



GRAPH PRODUCT OF GENERALIZED CAYLEY GRAPHS OVER POLYGROUPS

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ABSTRACT. In this paper, we introduce a suitable generalization of Cayley graphs that is defined over polygroups (GCP-graph) and give some examples and properties. Then, we mention a generalization of NEPS that contains some known graph operations and apply to GCP-graphs. Finally, we prove that the product of GCP-graphs is again a GCP-graph.

1. INTRODUCTION AND PRELIMINARIES

A graph product is a binary operation on graphs. So, many large graphs can be constructed from existing smaller graphs. In [10], Li et al. studied the properties of the lexicographic product of vertex-transitive and of edge-transitive graphs, and of the Cayley graphs. They proved that the lexicographic product of vertex-transitive (edge-transitive) graphs is a vertex-transitive (edge-transitive) graph and, in particular, the lexicographic product of Cayley graphs is a Cayley graph.

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The non-complete extended p-sum (NEPS) of graphs is very general graph operation. Many graph operations are special cases of NEPS, to name just the sum, product and strong product of graphs.

Definition 1.1. Let $B \subseteq \{0, 1\}^n \setminus \{(0, 0, \dots, 0)\}$ be a set of binary n -tuples. NEPS with basis B of graphs G_1, \dots, G_n is the graph whose vertex set is the Cartesian product of the vertex sets of graphs G_1, \dots, G_n in which two vertices, say (x_1, \dots, x_n) and (y_1, \dots, y_n) , are adjacent if and only if there exists an n -tuple $(\beta_1, \dots, \beta_n) \in B$ such that $x_i = y_i$ holds whenever $\beta_i = 0$, and x_i is adjacent to y_i (in G_i) whenever $\beta_i = 1$.

Klotz and Sander proved that the class of gcd-graphs and the class of NEPS of complete graphs coincide [14].

The theory of hypergroups which is a generalization of the concept of ordinary groups first was introduced by Marty [15]. Since then many researchers have worked on hypergroups and developed it. A short review of this theory appears in [5, 6, 8, 9, 17]. Application of hypergroups has mainly appeared in special subclasses. For example, polygroups which are certain subclasses of hypergroups are studied in [13] by Ioulidis and are used to study color algebra [3, 4]. Quasi-canonical hypergroups (called polygroups by Comer) were introduced in [2], as a generalization of canonical hypergroups, introduced in [16]. In [11] Heidari et al. studied the concept of topological polygroups as a generalization of topological groups.

Definition 1.2. [3, 9] A *polygroup* is a system $\langle P, \circ, 1, {}^{-1} \rangle$, where $1 \in P, {}^{-1}$ is a unitary operation on P , \circ maps $P \times P$ into the non-empty subsets of P , and the following axioms hold for all $x, y, z \in P$:

$$(P1) \quad x \circ (y \circ z) = (x \circ y) \circ z,$$

$$(P2) \quad 1 \circ x = x = x \circ 1,$$

$$(P3) \quad x \in y \circ z \text{ implies } y \in x \circ z^{-1} \text{ and } z \in y^{-1} \circ x.$$

If A, B are non-empty subsets of P , then $A \circ B$ is given by

$$A \circ B = \bigcup_{\substack{a \in A \\ b \in B}} a \circ b.$$

$x \circ A$ is used for $\{x\} \circ A$ and $A \circ x$ for $A \circ \{x\}$. Clearly, every group is a polygroup. The following elementary facts about polygroups follow easily from the axioms: $1 \in x \circ x^{-1} \cap x^{-1} \circ x$, $1^{-1} = 1$, $(x^{-1})^{-1} = x$, and $(x \circ y)^{-1} = y^{-1} \circ x^{-1}$, where $A^{-1} = \{a^{-1} \mid a \in A\}$. A polygroup in which every element has order 2 (*i.e.*, $x^{-1} = x$ for all x) is called symmetric. For more details about polygroups we refer the readers to [9].

Lemma 1.3. Let $\mathbf{P} = \langle P, \circ, 1, {}^{-1} \rangle$ be a polygroup and B be a non-empty subset of P . Then

$$\langle B \rangle = \{x \in P : x \in b_1 \circ b_2 \circ \dots \circ b_k, \text{ s.t. } k \in \mathbb{N}, b_i \in B \cup B^{-1}\}.$$

Proof. It is straightforward. \square

2. MAIN RESULTS

It is easy to see that the Cartesian, tensor and strong product of graphs are special types of NEPS but the lexicographic product is not. In this section, we mention a generalization of NEPS that contain some other known graph operations, for example the compliment and the lexicographic product of graphs. Firstly, we introduce a suitable generalization of Cayley graphs that is defined over polygroups.

