer. It was shown that for finite groups the normalizer condition is equivalent to nilpotency.

Theorem 2.5. Let G be a nilpotent group. Then

(i) every subgroup of G is subnormal,

(ii) G satisfies the normalizer condition,

(iii) every maximal subgroup of G is normal,

(iv) $G' \subseteq \varphi(G)$.

Proof. The proof is similar to the proof of [8, 5.2.4]. □

Lemma 2.6. Let G be a group such that G' is nilpotent and $\varphi(G) \cap G' = 1$. Then G' is abelian. In particular, if G' is finite, then G' is an elementary abelian group.

Proof. Theorem 2.5 (iv) implies $G'' \subseteq \varphi(G')$. Therefore G'' is a subgroup of $\varphi(G) \cap G'$, and hence G' is abelian. Now, let H be an arbitrary subgroup of finite group G'. Since G' is abelian, we have H is a normal subgroup and then $\varphi(H) \subseteq \varphi(G) \cap G'$. Therefore, $\varphi(H)$ is trivial and the result follows. □

Now, using Lemma 2.6, one can see that a group G in which $\varphi(G) \cap G' = 1$ is metabelian if G' is a nilpotent group.

It is easy to see that, if H is a subgroup of G such that $G = H\gamma_2(G)$, then $G = H\gamma_{i+1}(G)$ for all $i \geq 1$. A proof can be found in [5, Theorem 2.3]. This fact helps us to state the following proposition.

Proposition 2.7. Let G be a finite group such that $\varphi(G) \cap G' = 1$ and G' be a nilpotent group. Then G' is the smallest term of the lower central series of G.

Proof. By Lemma 2.6, G' is an elementary abelian group. Using [8, 5.2.13], there exists a subgroup H in G such that $G = HG'$ and $H \cap G' = 1$. On the other hand, we have $G = H\gamma_{i+1}(G)$ for all $i \geq 1$, by using McLain's result [5, Theorem 2.3]. Now, since $H \cap \gamma_{i+1}(G) = 1$ for all $i \geq 1$, we will have $G' = \gamma_{i+1}(G)$, as desirable. □

Here, we can show that a finite group G may be described by two of its abelian subgroups whenever $\varphi(G) \cap G' = 1$.

Theorem 2.8. Let G be a finite group such that $\varphi(G) \cap G' = 1$ and G' be nilpotent. Then $G = HG'$, where $H \cap G' = 1$, G' is elementary abelian and H is abelian. Moreover, H contains the center of G.

Proof. By Lemma 2.6, G' is an elementary abelian group. As you see in the process of the proof of Proposition 2.7, $G = HG'$ in which H is a subgroup of G and $H \cap G' = 1$. On the other hand, H' is a subgroup of $H \cap G'$, and hence H is abelian. Now suppose that z is an arbitrary element of $Z(G)$. Then $z = hg' \in Z(G)$, for some $g' \in G'$ and $h \in H$. Since $[hg', h_1] = [g', h_1] = 1$ for all $h_1 \in H$, we will have $g' \in Z(G)$. Therefore $Z(G) \subseteq H$, because of $g' \in G' \cap Z(G) = 1$. □

Let G be a group such that $\varphi(G) \cap G' = 1$. Note that, by Proposition 2.2, $\varphi(G) \subseteq Z(G)$, and so $\varphi(G)$ is abelian. Moreover, if Fitting subgroup $F(G)$ of G, the unique maximal nilpotent normal subgroup of G, is nilpotent, then it is not difficult to check that $F(G)$ is also abelian.

Theorem 2.9. Let G be a finite A-group such that G' is nilpotent. Then $\varphi(G) = (\varphi(G) \cap Z(G)) \times (\varphi(G) \cap G')$ and $F(G) = Z(G) \times G'$. Furthermore, $\varphi(G) \cap G' = 1$ if and only if $\varphi(G)$ is central.

Proof. We know [9, Theorem 5.3] that a normal abelian subgroup N of an A-group G can be written as $N = N_0 \times N_1 \times \ldots \times N_{n-1}$ where $N_i = N \cap Z(G^{(i)})$ and $G^{(n)} = 1$. Moreover, $\varphi(G)$, G' and $F(G)$ are finite nilpotent A-groups. So, one can conclude that they are abelian. Therefore, $Z(G') = G'$, $n = 2$ and also we will have $\varphi(G) = (\varphi(G) \cap Z(G)) \times (\varphi(G) \cap G')$ and $F(G) = (F(G) \cap Z(G)) \times (F(G) \cap G') = Z(G) \times G'$. This completes the proof. □

Theorem 2.10. Let q be a prime number and G be a finite group such that $\varphi(G) \cap G' = 1$. If G' is a q-group, then G is an A-group.

