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ON SOME CLASSES OF EXPANSIONS OF IDEALS IN MV-ALGEBRAS

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Abstract. In this paper, we introduce the notions of expansion of ideals in MV-algebras, $(\tau,\sigma)-$primary, (τ,σ)-obstinate and (τ,σ)-Boolean in $MV-$algebras. We investigate the relations of them. For example, we show that every (τ,σ)-obstinate ideal of an $MV-$ algebra is (τ,σ)-primary and (τ,σ)-Boolean. In particular, we define an expansion σ_y of ideals in an $MV-$algebra. A characterization of expansion ideal with respect to σ_y is given. Finally, we show that the class $C(\sigma_y)$ of all constant ideals relative to σ_y is a Heyting algebra.

1. Introduction

$MV-$algebras were introduced by C. C. Chang in [2] and [3] as an algebraic counterpart of the Łukasiewicz infinite valued propositional logic.
In order to keep the paper brief, we refer the reader to [2, 3, 4, 5, 8] for results on $MV-$algebras. In [7] results regarding expansions of filters in residuated lattices were obtained. In this paper, we illustrate how we can obtain various results for some types of expansions of ideals in $MV-$algebras.
That is, we introduce expansions of ideals in $MV-$algebras and study their some properties.

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Also, we introduce $(\tau, \sigma)-$primary, $(\tau, \sigma)-$obstinate, $(\tau, \sigma)-$Boolean and investigate the relations between $(\tau, \sigma)-$obstinate and the other expansions of ideals in an $MV-$algebra.
In particular, we show that every $(\tau, \sigma)-$obstinate ideal is $(\tau, \sigma)-$primary but in general, the converse of it is not true and every $(\tau, \sigma)-$obstinate is $(\tau, \sigma)-$Boolean ideal of A, but in general, the converse of it is not true. We define expansions ideal σ_y, and so we give a characterization of $\sigma_y(I)$. We show that $\sigma_y(I)$ is the smallest constant ideal relative to σ_y such that $I \subseteq \sigma_y(I)$. Finally, we show that class $C(\sigma_y)$ of all constant ideals relative to expansion σ_y is a complete Heyting algebra.

2. **Preliminaries**

We recollect some definitions and results which will be used in the following:

Definition 2.1. [2, 4, 8] An MV-algebra is a structure $(A, \oplus, ^*, 0)$ where $\oplus$ is a binary operation, * is a unary operation, and 0 is a constant such that the following axioms are satisfied for any $a, b \in A$:
$(MV1)$ $(A, \oplus, 0)$ is an abelian monoid,
$(MV2)$ $(a^*)^* = a$,
$(MV3)$ $0^* \oplus a = 0^*$,
$(MV4)$ $(a^* \oplus b)^* \oplus b = (b^* \oplus a)^* \oplus a$.

Note that $1 = 0^*$ and the auxiliary operation $\odot$ as follow: $x \odot y = (x^* \oplus y^*)^*$. We recall that the natural order determines a bounded distributive lattice structure such that $x \vee y = x \oplus (x^* \odot y) = y \oplus (x \odot y^*)$ and $x \wedge y = x \odot (x^* \oplus y) = y \odot (y^* \oplus x)$.

Lemma 2.2. [4, 8] *In each MV-algebra, the following relations hold for all $x, y, z \in A$:*
(1) $x \leq y$ *if and only if* $y^* \leq x^*$,
(2) *If* $x \leq y$, *then* $x \oplus z \leq y \oplus z$ *and* $x \odot z \leq y \odot z$,
(3) $x \leq y$ *if and only if* $x^* \oplus y = 1$ *if and if* $x \odot y^* = 0$,
(4) $x, y \leq x \oplus y$ *and* $x \odot y \leq x, y$, $x \leq nx = x \oplus x \oplus \cdots \oplus x$ *and* $x^n = x \odot x \odot \cdots \odot x \leq x$,
(5) $x \oplus x^* = 1$ *and* $x \odot x^* = 0$,
(6) *If* $x \leq y$ *and* $z \leq t$, *then* $x \oplus z \leq y \oplus t$,
(7) $(x \oplus y) \wedge z \leqslant (x \wedge z) \oplus (y \wedge z)$,
(8)$(x] \cap (y] = (x \wedge y]$.

Definition 2.3. [2, 8] An ideal of an MV-algebra A is a nonempty subset I of A satisfying the following conditions:
(1) If $x \in I$, $y \in A$ and $y \leq x$, then $y \in I$,
(2) If $x, y \in I$, then $x \oplus y \in I$.
We denote by $Id(A)$ the set of ideals of an MV-algebra A.

Definition 2.4. [4] Let I be an ideal of an MV-algebra A. I is a prime ideal of A if $I \neq A$ and $a \wedge b \in I, a \notin I$, then $b \in I$ for all $a, b \in A$.

• [6] An ideal I of an MV-algebra A is called Boolean ideal if $x \wedge x^* \in I$, for all $x \in A$.

•[5] A proper ideal of A is called an obstinate ideal A, if $x, y \notin I$ implies $x \odot y^* \in I$ and $y \odot x^* \in I$, for all $x, y \in A$.

Lemma 2.5. [5] *Let I be a proper ideal of A. Then I is an obstinate ideal if and only if $x \in I$ or $x^* \in I$, for all $x \in A$.*

Remark 2.6. [4] In an MV-algebra A, the distance function is
$d : A \times A \longrightarrow A \quad , \qquad d(x, y) := (x \odot y^*) \oplus (y \odot x^*)$.
Suppose that I is an ideal of an MV-algebra A. Define $x \sim_I y$ if and only if $d(x, y) \in I$ if and only if $x \odot y^* \in I$ and $y \odot x^* \in I$. Then $\sim_I$ is a congruence relation on A. The set of all congruence classes is denoted by A/I then $A/I = \{[x] : x \in A\}$, where $[x] = \{y \in A : x \sim_I y\}$. We can easily to see that $x \in I$ if and only if $x/I = 0/I$. The MV-algebra operations on A/I given by $x/I \oplus y/I = (x \oplus y)/I$ and $(x/I)^* = x^*/I$, are well defined. Hence $(A/I, \oplus, *, [0])$ becomes an MV-algebra [4, 8].

Definition 2.7. [8] A Heyting algebra is a lattice $(A, \vee, \wedge)$ with 0 such that for every $a, b \in A$, there exists an element $a \to b \in A$ such that for $x \in A$, $a \wedge x \leqslant b$ if and only if $x \leqslant a \to b$.

Note that in a Heyting algebra A, $x \odot x = x$, for $x \in A$, hence $x \odot y = x \wedge y = x \odot (x \to y)$, for all $x, y \in A$.