Definition 2.1. [12] Let $\mathbf{P} = \langle P, \circ, 1, ^{-1} \rangle$ be a polygroup and S , say the connection set, be a non-empty inverse closed subset (i.e. $S^{-1} = S$) of P . Then we define the *generalized Cayley graph* $GCP(\mathbf{P}; S)$ as a simple graph with vertex set P and edge set

$$E = \{\{x, y\} \mid x \neq y \text{ and } x \circ y^{-1} \cap S \neq \emptyset\}.$$

A graph Λ is called a GCP-graph if there exists a polygroup \mathbf{P} and a connection set S such that $\Lambda \cong GCP(\mathbf{P}; S)$.

In what follows, we provide some examples of generalized Cayley graphs with the given polygroups and connection sets.

Example 2.2. The generalized Cayley graph of the polygroup $\mathbf{P}_2 = \langle \{1, 2, 3, 4\}, \circ, 1, ^{-1} \rangle$ and the connection set $\{3, 4\}$ is shown in Figure 2.2.

\circ	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	$\{1, 3\}$	$\{2, 4\}$
4	4	3	$\{2, 4\}$	$\{1, 3\}$

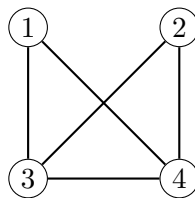


FIGURE 1. $GCP(\mathbf{P}_2; \{3, 4\})$

Example 2.3. The generalized Cayley graph of the polygroup $\mathbf{P}_3 = \langle \{1, 2, 3, 4, 5\}, \circ, 1,^{-1} \rangle$ and the connection set $\{3, 4\}$ is shown in Figure 2.3.

\circ	1	2	3	4	5
1	1	2	3	4	5
2	2	{1, 2}	3	4	5
3	3	3	{1, 2, 4}	{3, 5}	{4, 5}
4	4	4	{3, 5}	{1, 2, 4, 5}	{3, 4}
5	5	5	{4, 5}	{3, 4}	{1, 2, 3, 5}

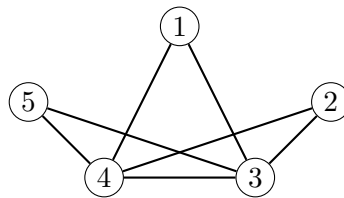


FIGURE 2. $\text{GCP}(\mathbf{P}_3; \{3, 4\})$

The necessary and sufficient condition for connectedness of a GCP-graph over a polygroup is same as that in Cayley graphs. In other words:

Theorem 2.4. [12] *Let $\mathbf{P} = \langle P, \circ, 1,^{-1} \rangle$ be a polygroup and S be a connection set. Then, the generalized Cayley graph $\text{GCP}(\mathbf{P}; S)$ is connected if and only if S generates P .*

Definition 2.5. Let G_1, G_2, \dots, G_n be graphs and $B \subseteq \{-1, 0, 1\}^n$. Then the product of G_1, G_2, \dots, G_n respect to the base B , denoted by $G = \mathbf{Pr}(G_1, \dots, G_n; B)$, is a graph with the vertex set $V(G) = V(G_1) \times \dots \times V(G_n)$ and two distinct vertices $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ are adjacent in G , if and only if there exists an n -tuple $\beta = (\beta_1, \dots, \beta_n) \in B$ such that $x_i = y_i$, whenever $\beta_i = 0$, x_i and y_i are adjacent in G_i , whenever $\beta_i = 1$ and x_i and y_i are non-adjacent in G_i , whenever $\beta_i = -1$.

In particular, for a simple graph $G = (V, E)$, we have:

$$\mathbf{Pr}(G; \emptyset) = \mathbf{Pr}(G; \{0\}) \cong \overline{K_{|V|}};$$

$$\mathbf{Pr}(G; \{1\}) = G;$$

$$\mathbf{Pr}(G; \{-1\}) = \overline{G}.$$

Also, some known product operations on graphs can be considered as the product graphs with suitable bases.

