



ON SOME CLASSES OF EXPANSIONS OF IDEALS IN MV -ALGEBRAS

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ABSTRACT. In this paper, we introduce the notions of expansion of ideals in MV -algebras, (τ, σ) -primary, (τ, σ) -obstinate and (τ, σ) -Boolean in MV -algebras. We investigate the relations of them. For example, we show that every (τ, σ) -obstinate ideal of an MV -algebra is (τ, σ) -primary and (τ, σ) -Boolean. In particular, we define an expansion σ_y of ideals in an MV -algebra. A characterization of expansion ideal with respect to σ_y is given. Finally, we show that the class $C(\sigma_y)$ of all constant ideals relative to σ_y is a Heyting algebra.

1. INTRODUCTION

MV -algebras were introduced by C. C. Chang in [2] and [3] as an algebraic counterpart of the Łukasiewicz infinite valued propositional logic.

In order to keep the paper brief, we refer the reader to [2, 3, 4, 5, 8] for results on MV -algebras. In [7] results regarding expansions of filters in residuated lattices were obtained. In this paper, we illustrate how we can obtain various results for some types of expansions of ideals in MV -algebras.

That is, we introduce expansions of ideals in MV -algebras and study their some properties.

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Also, we introduce (τ, σ) -primary, (τ, σ) -obstinate, (τ, σ) -Boolean and investigate the relations between (τ, σ) -obstinate and the other expansions of ideals in an MV -algebra.

In particular, we show that every (τ, σ) -obstinate ideal is (τ, σ) -primary but in general, the converse of it is not true and every (τ, σ) -obstinate is (τ, σ) -Boolean ideal of A , but in general, the converse of it is not true. We define expansions ideal σ_y , and so we give a characterization of $\sigma_y(I)$. We show that $\sigma_y(I)$ is the smallest constant ideal relative to σ_y such that $I \subseteq \sigma_y(I)$. Finally, we show that class $C(\sigma_y)$ of all constant ideals relative to expansion σ_y is a complete Heyting algebra.

2. Preliminaries

We recollect some definitions and results which will be used in the following:

Definition 2.1. [2, 4, 8] An MV -algebra is a structure $(A, \oplus, *, 0)$ where \oplus is a binary operation, $*$ is a unary operation, and 0 is a constant such that the following axioms are satisfied for any $a, b \in A$:

- (MV1) $(A, \oplus, 0)$ is an abelian monoid,
- (MV2) $(a^*)^* = a$,
- (MV3) $0^* \oplus a = 0^*$,
- (MV4) $(a^* \oplus b)^* \oplus b = (b^* \oplus a)^* \oplus a$.

Note that $1 = 0^*$ and the auxiliary operation \odot as follow: $x \odot y = (x^* \oplus y^*)^*$. We recall that the natural order determines a bounded distributive lattice structure such that $x \vee y = x \oplus (x^* \odot y) = y \oplus (x \odot y^*)$ and $x \wedge y = x \odot (x^* \oplus y) = y \odot (y^* \oplus x)$.

Lemma 2.2. [4, 8] *In each MV -algebra, the following relations hold for all $x, y, z \in A$:*

- (1) $x \leq y$ if and only if $y^* \leq x^*$,
- (2) If $x \leq y$, then $x \oplus z \leq y \oplus z$ and $x \odot z \leq y \odot z$,
- (3) $x \leq y$ if and only if $x^* \oplus y = 1$ if and if $x \odot y^* = 0$,
- (4) $x, y \leq x \oplus y$ and $x \odot y \leq x, y$, $x \leq nx = x \oplus x \oplus \cdots \oplus x$ and $x^n = x \odot x \odot \cdots \odot x \leq x$,
- (5) $x \oplus x^* = 1$ and $x \odot x^* = 0$,
- (6) If $x \leq y$ and $z \leq t$, then $x \oplus z \leq y \oplus t$,
- (7) $(x \oplus y) \wedge z \leq (x \wedge z) \oplus (y \wedge z)$,
- (8) $(x] \cap (y] = (x \wedge y]$.