Proposition 2.8. [8] *$(Id(A), \wedge, \vee, \to, \{0\}, A)$ is a complete Heyting algebra, where for all $I, J \in Id(A)$:*
$I \wedge J = I \cap J$, $I \vee J = (I \cup J]$ and $I \to J = \{x \in A \mid I \cap (x] \subseteq J\}$. Hence $I \wedge (\vee_\lambda J_\lambda) = \vee_\lambda (I \wedge J_\lambda)$.

Definition 2.9. [4] Let $X \subseteq A$. The ideal of A generated by X will be denoted by $(X]$. We have

$$(X] = \{a \in A \mid a \leqslant x_1 \oplus x_2 \oplus ...x_n,\ for\ some\ n \in \mathbb{N}\ and\ x_i \in X,\ 1 \leqslant i \leqslant n\}$$

In particular, $(a] = \{x \in A \mid x \leqslant na,\ for\ some\ n \in \mathbb{N}\}$.

Lemma 2.10. [8] *Let A be a Heyting algebra. Then we have the following rules, for $x, y, z \in A$:*

1) $1 \to x = x$, $x \to x = 1$,

2) $x \leqslant y$ *implies* $z \to x \leqslant z \to y$,

3) $x \to (y \to z) = (x \odot y) \to z = y \to (x \to z)$,

4) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$.

3. Expansions of ideals in $MV-$algebras

Definition 3.1. An expansion of ideals in A is defined to be a function
$\sigma : Id(A) \longrightarrow Id(A)$ such that
f_1) $(\forall I \in Id(A))(I \subseteq \sigma(I))$,
f_2) $(\forall I, J \in Id(A))(I \subseteq J \Longrightarrow \sigma(I) \subseteq \sigma(J))$.

Example 3.2. (1) The function $\sigma_0 : Id(A) \longrightarrow Id(A)$ defined by $\sigma_0(I) = I$ is an expansion of ideals in A.
(2) $C = \{0, c, 2c, 3c, ..., 1-2c, 1-c, 1\}$ be the $MV-$algebra defined in [8, Example 2.6] with operations as follows:
if $x = nc$ and $y = mc$, then $x \oplus y := (m+n)c$,
if $x = 1-nc$ and $y = 1-mc$, then $x \oplus y := 1$,
if $x = nc$ and $y = 1-mc$ and $m \leq n$, then $x \oplus y := 1$,
if $x = nc$ and $y = 1-mc$ and $n < m$, then $x \oplus y := 1-(m-n)c$,
if $x = 1-mc$ and $y = nc$ and $m \leq n$, then $x \oplus y := 1$,
if $x = 1-mc$ and $y = nc$ and $n < m$, then $x \oplus y := 1-(m-n)c$,
if $x = nc$, then $x^* := nc$.
It has only three ideals: $I_0 = \{0\}, I_1 = \{0, c, 2c, 3c, ...\}$ and $I_2 = C$.

$$\sigma : Id(C) \longrightarrow Id(C)$$

$$I_0 \longmapsto I_1,\ I_1 \longmapsto I_2,\ I_2 \longmapsto I_2.$$

The function σ is an expansion of ideals in C.
(3) Let $\mathbb{R}^*$ be a nonstandard model of real numbers with natural order and ε be a positive infinitesimal element of $\mathbb{R}^*$. Let $\varepsilon^2 = \varepsilon.\varepsilon, ..., \varepsilon^n = \varepsilon.\varepsilon...\varepsilon(n-times)$, where $\cdot$ is the ususl product in the filed $\mathbb{R}^*$; then $\varepsilon^i > 0$, for any $i \in \mathbb{N}$ and $\varepsilon^i \leqslant \varepsilon^j$, for $i > j$.
The unite interval $[0,1]^* \subseteq \mathbb{R}^*$ is an $MV-$algebra under the operations: $x \oplus y = min\{1, x+y\}, x^* = 1-x$. Let $\mathbb{N}$ be the ordered set of positive natural numbers. For every $n \in \mathbb{N}$, let E_n be the subalgebra of $[0,1]^*$ generated by $\{\varepsilon, \varepsilon^2, ..., \varepsilon^n\}$ and E be the subalgebra $\bigcup_{n\in\mathbb{N}} E_n$. We recall that ,[1] that $E =< (\varepsilon^i) \mid i \in \mathbb{N} >$. The set of all ideals of E is $\{0\}, < \varepsilon >, ..., < \varepsilon^i > ...$,where $i \in \mathbb{N}$ and $< \varepsilon^i > \subseteq < \varepsilon^j >$, for any $i > j$. The function $\sigma : Id(E) \longrightarrow Id(E)$ defined by $\sigma(\{0\}) = \{0\}$ and $\sigma(< \varepsilon^i >) =< \varepsilon^{i+1} >$, for every $i \in \mathbb{N}$. Obviously, σ does not hold in (f_1). Hence σ is not expansion of ideals in E.

Definition 3.3. Let τ and σ be expansions of ideals. Then an ideal I of A is said to be $(\tau, \sigma)-$primary, if

$$(\forall a, b \in A)(a \wedge b \in I, a \notin \tau(I) \Rightarrow b \in \sigma(I)).$$

Example 3.4. Let $A = \{0, a, b, c, d, 1\}$, where $0 < a, c < d < 1$ and $0 < a < b < 1$. Define $\odot$, $\oplus$ and $*$ as follows:

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	a	0	0	a
b	0	a	b	0	a	b
c	0	0	0	c	c	c
d	0	0	a	c	c	d
1	0	a	b	c	d	1

$\oplus$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	b	b	d	1	1
b	b	b	b	1	1	1
c	c	d	1	c	d	1
d	d	1	1	d	1	1
1	1	1	1	1	1	1

$*$	0	a	b	c	d	1
	1	d	c	b	a	0

Then $(A, \oplus, \odot, *, 0, 1)$ is an MV-algebra [8]. It has only four ideals: $I_0 = \{0\}, I_1 = \{0, c\}, I_2 = \{0, a, b\}, I_3 = A$.

$$\begin{array}{ll} \sigma : Id(A) \longrightarrow Id(A) & \delta : Id(A) \longrightarrow Id(A) \\ I_0 \longmapsto I_0 & I_0 \longmapsto I_0 \\ I_1 \longmapsto I_1 & I_1 \longmapsto I_3 \\ I_2 \longmapsto I_3 & I_2 \longmapsto I_3 \\ I_3 \longmapsto I_3. & I_3 \longmapsto I_3. \end{array}$$

Are expansions of ideals A.

Obviously, I_1 is $(\sigma, \delta)-$primary and since $a \wedge c = 0$ but $a \notin I_0$ and $c \notin I_0$, hence I_0 is not $(\sigma, \delta)-$primary.