Proof. Using Theorem 2.8, $G = HG'$, where $H \cap G' = 1$, H and G' are abelian. Let P_G be a Sylow p-subgroup of G such that $p \neq q$. Since $G' \cap P_G = 1$, we have $P_G' = 1$. So P_G is abelian. Now, let $p = q$ and P_H be a Sylow p-subgroup of H. Then $H_0 = P_H G'$ is a normal Sylow p-subgroup of G because of $P_H \cap G' = 1$. Therefore $H_0' \subseteq \varphi(H_0) \cap H_0' = \varphi(G) \cap G' = 1$. Thus H_0 is abelian and the result follows. □

It is known [8, 1.6.18] that each Sylow subgroup of a normal subgroup N of G, P_N, is as $N \cap P$ in which P is a Sylow subgroup of G.

Theorem 2.11. Let G be a finite group such that $\varphi(G) \cap G' = 1$ and G' be nilpotent. Then G is an A-group.

Proof. Let P_G be a Sylow p-subgroup of G. If $p \nmid |G'|$, then as seen in the process of the proof of Theorem 2.10, P_G is abelian. Now, let $p \mid |G'|$. By Lemma 2.6, we have G' is a finite elementary abelian group. Since all Sylow subgroups of G' are characteristic in G', we will have every Sylow subgroup of G' is normal in G. Moreover, G' can be written as a direct product of its Sylow subgroups. So, there is a normal subgroup K of G in G' such that $G' = K \times P_{G'}$. In what follows, we will try to show that the factor group G/K is an A-group. Since $(G/K)' \cong P_{G'}$, we have $P_{\varphi(G/K)} = \cap_{gK \in G/K}(\varphi((P_G K/K)^{gK}))$. Therefore, $P_{\varphi(G/K)} = (\cap_{g \in G}\varphi((P_G)^g))K/K = P_{\varphi(G)}K/K$. Now, since $(G/K)'$ is a p-group, we get

$$\varphi(G/K) \cap (G/K)' \subseteq P_{\varphi(G/K)} \cap (G/K)' = (P_{\varphi(G)} \cap G')K/K = 1,$$

because of $(P_{\varphi(G)} \cap G')K/K \subseteq (\varphi(G) \cap G')K/K$. Therefore, G/K is an A-group, using Theorem 2.10. On the other hand, K is a p' subgroup, so $P_G \cap K = 1$. Thus P_G is abelian as required. □

Proof of Theorem A.
The result holds by Theorems 2.9 and 2.11.

Lemma 2.12. Let G be a group such that $\varphi(G) \cap G' = 1$. Then each stem group that is isoclinic to G is centerless.

Proof. Let S be an stem group that is isoclinic to G. Then $G' \cap Z(G) \cong Z(S)$. Since $G' \cap Z(G) \subseteq \varphi(G) \cap G' = 1$, we have $Z(S)$ is trivial, as required. □

Lemma 2.13. Let G be a group such that $\varphi(G) \cap G' = 1$. Then $G \sim G/Z(G)$.

Proof. We know [5, Lemma 3.5] that $G/N \sim G/(N \cap G')$, for each normal subgroup N of G. So the result holds, because of $Z(G) \cap G' = 1$. □

One can obtain the following result by combining Lemmas 2.1, 2.12 and 2.13.

Corollary 2.14. Let G be a group such that $\varphi(G) \cap G' = 1$. Then an stem group S is isoclinic to G, if and only if S is an n-stem group that is n-isoclinic to G.

Here we can say that every n-isoclinism family of such groups contains only one isoclinism family.

Theorem 2.15. Let G_i be a group such that $\varphi(G_i) \cap G_i' = 1$, for $i = 1, 2$. Then $G_1 \sim G_2$ if and only if $G_1 \sim_n G_2$.

Theorem 2.16. Let G and H be two isoclinic groups such that $\varphi(G) \cap G' = 1$. Then $Z(H)$ is the hypercenter of $H, \varphi(H/Z(H))$ is trivial and so $\varphi(H) \subseteq Z(H)$. Moreover, $\varphi(H) \cap H' = 1$ and $G/\varphi(G) \sim H/\varphi(H)$.

Proof. It is easy to see that, if N is an intersection of some non-normal maximal subgroups of G, then $\varphi(G/N)$ is trivial. So, using Proposition 2.2, we have $\varphi(G/Z(G)) = 1$. Since $G \sim H$, we have $G/Z(G) \cong H/Z(H)$ and $G' \cap Z(G) \cong H' \cap Z(H)$. Therefore, $\varphi(H/Z(H)) = 1$ and we get $\varphi(H) \cap H' = 1$. The proof is completed. □

According to Theorem 2.16, we may say that the intersection of Frattini and derived subgroup is a family invariant among such groups.