Definition 2.3. [2, 8] An ideal of an MV -algebra A is a nonempty subset I of A satisfying the following conditions:

- (1) If $x \in I$, $y \in A$ and $y \leq x$, then $y \in I$,
- (2) If $x, y \in I$, then $x \oplus y \in I$.

We denote by $Id(A)$ the set of ideals of an MV -algebra A .

Definition 2.4. [4] Let I be an ideal of an MV -algebra A . I is a prime ideal of A if $I \neq A$ and $a \wedge b \in I, a \notin I$, then $b \in I$ for all $a, b \in A$.

- [6] An ideal I of an MV -algebra A is called Boolean ideal if $x \wedge x^* \in I$, for all $x \in A$.
- [5] A proper ideal of A is called an obstinate ideal A , if $x, y \notin I$ implies $x \odot y^* \in I$ and $y \odot x^* \in I$, for all $x, y \in A$.

Lemma 2.5. [5] Let I be a proper ideal of A . Then I is an obstinate ideal if and only if $x \in I$ or $x^* \in I$, for all $x \in A$.

Remark 2.6. [4] In an MV -algebra A , the distance function is

$$d : A \times A \longrightarrow A, \quad d(x, y) := (x \odot y^*) \oplus (y \odot x^*).$$

Suppose that I is an ideal of an MV -algebra A . Define $x \sim_I y$ if and only if $d(x, y) \in I$ if and only if $x \odot y^* \in I$ and $y \odot x^* \in I$. Then \sim_I is a congruence relation on A . The set of all congruence classes is denoted by A/I then $A/I = \{[x] : x \in A\}$, where $[x] = \{y \in A : x \sim_I y\}$. We can easily to see that $x \in I$ if and only if $x/I = 0/I$. The MV -algebra operations on A/I given by $x/I \oplus y/I = (x \oplus y)/I$ and $(x/I)^* = x^*/I$, are well defined. Hence $(A/I, \oplus, *, [0])$ becomes an MV -algebra [4, 8].

Definition 2.7. [8] A Heyting algebra is a lattice (A, \vee, \wedge) with 0 such that for every $a, b \in A$, there exists an element $a \rightarrow b \in A$ such that for $x \in A$, $a \wedge x \leq b$ if and only if $x \leq a \rightarrow b$.

Note that in a Heyting algebra A , $x \odot x = x$, for $x \in A$, hence $x \odot y = x \wedge y = x \odot (x \rightarrow y)$, for all $x, y \in A$.

Proposition 2.8. [8] $(Id(A), \wedge, \vee, \rightarrow, \{0\}, A)$ is a complete Heyting algebra, where for all $I, J \in Id(A)$:

$$I \wedge J = I \cap J, \quad I \vee J = (I \cup J) \text{ and } I \rightarrow J = \{x \in A \mid I \cap (x) \subseteq J\}. \text{ Hence } I \wedge (\vee_\lambda J_\lambda) = \vee_\lambda (I \wedge J_\lambda).$$

Definition 2.9. [4] Let $X \subseteq A$. The ideal of A generated by X will be denoted by (X) . We have

$$(X) = \{a \in A \mid a \leq x_1 \oplus x_2 \oplus \dots \oplus x_n, \text{ for some } n \in \mathbb{N} \text{ and } x_i \in X, 1 \leq i \leq n\}$$

In particular, $(a) = \{x \in A \mid x \leq na, \text{ for some } n \in \mathbb{N}\}$.

Lemma 2.10. [8] Let A be a Heyting algebra. Then we have the following rules, for $x, y, z \in A$:

- 1) $1 \rightarrow x = x$, $x \rightarrow x = 1$,
- 2) $x \leq y$ implies $x \rightarrow z \leq y \rightarrow z$,
- 3) $x \rightarrow (y \rightarrow z) = (x \odot y) \rightarrow z = y \rightarrow (x \rightarrow z)$,
- 4) $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$.