Lemma 3.5. *Let $I \in Id(A), \tau_0$ and σ_0 be the functions in Example 3.2 (1). Then I is $(\tau_0, \sigma_0)-$primary if and only if it is a prime ideal of A.*

Proof. Let I be a $(\tau_0, \sigma_0)-$primary, and $a \wedge b \in I$ such that $a \notin I$. So $a \notin \tau_0(I)$. Hence $b \in \sigma_0(I) = I$. Therefore I is a prime ideal of A.

Conversely, let I be a prime ideal of $A, a \wedge b \in I$ and $a \notin \tau_0(I) = I$. So $b \in I$. Hence $b \in \sigma_0(I) = I$. Therefore I is a $(\tau_0, \sigma_0)-$primary. □

Theorem 3.6. *Let σ and τ be expansions of ideals in A. Then I is a $(\tau, \sigma)-$primary if and only if I is $(\sigma, \tau)-$primary.*

Proof. Let I be $(\tau,\sigma)-$primary, $a \wedge b \in I$ and $a \notin \sigma(I)$. We must show that $b \in \tau(I)$. Let $b \notin \tau(I)$. By the fact that I is a $(\tau,\sigma)-$primary, we get $a \in \sigma(I)$, which is a contradiction. So $b \in \tau(I)$, i.e. I is a $(\sigma,\tau)-$primary. The proof of the converse part is similar. □

Theorem 3.7. *Let τ and σ be expansion of ideals in A. Then every prime ideal of A is $(\tau,\sigma)-$primary.*

Proof. Let I be prime ideal of A, $a \wedge b \in I$ and $a \notin \tau(I)$. By the fact that $I \subseteq \tau(I)$, we get $a \notin I$. I is prime ideal of A, so $b \in I$. By $I \subseteq \sigma(I)$, we get $b \in \sigma(I)$. Therefore I is a $(\tau,\sigma)-$primary. □

The following example show that the coverse of the above theorem my not be true, in general.

Example 3.8. Let $A = \{0, a, b, 1\}$, where $0 < a, b < 1$. Define $\odot$, $\oplus$ and $*$ as follows:

$\odot$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\oplus$	0	a	b	1
0	0	a	b	1
a	a	a	1	1
b	b	1	b	1
1	1	1	1	1

$*$	0	a	b	1
	1	b	a	0

Then $(A, \oplus, \odot, *, 0, 1)$ is an MV-algebra [8].
$I_0 = \{0\}, I_1 = \{0, a\}, I_2 = \{0, b\}$ and $I_3 = A$ are only ideals of A. Define

$$\sigma : Id(A) \longrightarrow Id(A)$$

$$I_0 \longmapsto I_2$$

$$I_1 \longmapsto I_3$$

$$I_2 \longmapsto I_3$$

$$I_3 \longmapsto I_3.$$

Obviously, I_0 is $(\tau_0, \sigma)-$primary but I_0 is not prime ideal of A, because $a \wedge b = 0 \in I_0$ but $a \notin I_0$ and $b \notin I_0$.

Theorem 3.9. *Let τ be an expansion of ideals A. If $\tau(I)$ is a prime ideal, then I is a $(\tau,\tau)-$primary.*

Proof. Let $\tau(I)$ be a prime ideal, $a \wedge b \in I$ and $a \notin \tau(I)$. We know $I \subseteq \tau(I)$, so $a \wedge b \in \tau(I)$. Since $\tau(I)$ is a prime ideal, then $b \in \tau(I)$. Therefore I is $(\tau, \tau)-$primary. □

Corollary 3.10. *Let τ and σ be expansions of ideals in A and $I \in Id(A)$ and $\tau(I)$ be a prime ideal. Then $\tau(I)$ is a $(\tau, \sigma)-$primary.*

Proof. Let $\tau(I)$ be a prime ideal, $a \wedge b \in \tau(I)$ and $a \notin \tau(\tau(I))$. We have $\tau(I) \subseteq \tau(\tau(I))$, so $a \notin \tau(I)$. By the fact that $\tau(I)$ is a prime ideal of A, then $b \in \tau(I)$. Then by $\tau(I) \subseteq \sigma(\tau(I))$, we get $b \in \sigma(\tau(I))$. Therefore $\tau(I)$ is a $(\tau, \sigma)-$primary. □

Theorem 3.11. *Let τ, σ, ψ and δ be expansions of ideals in A such that $\tau \subseteq \psi$ and $\sigma \subseteq \delta$. Then every $(\tau, \sigma)-$primary ideal is $(\psi, \delta)-$primary.*

Proof. Let I be $(\tau, \sigma)-$primary, $a \wedge b \in I$ and $a \notin \psi(I)$. We have $\tau \subseteq \psi$, so $a \notin \tau(I)$. Then by hypothesis, we get $b \in \sigma(I)$. Hence by the fact that $\sigma \subseteq \delta$, we get $b \in \delta(I)$, i.e. I is a $(\psi, \delta)-$primary. □

Corollary 3.12. *Let τ and σ be expansions of ideals in A. Then every $(\tau, \tau_0)-$primary is $(\tau, \sigma)-$primary.*

Proof. Let I be a $(\tau, \tau_0)-$primary , $a \wedge b \in I$ and $a \notin \tau(I)$. By hypothesis we get $b \in \tau_0(I) = I$. We know $I \subseteq \sigma(I)$, so $b \in \sigma(I)$. Therefore I is a $(\tau, \sigma)-$primary. □

Let $y \in Id(A)$. The function $\sigma_y : Id(A) \longrightarrow Id(A)$ defined by

$$(\forall I \in Id(A))(\sigma_y(I) = \{x \in A \mid x \wedge y \in I\}).$$

Lemma 3.13. *Let $I, J \in Id(A)$. Then*

1)$\sigma_y(I) \in Id(A)$,

2)$I \subseteq \sigma_y(I)$,

3)$I \subseteq J \Longrightarrow \sigma_y(I) \subseteq \sigma_y(J)$,

4)$\sigma_{\frac{y}{J}}(\frac{I}{J}) = \frac{\sigma_y(I)}{J}$, *where* $J \subseteq I$,

5)$\sigma_y(\sigma_y(I)) = \sigma_y(I)$,

6)$\sigma_y(I) \longrightarrow \sigma_y(J) \subseteq I \longrightarrow \sigma_y(J)$,

7)$\sigma_y(I \longrightarrow J) \subseteq \sigma_y(I \longrightarrow \sigma_y(J))$,

8)$\sigma_y(I)$ *is prime ideal if and only if* $\sigma_y(I)$ *is* $(\sigma_y, \sigma_y)-$*primary,*