Lemma 2.6. *Let G_1, \dots, G_n be simple graphs and $*$ be the Cartesian (\square), tensor (\times), strong (\boxtimes) or lexicographic product (\circ) of graphs. Then there exists a base $B \subseteq \{-1, 0, 1\}^n$ such that $G_1 * \dots * G_n = \mathbf{Pr}(G_1, \dots, G_n; B)$.*

Proof. For every $i \in \{1, 2, \dots, n\}$ put

$$\Delta_i = \{0\}^{i-1} \times \{1\} \times \{0\}^{n-i} \text{ and } \Sigma_i = \{0\}^{i-1} \times \{1\} \times \{-1, 0, 1\}^{n-i}.$$

Then we have

$$G_1 \square \dots \square G_n = \mathbf{Pr}(G_1, \dots, G_n; B_1);$$

$$G_1 \times \dots \times G_n = \mathbf{Pr}(G_1, \dots, G_n; B_2);$$

$$G_1 \boxtimes \dots \boxtimes G_n = \mathbf{Pr}(G_1, \dots, G_n; B_3);$$

$$G_1 \circ \dots \circ G_n = \mathbf{Pr}(G_1, \dots, G_n; B_4);$$

where $B_1 = \{(1, 1, \dots, 1)\}$, $B_2 = \bigcup_{i=1}^n \Delta_i$, $B_3 = (\bigcup_{i=1}^n \Delta_i) \cup \{(1, 1, \dots, 1)\}$ and $B_4 = \bigcup_{i=1}^n \Sigma_i$. \square

In the following theorem we prove that the product of GCP-graphs is a GCP-graph.

Theorem 2.7. *Let G_1, G_2, \dots, G_n be GCP-graphs and $B \subseteq \{-1, 0, 1\}^n$. Then $\mathbf{Pr}(G_1, G_2, \dots, G_n; B)$ is a GCP-graph.*

Proof. Suppose that G_1, G_2, \dots, G_n are GCP-graphs, $B \subseteq \{-1, 0, 1\}^n$ and $G = \mathbf{Pr}(G_1, G_2, \dots, G_n; B)$. Thus, there exist polygroups

$$\mathbf{P}_1 = \langle P_1, \circ_1, e_1, I_1 \rangle, \mathbf{P}_2 = \langle P_2, \circ_2, e_2, I_2 \rangle, \dots, \mathbf{P}_n = \langle P_n, \circ_n, e_n, I_n \rangle$$

and connection sets $S_1 \subseteq P_1, S_2 \subseteq P_2, \dots, S_n \subseteq P_n$ such that

$$G_i \cong \Gamma_i = \text{GCP}(\mathbf{P}_i; S_i),$$

for every $i = 1, 2, \dots, n$.

Let $\Gamma = \mathbf{Pr}(\Gamma_1, \Gamma_2, \dots, \Gamma_n; B)$ and $\Lambda = \text{GCP}(\mathbf{P}_1 \times \mathbf{P}_2 \times \dots \times \mathbf{P}_n, S)$ such that

$$S = \bigcup_{\beta \in B} (S_1^{(\beta_1)} \times S_2^{(\beta_2)} \times \dots \times S_n^{(\beta_n)}),$$

where $S_i^{(0)} = \{e_i\}$, $S_i^{(1)} = S_i$ and $S_i^{(-1)} = P_i \setminus S_i$ for all $i = 1, 2, \dots, n$.

We claim that $\Gamma = \Lambda$. Since, by the definition, $V(\Gamma) = V(\Lambda) = P_1 \times P_2 \times \cdots \times P_n$ as desired. On the other hand,

$$\begin{aligned}
E(\Gamma) &= \{((x_1, \dots, x_n), (y_1, \dots, y_n)) \mid \exists \beta \in B, x_i = y_i \Leftrightarrow \beta_i = 0, \\
&\quad x_i \sim y_i \Leftrightarrow \beta_i = 1, x_i \neq y_i, x_i \approx y_i \Leftrightarrow \beta_i = -1, \forall i \in \{1, 2, \dots, n\}\} \\
&= \{((x_1, \dots, x_n), (y_1, \dots, y_n)) \mid \exists \beta \in B, e_i \in x_i \circ_i y_i^{I_i} \Leftrightarrow \beta_i = 0, \\
&\quad (x_i \circ_i y_i^{I_i}) \cap S_i \neq \emptyset \Leftrightarrow \beta_i = 1, (x_i \circ_i y_i^{I_i}) \cap S_i = \emptyset \Leftrightarrow \beta_i = -1, \\
&\quad \forall i \in \{1, 2, \dots, n\}\} \\
&= \{((x_1, \dots, x_n), (y_1, \dots, y_n)) \mid \exists \beta \in B, \\
&\quad ((x_1 \circ_1 y_1^{I_1}) \times \cdots \times (x_n \circ_n y_n^{I_n})) \cap (S_1^{(\beta_1)} \times \cdots \times S_n^{(\beta_n)}) \neq \emptyset\} \\
&= E(\Lambda).
\end{aligned}$$

Therefore, $G \cong \Gamma = \Lambda$ so the proof is complete since, Λ is a GCP-graph. \square

Corollary 2.8. *The Cartesian, tensor, strong and lexicographic product of GCP-graphs are GCP-graphs.*

Finally, from Theorem 2.7 we can obtain the connection sets for the Cartesian, tensor, strong and lexicographic product of two GCP-graphs.