Proof of Theorem B.

(i) The result follows from Proposition 2.2.

(ii) The result follows from Lemma 2.1, and Lemma 2.16.

(iii) By part (ii), we have $G/Z(G)$ is a finite group with trivial Frattini subgroup and $(G/Z(G))' \cong G'$. Also, Theorem 2.8 implies $G/Z(G) = (H/Z(G))(G'Z(G)/Z(G))$, where $(H/Z(G)) \cap (G'Z(G)/Z(G)) = 1$, G' is elementary abelian and $H/Z(G)$ is abelian. But $(H/Z(G)) \cap (G'Z(G)/Z(G)) = (H \cap G')Z(G)/Z(G) = 1$ and then $H \cap G' \subseteq Z(G) \cap G' = 1$, and H is abelian.

(iv) Suppose that S is an stem group that is isoclinic to G. By Lemma 2.12, we have $S \cong G/Z(G)$. Now, since $G/Z(G)$ is an A-group, by part (ii), and Theorem 2.11, the result follows.

Proof of Theorem C. Let G be a group such that $\varphi(G) \cap G' = 1$. Lemma 2.1 and Proposition 2.2 imply that $Z(G/Z(G)) = \varphi(G/Z(G)) = 1$. Moreover, it is known that [1, Theorem 0.4], for a finite non-abelian group G such that $\varphi(G) = 1$, then $|G/Z(G)| < |G'||G^{\mathcal{N}}|$. Also, there is a simple lower bound for $|G : Z(G)|$ in terms of $|G'|$, whenever $G' \cap Z(G) = 1$. In fact, we have $|G'| \leq |G : Z(G)|$. Furthermore, by Proposition 2.7, we have $(G/Z(G))^{\mathcal{N}} \cong G^{\mathcal{N}} = G'$. Now, by considering the finite group $G/Z(G)$ in the above inequalities, one can obtain the result.

The alternating group $A_n (n \geq 5)$ asserts that the inequality in Theorem C is sharp and that the bound we obtained there is the best possible one.

The following example shows that the condition $\varphi(G) \cap G' = 1$ is necessary and cannot be omitted.

Example 2.17. Let G be an infinitely generated group by x_i, y_i and z, subject to the relations $x_i^p = y_i^p = z^p = 1$, $[x_i, x_j] = [y_i, y_j] = 1$, and $[x_i, y_i] = z, [x_i, y_j] = 1$, for all $i \neq j$, and $[z, x_i] = [z, y_i] = 1$, for all i. Then $Z(G) = G' = \langle z \rangle$ is finite and $\varphi(G) \cap G' = G'$, but $G/Z(G)$ is infinite.

References

[1] X. Guo and L. Gong, *A note on the size of the nilpotent residual in finite groups*, Arch. Math. Vol. 99 (2012), pp. 413–416. DOI: 10.1007/s00013-012-0452-5

[2] Z. Halasi and K. Podoski, *Bounds in groups with trivial Frattini subgroup*, J. Algebra Vol. 319 (2008), pp. 893–896. DOI: 10.1016/j.jalgebra.2007.02.053

[3] P. Hall, *The classification of prime power groups*, J. Reine Angew. Math. Vol. 182 (1940), pp. 130–141.

[4] P. Hall, *The construction of soluble groups*, Journal fr die reine und angewandte Mathematik Vol. 182 (1940), pp. 206214,

[5] N.S. Hekster, *On the structure of n-isoclinism classes of groups*, J. Pure Appl. Algebra Vol. 40 (1986), pp. 63–85. DOI.10.1016/0022-4049(86)90030-7

[6] M. Hezog, G. Kaplan and A. Lev, *On the commutator and the center of finite groups*, J. Algebra Vol. 278 (2004), pp. 494–501. DOI.10.1016/j.jalgebra.2004.03.021

[7] A.Y.E. Ol'shanskii, *Varieties of finitely approximable groups*, Izvestiya Rossiiskoi Akademii Nauk. Vol. 33 (1969), pp. 915-927.

[8] D.J.S. Robinson, *A Course in the Theory of Groups*, Springer-Verlag, New York, (1982).

[9] D.R. Taunt, *On A-groups*, Mathematical proceedings of the cambridge philosophical society Vol. 45 (1949), pp. 24–42. DOI: 10.1017/S0305004100000414 Published online: 24 October 2008

[10] J.H. Walter, *The characterization of finite groups with abelian Sylow 2-subgroups*, Annals of Mathematics Vol. 89 No. 3 (1969), pp. 405-514.

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