9) *If I is $(\sigma_{x \vee y}, \sigma_0)-$primary, then I is (σ_x, σ_0) and $(\sigma_y, \sigma_0)-$ primary,*

10) *σ_y is an expansion of ideals in A.*

Proof. 1) We have $0 \wedge y = 0 \in I$, so $0 \in \sigma_y(I)$. Now let $x \leq z$ and $z \in \sigma_y(I)$. Hence $z \wedge y \in I$. We have $x \wedge y \leqslant z \wedge y$. Then by ideal property, we get $x \wedge y \in I$, i.e $x \in \sigma_y(I)$. Let $\alpha, \beta \in \sigma_y(I)$. Then $\alpha \wedge y \in I$ and $\beta \wedge y \in I$. By Lemma 2.2 (7), we have $(\alpha \oplus \beta) \wedge y \leqslant (\alpha \wedge y) \oplus (\beta \wedge y) \in I$. Therefore $(\alpha \oplus \beta) \wedge y \in I$. Hence $\alpha \oplus \beta \in \sigma_y(I)$. Therefore $\sigma_y(I) \in Id(A)$.

2) Let $x \in I$. We know $x \wedge y \leq x$, so $x \wedge y \in I$. Hence $x \in \sigma_y(I)$, i.e. $I \subseteq \sigma_y(I)$.

3) Let $x \in \sigma_y(I)$. Then $x \wedge y \in I$, Hence $x \wedge y \in J$, Therefore $x \in \sigma_y(J)$.

4) By Lemma 2.2, Remark 2.6, and since I, J are ideals of A such that $J \subseteq I$, we have: $\frac{x}{J} \in \sigma_{\frac{y}{J}}(\frac{I}{J})$ so $\frac{x}{J} \wedge \frac{y}{J} \in \frac{I}{J}$, and so $\frac{x \wedge y}{J} \in \frac{I}{J}$, d(x$\wedge y, \alpha) \in Jthusthereexist\alpha \in I$ such that $\frac{x \wedge y}{J} = \frac{\alpha}{J}$ and so $d(x \wedge y, \alpha) \in J$. So

$$\alpha^* \odot (x \wedge y) \leqslant (x \wedge y) \odot \alpha^* \oplus \alpha \odot (x \wedge y)^* \in J,$$

$$\Longrightarrow \alpha^* \odot (x \wedge y) \in I \ , \ \alpha \in I,$$

$$\Longrightarrow \alpha \oplus (\alpha^* \odot (x \wedge y)) \in I,$$

$$\Longrightarrow x \wedge y \leqslant \alpha \vee (x \wedge y) \in I,$$

$$\Longrightarrow x \wedge y \in I,$$

$$\Longrightarrow x \in \sigma_y(I),$$

$$\Longrightarrow \frac{x}{J} \in \frac{\sigma_y(I)}{J}.$$

Therefore $\sigma_{\frac{y}{J}}(\frac{I}{J}) \subseteq \frac{\sigma_y(I)}{J}$. Let $\frac{x}{J} \in \frac{\sigma_y(I)}{J}$, so there exist $t \in \sigma_y(I)$ such that $\frac{x}{J} = \frac{t}{J}$ and so $d(x, t) \in J$. Therefore

$$x \odot t^* \leq (x \odot t^*) \oplus (t \odot x^*) \in J \subseteq I \subseteq \sigma_y(I),$$

$$\Longrightarrow x \odot t^* \in \sigma_y(I) \ , \ t \in \sigma_y(I),$$

$$\Longrightarrow t \oplus (t^* \odot x) \in \sigma_y(I),$$

$$\Longrightarrow x \leqslant t \vee x \in \sigma_y(I),$$

$$\Longrightarrow x \in \sigma_y(I),$$

$$\Longrightarrow x \wedge y \in I,$$

$$\Longrightarrow \frac{x}{J} \wedge \frac{y}{J} \in \frac{I}{J},$$

$$\Longrightarrow \frac{x}{J} \in \sigma_{\frac{y}{J}}(\frac{I}{J}).$$

Hence $\dfrac{\sigma_y(I)}{J} \subseteq \sigma_{\frac{y}{J}}(\dfrac{I}{J})$. Therefore $\sigma_{\frac{y}{J}}(\dfrac{I}{J}) = \dfrac{\sigma_y(I)}{J}$.

5) We have $\sigma_y(I) \subseteq \sigma_y(\sigma_y(I))$. Now let $x \in \sigma_y(\sigma_y(I))$. Then $x \wedge y \in \sigma_y(I)$, hence $x \wedge y \wedge y \in I$, and so $x \wedge y \in I$, i.e. $x \in \sigma_y(I)$.

6) Let $x \in \sigma_y(I) \to \sigma_y(J)$. Hence $(x] \cap I \subseteq (x] \cap \sigma_y(I) \subseteq \sigma_y(J)$. Thus $x \in I \to \sigma_y(J)$.

7) Let $x \in \sigma_y(I \to J)$. Then $x \wedge y \in I \to J$. It follows that $(x \wedge y] \cap I \subseteq J \subseteq \sigma_y(J)$. Hence $x \wedge y \in I \to \sigma_y(J)$. Thus $x \in \sigma_y(I \to \sigma_y(J))$.

8) Let $\sigma_y(I)$ be a prime ideal, $a \wedge b \in \sigma_y(I)$ and $a \notin \sigma_y(\sigma_y(I))$. By part (5), we have $\sigma_y(\sigma_y(I)) = \sigma_y(I)$. So, $a \wedge b \in \sigma_y(I)$ and $a \notin \sigma_y(I)$. Hence by hypothesis we get $b \in \sigma_y(I)$. Thus $b \in \sigma_y(\sigma_y(I))$, i.e. $b \in \sigma_y(I)$. Therefore $\sigma_y(I)$ is a $(\sigma_y, \sigma_y)-$primary.

Conversely, let $\sigma_y(I)$ be a $(\sigma_y, \sigma_y)-$primary , $a \wedge b \in \sigma_y(I)$ and $a \notin \sigma_y(I)$. Hence by part (5) , we have $a \notin \sigma_y(\sigma_y(I))$. Then by hypothesis we get $b \in \sigma_y(\sigma_y(I))$, i.e. $b \in \sigma_y(I)$. Therefore $\sigma_y(I)$ is a prime ideal of A.

9) Let I be $(\sigma_{x\vee y}, \sigma_0)-$primary, $a \wedge b \in I$ and $a \notin \sigma_y(I)$. Then $a \wedge y \notin I$. We know $a \wedge y \leq a \wedge (x \vee y)$, so $a \wedge (x \vee y) \notin I$. Hence $a \notin \sigma_{x\vee y}$. So by hypothesis we get $b \in \sigma_0(I) = I$, i.e. I is a $(\sigma_y, \sigma_0)-$primary. Similarly, we can prove that I is a $(\sigma_x, \sigma_0)-$primary.