Lemma 2.9. *Let $\mathbf{P}_1 = \langle P_1, \circ_1, e_1, I_1 \rangle$ and $\mathbf{P}_2 = \langle P_2, \circ_2, e_2, I_2 \rangle$ be two polygroups and $\mathbf{P} = \mathbf{P}_1 \times \mathbf{P}_2$. Then for every connection sets $S_1 \subseteq P_1$ and $S_2 \subseteq P_2$ the following assertions hold:*

- (1) $\text{GCP}(\mathbf{P}_1, S_1) \square \text{GCP}(\mathbf{P}_2; S_2) = \text{GCP}(\mathbf{P}; S_1 \times S_2)$;
- (2) $\text{GCP}(\mathbf{P}_1, S_1) \times \text{GCP}(\mathbf{P}_2; S_2) = \text{GCP}(\mathbf{P}; S_1 \times \{e_2\} \cup \{e_1\} \times S_2)$;
- (3) $\text{GCP}(\mathbf{P}_1, S_1) \boxtimes \text{GCP}(\mathbf{P}_2; S_2) = \text{GCP}(\mathbf{P}; S_1 \times S_2 \cup S_1 \times \{e_2\} \cup \{e_1\} \times S_2)$;
- (4) $\text{GCP}(\mathbf{P}_1, S_1) \circ \text{GCP}(\mathbf{P}_2; S_2) = \text{GCP}(\mathbf{P}; S_1 \times P_2 \cup \{e_1\} \times S_2)$.

In spite of the fact that every Cayley graph is a GCP-graph and so the following proposition holds as a result of Lemma 2.9, we prove it independently for the lexicographic product. This proof is modified the proof of Theorem 2.4 in [10].

Proposition 2.10. *The Cartesian, tensor, strong and lexicographic product of Cayley graphs are Cayley graphs.*

Proof. Let $\Gamma_1 = \text{Cay}(G_1, S_1)$ and $\Gamma_2 = \text{Cay}(G_2, S_2)$ be two Cayley graphs. Consider $\Gamma = \Gamma_1 \circ \Gamma_2$ and $\Lambda = \text{Cay}(G_1 \times G_2, S)$ where, $S = S_1 \times G_2 \cup \{e_1\} \times S_2$. Then $\Gamma = \Lambda$ since, $V(\Gamma) = V(\Lambda) = G_1 \times G_2$ and

$$\begin{aligned}
E(\Gamma) &= \{((x_1, x_1), (y_1, y_2)) \mid x_1 \sim y_1 \text{ or } (x_1 = y_1 \text{ and } x_2 \sim y_2)\} \\
&= \{((x_1, x_1), (y_1, y_2)) \mid x_1 y_1^{-1} \in S_1 \text{ or } x_1 y_1^{-1} = e_1 \text{ and } x_2 y_2^{-1} \in S_2\} \\
&= \{((x_1, x_1), (y_1, y_2)) \mid (x_1, x_2)(y_1, y_2)^{-1} \in S\} = E(\Lambda).
\end{aligned}$$

Therefore, the proof is complete. \square

Example 2.11. Let K_2 be the complete graph on two vertices and H be the chair graph. Then $K_2 \cong \text{Cay}(\mathbb{Z}_2, \{1\})$ and $H \cong \text{GCP}(\mathbf{P}_3, \{5\})$, where \mathbf{P}_3 is the polygroup considered in Example 2. The Cartesian product of K_2 and H is $G = \text{GCP}(\mathbb{Z}_2 \times P_3, \{(1, 5)\})$. See Figure 2.11.

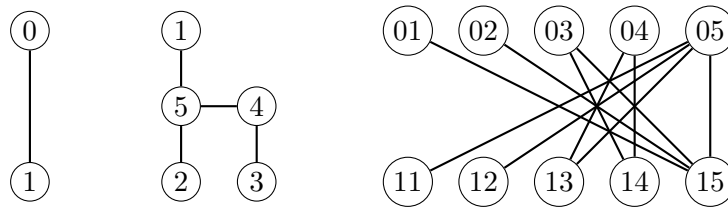


FIGURE 3. $K_2 \times H \cong G$

3. ACKNOWLEDGMENTS

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