10) By parts (1) , (2) , (3) , the proof is clear. □

Proposition 3.14. *Let $\{I_j\}_{j\in J}$ be a family of ideals of A. Then*

1) $\bigcap_{j\in J} \sigma_y(I_j) = \sigma_y(\bigcap_{j\in J} I_j)$.

2) If $\{I_j\}_{j\in J}$ is a chain of ideals, then $\bigcup_{j\in J} \sigma_y(I_j) = \sigma_y(\bigcup_{j\in J} I_j)$.

Proof. 1) We have

$$\begin{aligned} a \in \bigcap_{j\in J} \sigma_y(I_j) &\Longleftrightarrow a \in \sigma_y(I_j),\ \forall j \in J,\\ &\Longleftrightarrow a \wedge y \in I_j,\ \forall j \in J,\\ &\Longleftrightarrow a \wedge y \in \bigcap_{j\in J} I_j,\\ &\Longleftrightarrow a \in \sigma_y(\bigcap_{j\in J} I_j). \end{aligned}$$

2) Let $\{I_j\}_{j\in J}$ be a chain of ideals. It is easy to see that $\bigcup_{j\in J} I_j$ is an ideal. Let $x \in \bigcup_{j\in J} \sigma_y(I_j)$. Then there exists $j \in J$ such that $x \in \sigma_y(I_j)$, i.e. there exists $j \in J$ such that $x \wedge y \in I_j$. Then $x \wedge y \in \bigcup_{j\in J} I_j$ and so $x \in \sigma_y(\bigcup_{j\in J} I_j)$ therefore $\bigcup_{j\in J} \sigma_y(I_j) \subseteq \sigma_y(\bigcup_{j\in J} I_j)$.

Conversely, let $x \in \sigma_y(\bigcup_{j\in J} I_j)$. Then $x \wedge y \in \bigcup_{j\in J} I_j$. since $\{I_j\}_{j\in J}$ is a chain of ideals, there exists $j \in J$ such that $x \wedge y \in I_j$. Hence $x \in \sigma_y(I_j)$, for some $j \in J$ and so $x \in \bigcup_{j \in J} \sigma_y(I_j)$. Therefore $\sigma_y(\bigcup_{j\in J} I_j) \subseteq \bigcup_{j\in J} \sigma_y(I_j)$. Hence $\bigcup_{j\in J} \sigma_y(I_j) = \sigma_y(\bigcup_{j\in J} I_j)$. □

Remark 3.15. If I is a prime ideal, then so is $\sigma_y(I)$.

Proof. Let I be a prime ideal and $a \wedge b \in \sigma_y(I)$. Hence $(a \wedge b) \wedge y \in I$. Since I is a prime ideal, $a \in I$ or $b \wedge y \in I$. Then $a \wedge y \leqslant a \in I$ or $b \wedge y \in I$. Thus $a \in \sigma_y(I)$ or $b \in \sigma_y(I)$. Therefore $\sigma_y(I)$ is a prime ideal. □

Lemma 3.16. *Let I be an ideal. Then we have $\sigma_y(I) = (y] \to I$ in the complete Heyting algebra $(Id(A), \wedge, \vee, \to, \{0\}, A)$.*

Proof. Let $x \in \sigma_y(I)$. We prove that $x \in (y] \to I$. It is sufficient to show that $(x] \cap (y] \subseteq I$. By Lemma 2.2 (8), we have $(x \wedge y] = (x] \cap (y] \subseteq I$. Suppose that $t \in (x \wedge y]$, then there exists $n \in \mathbb{N}$ such that $t \leqslant n(x \wedge y)$. Since $x \in \sigma_y(I)$, so $x \wedge y \in I$. Hence we have $t \leqslant n(x \wedge y) \in I$. Thus $t \in I$. Therefore $\sigma_y(I) \subseteq (y] \to I$.
Conversely, let $x \in (y] \to I$. Hence $(x \wedge y] = (x] \cap (y] \subseteq I$. We obtain that $x \wedge y \in I$. Thus $x \in \sigma_y(I)$. We get $\sigma_y(I) = (y] \to I$ in the Heyting algebra $Id(A)$. □

Definition 3.17. An ideal I is called constant relative to expansion σ_y if $I = \sigma_y(I)$.

Lemma 3.18. *Let I be an ideal of A. Then $\sigma_y(I)$ is the smallest constant ideal relative to expansion σ_y such that $I \subseteq \sigma_y(I)$.*

Proof. By Lemma 3.13 (5), we have $\sigma_y(\sigma_y(I)) = \sigma_y(I)$. Thus $\sigma_y(I)$ is constant ideal relative to expansion σ_y. Let J be constant ideal relative to expansion σ_y such that $I \subseteq J$. It follows from Lemma 3.13 (3) that $\sigma_y(I) \subseteq \sigma_y(J) = J$. □

Corollary 3.19. *By Lemma 3.18, we get $\sigma_y(I)$ is constant ideal relative to σ_y. Hence every contstant ideal relative to σ_y is in $C(\sigma_y)$, where $C(\sigma_y) = \{\sigma_y(I) \mid I \in Id(A)\}$. For all elements $\sigma_y(I), \sigma_y(J) \in C(\sigma_y)$, we define two operations $\sqcap$ and $\sqcup$ as follows:*

$$\sigma_y(I) \sqcap \sigma_y(J) = \sigma_y(I \wedge J) \text{ and } \sigma_y(I) \sqcup \sigma_y(J) = \sigma_y(I \vee J).$$

Where $\sigma_y(I \wedge J)$ (or $\sigma_y(I \vee J)$) is infimum (supremum) of $\{\sigma_y(I), \sigma_y(J)\}$ in $C(\sigma_y)$. It is easy to show that $\sigma_y(I) \sqcap \sigma_y(J) = \sigma_y(I \wedge J)$. We show that $\sigma_y(I \vee J)$ is a supremum of $\sigma_y(I), \sigma_y(J)$ in $C(\sigma_y)$. By Lemma 3.13 (3), we get $\sigma_y(I), \sigma_y(J) \subseteq \sigma_y(I \vee J)$. For any constant ideal relative to σ_y, $\sigma_y(K)$ such that $\sigma_y(I), \sigma_y(J) \subseteq \sigma_y(K)$, we prove that $\sigma_y(I \vee J) \subseteq \sigma_y(K)$. Let $x \in \sigma_y(I \vee J)$.

Then $x \wedge y \in I \vee J$. *Hence* $x \wedge y \leqslant a \oplus c$, *for some* $a \in I \subseteq \sigma_y(I)$ *and* $c \in J \subseteq \sigma_y(J)$. *We obtain* $x \wedge y \in \sigma_y(I) \vee \sigma_y(J) \subseteq \sigma_y(K)$. *Thus* $x \in \sigma_y(\sigma_y(K)) = \sigma_y(K)$ *(by Lemma 3.13 (5)). This means that* $\sigma_y(I \vee J)$ *is the supremum of* $\{\sigma_y(I), \sigma_y(J)\}$ *in* $C(\sigma_y)$. *Thus* $(C(\sigma_y), \sqcap, \sqcup)$ *is a lattice.*

Theorem 3.20. $(C(\sigma_y), \sqcap, \sqcup, \rightarrow, \sigma_y(\{0\}), A)$ *is a complete Heyting algebra.*

Proof. By Remark 3.19 and Proposition 3.14, we have $\bigwedge_{j\in J}\sigma_y(I_j) = \sigma_y(\bigwedge_{j\in J} I_j)$ and so $C(\sigma_y)$ is complete. We only show that for $\sigma_y(I), \sigma_y(J), \sigma_y(K) \in C(\sigma_y)$,

1) $\sigma_y(I) \rightarrow \sigma_y(J) \in C(\sigma_y)$,

2) $\sigma_y(I) \sqcap \sigma_y(J) \subseteq \sigma_y(K) \Leftrightarrow \sigma_y(I) \leqslant \sigma_y(J) \rightarrow \sigma_y(K)$.

By Lemma 3.16 and Lemma 2.10, we obtain

$$\begin{aligned} \sigma_y(I) \rightarrow \sigma_y(J) &= ((y] \rightarrow I) \rightarrow ((y] \rightarrow J) \\ &= (y] \rightarrow [((y] \rightarrow I) \rightarrow J] \\ &= [(y] \odot ((y] \rightarrow I)] \rightarrow J \\ &= ((y] \wedge I) \rightarrow J \\ &= ((y] \odot I) \rightarrow J \\ &= (y] \rightarrow (I \rightarrow J) \in C(\sigma_y). \end{aligned}$$

For the case (2), it follows from Lemma 3.16 and Lemma 2.10 that

$$\begin{aligned} \sigma_y(I) \sqcap \sigma_y(J) \subseteq \sigma_y(K) &\Leftrightarrow \sigma_y(I \wedge J) \subseteq \sigma_y(K), \\ &\Leftrightarrow (y] \rightarrow I \wedge J \subseteq (y] \rightarrow K, \\ &\Leftrightarrow (y] \wedge ((y] \rightarrow I \wedge J) \subseteq K, \\ &\Leftrightarrow (y] \wedge I \wedge J \subseteq K, \\ &\Leftrightarrow (y] \wedge I \subseteq J \rightarrow K, \\ &\Leftrightarrow (y] \rightarrow ((y] \wedge I) \subseteq (y] \rightarrow (J \rightarrow K), \\ &\Leftrightarrow ((y] \rightarrow (y]) \wedge ((y] \rightarrow I) \subseteq (y] \rightarrow (J \rightarrow K), \\ &\Leftrightarrow (y] \rightarrow I \subseteq (y] \rightarrow (J \rightarrow K), \\ &\Leftrightarrow \sigma_y(I) \subseteq \sigma_y(J) \rightarrow \sigma_y(K). \end{aligned}$$

Thus, $C(\sigma_y)$ of all constant ideals relative to expansion σ_y is the Heyting algebra. □

Lemma 3.21. *Let* $I, J \in Id(A)$ *such that* $J \subseteq I$. *If* I *is prime, then* $\sigma_y(J) \subseteq I$, *for all* $y \in A - I$.

Proof. Let $x \in \sigma_y(J)$. Then $x \wedge y \in J \subseteq I$. By the fact that I is a prime ideal, we get $x \in I$ or $y \in I$. Since $y \in A - I$, we have $x \in I$, i.e. $\sigma_y(J) \subseteq I$. □

Theorem 3.22. *Let $I \in Id(A)$. Then I is prime if and only if $\sigma_y(I) = I$, for all $y \in A - I$.*

Proof. Let I be a prime ideal of A and $x \in \sigma_y(I)$. Then $x \wedge y \in I$. Hence $x \in I$ *or* $y \in I$. We have $y \in A - I$, so $x \in I$. Therefore $\sigma_y(I) \subseteq I$. Thus $\sigma_y(I) = I$, for all $y \in A - I$.
Conversely, let $\sigma_y(I) = I$, for all $y \in A - I$. Also $x \wedge y \in I$ and $y \notin I$. Then $x \in \sigma_y(I)$. So by hypothesis, we get $x \in I$, i.e. I is a prime ideal of A. □

Corollary 3.23. *An ideal I of A is prime if and only if I is $(\sigma_y, \sigma_0)-$primary for all $y \in A-I$.*

Proof. Let I be a prime ideal of $A, a \wedge b \in I$ and $a \notin \sigma_y(I)$. By Theorem 3.22, $a \notin I$. Since I is a prime ideal, hence $b \in I$. Therefore $b \in \sigma_0(I)$.
Conversely, let I be a $(\sigma_y, \sigma_0)-$primary for all $y \in A - I$. By Theorem 3.22, it is enough to show that $\sigma_y(I) \subseteq I$, for all $y \in A - I$. Let $z \in \sigma_y(I)$. Then $z \wedge y \in I$. We have $y \wedge y = y \notin I$, so $y \notin \sigma_y(I)$. Hence by hypothesis we get $z \in \sigma_0(I) = I$. □

Definition 3.24. Let $I, J \in Id(A)$. Then (I, J) is defined to be

$$(I, J) = \bigcap_{y \in J} \sigma_y(I) = \{x \in A \mid x \wedge y \in I, \forall y \in J\}.$$

Theorem 3.25. *Let $I, J \in Id(A)$. Then $(I, J) \in Id(A)$.*

Proof. Let $z, t \in (I, J)$. We must show that $z \oplus t \in (I, J)$. We get $z \wedge y, t \wedge y \in I$, for all $y \in J$. So $(z \wedge y) \oplus (t \wedge y) \in I$. By Lemma 2.2 (7), we have $y \wedge (z \oplus t) \leqslant (y \wedge z) \oplus (y \wedge t) \in I$, for all $y \in J$. So $z \oplus t \in (I, J)$.
Now let $z \in (I, J)$ and $t \leqslant z$. Then $z \wedge y \in I$, for all $y \in J$, and also $t \wedge y \leqslant z \wedge y$ Hence $t \wedge y \in I$, for all $y \in J$. Therefore $t \in (I, J)$, and so $(I, J) \in Id(A)$. □

Theorem 3.26. *Let $I, J \in Id(A)$ and I be $(\sigma, \sigma_0)-$primary. Then*
1) *If J is not contained in $\sigma(I)$, then $(I, J) = I$,*
2)*(I, J) is $(\sigma, \sigma_0)-$primary.*

Proof. 1) Let $x \in I$. Then $x \wedge y \in I$, for all $y \in J$. So $x \in (I, J)$, i.e. $I \subseteq (I, J)$. Now let $z \in (I, J)$. Then $z \wedge y \in I$, for all $y \in J$. We have $J \nsubseteq \sigma_y(I)$. So there exists $y_0 \in J$ such that $y_0 \notin \sigma(I)$. By the fact that $y_0 \in J$, we get $z \wedge y_0 \in I$. Then by $y_0 \notin \sigma(I)$. Since I is $(\sigma, \sigma_0)-$primary, we get $z \in \sigma_0(I) = I$. Therefore $(I, J) \subseteq I$. Hence $(I, J) = I$.
2) Let $a \wedge b \in (I, J)$ and $a \notin \sigma((I, J))$. We have $I \subseteq (I, J)$, so $\sigma(I) \subseteq \sigma((I, J))$. Hence $a \notin \sigma(I)$.

Also by $a \wedge b \in (I, J)$, we get $a \wedge b \wedge y \in I$, for all $y \in J$. By the fact that $a \wedge b \wedge y \in I$ and $a \notin \sigma(I)$, we get $b \wedge y \in I$, for all $y \in J$, because I is $(\sigma, \sigma_0)-$primary. Therefore $b \in (I, J)$. So the proof is complete. □

4. $(\tau, \sigma)-$obstinate and $(\tau, \sigma)-$Boolean ideals in $MV-$algebras

Definition 4.1. Let τ and σ be expansions of ideals. Then an ideal I of A is said to be $(\tau, \sigma)-$obstinate if

$$(\forall a \in A)(a \in \tau(I) \text{ or } a^* \in \sigma(I)).$$

Example 4.2. Consider $MV-$algebra A of Example 3.4, Let:

$$\begin{aligned} \tau : & Id(A) \longrightarrow Id(A) \\ & I_0 \longmapsto I_1 \\ & I_1 \longmapsto I_1 \\ & I_2 \longmapsto I_3 \\ & I_3 \longmapsto I_3. \end{aligned}$$

Obviously, I_2 is $(\tau, \sigma_0)-$obstinate and I_1 is not $(\tau, \sigma_0)-$obstinate ideal of A, because $a \notin \tau(I_1) = I_1$ and $a^* = d \notin \sigma_0(I_1) = I_1$.

Lemma 4.3. *Let $I \in Id(A)$. I is $(\tau_0, \sigma_0)-$obstinate ideal if and only if it is an obstinate ideal of A.*

Proof. By Lemma 2.5, it is clear. □

Theorem 4.4. *Let σ and τ be expansions of ideals in A. Then I is $(\tau, \sigma)-$obstinate ideal of A if and only if I is (σ, τ)-obstinate ideal of A.*

Proof. Let I be an $(\tau, \sigma)-$obstinate and $\forall a \in A$, $a \notin \sigma(I)$. We have $(a^*)^* = a \notin \sigma(I)$ and I is $(\tau, \sigma)-$obstinate. Therefore $a^* \in \tau(I)$.
Conversely, it is similar. □

Theorem 4.5. *Let σ and τ be expansions of ideals in A. Then every obstinate ideal of A is $(\tau, \sigma)-$obstinate.*

Proof. Let I be obstinate ideal and for all $a \in A, a \notin \tau(I)$. By the fact that $I \subseteq \tau(I)$, we get $a \notin I$. Since I is an obstinate ideal of A, so $a^* \in I$. By $I \subseteq \sigma(I)$, we get $a^* \in \sigma(I)$. Therefore I is an $(\tau, \sigma)-$obstinate. □

The following example shows that the converse of the above theorem my not be true, in general.

Example 4.6. 1) Let $\Omega = \{1, 2\}$ and $A = P(\Omega) = \{\{1\}, \{2\}, \{1, 2\}, \emptyset\}$, which is an $MV-$algebra with $\oplus = \cup, \odot = . = \cap$ and $D^* = \Omega - D$ for any $D \in A$. It is clear that $I_0 = \{\emptyset\}, I_1 = \{\{1\}, \emptyset\}, I_2 = \{\{2\}, \emptyset\}$ and $I_3 = A$ are ideals of A. Let $\tau(I) = A$ for all $I \in Id(A)$. Obviously I_0 is $(\tau, \sigma_0)-$obstinate ideal but I_0 is not obstinate ideal of A, because $\{1\} \notin I_0$ and $\{1\}^* = \{2\} \notin I_0$.
2) In Example 3.4, Let $\tau(I) = A$ for all $I \in Id(A)$. Obviously I_0 is $(\tau, \sigma_0)-$
obstinate ideal but I_0 is not obstinate ideal of A, because $a \notin I_0$ and $a^* = d \notin I_0$.

Theorem 4.7. *Let τ be an expansion of ideals in A. If $\tau(I)$ is an obstinate ideal, then I is an $(\tau, \tau)-$obstinate ideal of A.*

Proof. Let $\tau(I)$ be an obstinate ideal and for all $a \in A, a \notin \tau(I)$. $\tau(I)$ is an obstinate ideal, Then $a^* \in \tau(I)$. Therefore I is an $(\tau, \tau)-$obstinate. □

Corollary 4.8. *Let τ and σ be expansions of ideals in A and $I \in Id(A)$. If $\tau(I)$ is an obstinate ideal, then $\tau(I)$ is an $(\tau, \sigma)-$obstinate ideal of A.*

Proof. Let $\tau(I)$ be an obstinate ideal and for all $a \in A, a \notin \tau(\tau(I))$. We have $\tau(I) \subseteq \tau(\tau(I))$, so $a \notin \tau(I)$. By the fact that $\tau(I)$ is an obstinate ideal, then $a^* \in \tau(I)$. Hence by $\tau(I) \subseteq \sigma(\tau(I))$, we get $a^* \in \sigma(\tau(I))$. Therefore $\tau(I)$ is an $(\tau, \sigma)-$obstinate ideal of A. □

Theorem 4.9. *Let τ, σ, ψ and δ be expansions of ideals in A such that $\tau \subseteq \psi$ and $\sigma \subseteq \delta$. Then every $(\tau, \sigma)-$obstinate ideal is $(\psi, \delta)-$obstinate.*

Proof. Let I be an $(\tau, \sigma)-$obstinate and for all $a \in A, a \notin \psi(I)$. We have $\tau \subseteq \psi$, so $a \notin \tau(I)$. Then by hypothesis, we get $a^* \in \sigma(I)$. Hence by the fact that $\sigma \subseteq \delta$, we get $a^* \in \delta(I)$, i.e. I is an $(\psi, \delta)-$obstinate ideal of A. □

Corollary 4.10. *Let τ and σ be expansions of ideals in A. Then every $(\tau, \tau_0)-$obstinate ideal is an $(\tau, \sigma)-$obstinate ideal of A.*

Proof. Let I be an $(\tau, \tau_0)-$obstinate and for all $a \in A, a \notin \tau(I)$. By hypothesis, we get $a^* \in \tau_0(I) = I$. We know $I \subseteq \sigma(I)$, so $a^* \in \sigma(I)$. Therefore I is an $(\tau, \sigma)-$obstinate ideal of A. $\square$

Lemma 4.11. *Let $I \in Id(A)$. Then $\sigma_y(I)$ is an obstinate ideal if and only if $\sigma_y(I)$ is an $(\sigma_y, \sigma_y)-$obstinate ideal of A.*

Proof. Let $\sigma_y(I)$ be an obstinate ideal and for all $a \in A, a \notin \sigma_y(\sigma_y(I))$. By part (5) of Lemma 3.13, we have $\sigma_y(\sigma_y(I)) = \sigma_y(I)$. So $a \notin \sigma_y(I)$. Hence by hypothesis, we get $a^* \in \sigma_y(I)$. Thus $a^* \in \sigma_y(\sigma_y(I))$, i.e. $\sigma_y(I)$ is an $(\sigma_y, \sigma_y)-$obstinate ideal of A.
Conversely, let $\sigma_y(I)$ be an $(\sigma_y, \sigma_y)-$obstinate ideal and for all $a \in A$, $a \notin \sigma_y(I)$. Hence by part (5) of Lemma 3.13, we have $a \notin \sigma_y(\sigma_y(I))$. Then by hypothesis, we get $a^* \in \sigma_y(\sigma_y(I))$, i.e. $a^* \in \sigma_y(I)$. Therefore $\sigma_y(I)$ is an obstinate ideal of A. $\square$

Theorem 4.12. *Let τ and σ be expansions of ideals A. Then every $(\tau, \sigma)-$obstinat is $(\tau, \sigma)-$primary ideal of A.*

Proof. Let $a \wedge b \in I$ and $a \notin \tau(I)$. Since I is an $(\tau, \sigma)-$obstinat and $\sigma(I)$ is an ideal of A, if $a^* \in \sigma(I)$, then $a^* \oplus (a \wedge b) \in \sigma(I)$, so $a^* \oplus (a \odot (a^* \oplus b)) \in \sigma(I)$, and so , $a^* \oplus b \leqslant a^* \vee (a^* \oplus b) \in \sigma(I)$. Therefore $b \leqslant a^* \oplus b \in \sigma(I)$ and so $b \in \sigma(I)$. $\square$

The following example shows that the converse of the above theorem may not be true in general.

Example 4.13. Consider $MV-$algebra A in Example 3.4. Define expansion τ of ideals as follows:

$$\begin{aligned} \tau :& Id(A) \longrightarrow Id(A) \\ & I_0 \longmapsto I_0 \\ & I_1 \longmapsto I_1 \\ & I_2 \longmapsto I_3 \\ & I_3 \longmapsto I_3. \end{aligned}$$

Obviously, $I_1 = \{0, c\}$ is a prime ideal of A. By Theorem 3.7, I_1 is $(\tau, \sigma_0)-$primary but I_1 is not $(\tau, \sigma_0)-$obstinate because $d \notin \tau(I_1)$ *and* $d^* = a \notin \sigma_0(I_1) = I_1$.

Theorem 4.14. *Let I be Boolean ideal of A and $(\tau, \sigma)-$primary ideal of A. Then I is $(\tau, \sigma)-$obstinate ideal of A.*

Proof. Let for all $a \in A, a \notin \tau(I)$. Since I is Boolean ideal, hence $a \wedge a^* \in I$ and I is $(\tau, \sigma)-$primary ideal, therefore $a^* \in \sigma(I)$. Then I is $(\tau, \sigma)-$ obstinate ideal of A. □

Definition 4.15. Let τ and σ be expansions of ideals A. Then ideal I of A is said to be (τ, σ)-Boolean if

$$(\forall a \in A)(a \wedge a^* \in \tau(I) \ or \ a \wedge a^* \in \sigma(I)).$$

Example 4.16. Consider $MV-$algebra A in Example 3.4. Define expansion σ of ideals as follows:

$$\begin{aligned} \sigma :&Id(A) \longrightarrow Id(A) \\ &I_0 \longmapsto I_1 \\ &I_1 \longmapsto I_1 \\ &I_2 \longmapsto I_3 \\ &I_3 \longmapsto I_3. \end{aligned}$$

I_2 is $(\tau_0, \sigma)-$Boolean and I_0 is not $(\tau_0, \sigma)-$Boolean ideal of A, becasue $a \wedge a^* = a \wedge d = a, a \notin \tau_0(I_0) = I_0$ and $a \notin \sigma(I_0) = I_1$.

Theorem 4.17. *Let τ and σ be expansions of ideals. Every $(\tau, \sigma)-$obstinate ideal is $(\tau, \sigma)-$Boolean ideal of A.*

Proof. Let I be an $(\tau, \sigma)-$obstinate and for all $x \in A, \ x \wedge x^* \notin \tau(I)$ and $x \wedge x^* \notin \sigma(I)$. We have $x \wedge x^* \leqslant x$ and $x \wedge x^* \leqslant x^*$, so $x \notin \tau(I)$ and $x^* \notin \sigma(I)$, that is a contradiction. Therefore for all $x \in A, \ x \wedge x^* \in \tau(I)$ or $x \wedge x^* \in \sigma(I)$. Hence I is a $(\tau, \sigma)-$Boolean ideal of A. □

The following example shows that the converse of the above theorem may not be true, in general.

Example 4.18. Consider $MV-$algebra A in Example 3.8. Define expansion σ of ideals as follows:

$$\sigma : Id(A) \longrightarrow Id(A)$$

$$I_0 \longmapsto I_0$$

$$I_1 \longmapsto I_1$$

$$I_2 \longmapsto I_3$$

$$I_3 \longmapsto I_3.$$

Obviously, I_0 is a $(\tau_0, \sigma)-$Boolean but I_0 is not an $(\tau_0, \sigma)-$obstinate ideal of A, because $a \notin \tau_0(I_0) = I_0$ and $a^* = b \notin \sigma(I_0) = I_0$.

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