3. EXPANSIONS OF IDEALS IN MV -ALGEBRAS

Definition 3.1. An expansion of ideals in A is defined to be a function

$\sigma : Id(A) \longrightarrow Id(A)$ such that

$f_1) (\forall I \in Id(A))(I \subseteq \sigma(I)),$

$f_2) (\forall I, J \in Id(A))(I \subseteq J \implies \sigma(I) \subseteq \sigma(J)).$

Example 3.2. (1) The function $\sigma_0 : Id(A) \longrightarrow Id(A)$ defined by $\sigma_0(I) = I$ is an expansion of ideals in A .

(2) $C = \{0, c, 2c, 3c, \dots, 1 - 2c, 1 - c, 1\}$ be the MV -algebra defined in [8, Example 2.6] with operations as follows:

if $x = nc$ and $y = mc$, then $x \oplus y := (m + n)c$,

if $x = 1 - nc$ and $y = 1 - mc$, then $x \oplus y := 1$,

if $x = nc$ and $y = 1 - mc$ and $m \leq n$, then $x \oplus y := 1$,

if $x = nc$ and $y = 1 - mc$ and $n < m$, then $x \oplus y := 1 - (m - n)c$,

if $x = 1 - mc$ and $y = nc$ and $m \leq n$, then $x \oplus y := 1$,

if $x = 1 - mc$ and $y = nc$ and $n < m$, then $x \oplus y := 1 - (m - n)c$,

if $x = nc$, then $x^* := nc$.

It has only three ideals: $I_0 = \{0\}$, $I_1 = \{0, c, 2c, 3c, \dots\}$ and $I_2 = C$.

$$\sigma : Id(C) \longrightarrow Id(C)$$

$$I_0 \longmapsto I_1, I_1 \longmapsto I_2, I_2 \longmapsto I_2.$$

The function σ is an expansion of ideals in C .

(3) Let \mathbb{R}^* be a nonstandard model of real numbers with natural order and ε be a positive infinitesimal element of \mathbb{R}^* . Let $\varepsilon^2 = \varepsilon \cdot \varepsilon, \dots, \varepsilon^n = \varepsilon \cdot \varepsilon \dots \varepsilon$ (n -times), where \cdot is the usual product in the field \mathbb{R}^* ; then $\varepsilon^i > 0$, for any $i \in \mathbb{N}$ and $\varepsilon^i \leq \varepsilon^j$, for $i > j$.

The unit interval $[0, 1]^* \subseteq \mathbb{R}^*$ is an MV -algebra under the operations: $x \oplus y = \min\{1, x + y\}$, $x^* = 1 - x$. Let \mathbb{N} be the ordered set of positive natural numbers. For every $n \in \mathbb{N}$, let E_n be the subalgebra of $[0, 1]^*$ generated by $\{\varepsilon, \varepsilon^2, \dots, \varepsilon^n\}$ and E be the subalgebra $\bigcup_{n \in \mathbb{N}} E_n$. We recall

that [1] that $E = \langle \varepsilon^i \mid i \in \mathbb{N} \rangle$. The set of all ideals of E is $\{0\}, \langle \varepsilon \rangle, \dots, \langle \varepsilon^i \rangle, \dots$, where $i \in \mathbb{N}$ and $\langle \varepsilon^i \rangle \subseteq \langle \varepsilon^j \rangle$, for any $i > j$. The function $\sigma : Id(E) \longrightarrow Id(E)$ defined by $\sigma(\{0\}) = \{0\}$ and $\sigma(\langle \varepsilon^i \rangle) = \langle \varepsilon^{i+1} \rangle$, for every $i \in \mathbb{N}$. Obviously, σ does not hold in (f_1) .

Hence σ is not expansion of ideals in E .

Definition 3.3. Let τ and σ be expansions of ideals. Then an ideal I of A is said to be (τ, σ) -primary, if

$$(\forall a, b \in A)(a \wedge b \in I, a \notin \tau(I) \implies b \in \sigma(I)).$$

Example 3.4. Let $A = \{0, a, b, c, d, 1\}$, where $0 < a, c < d < 1$ and $0 < a < b < 1$. Define \odot , \oplus and $*$ as follows:

\odot	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	a	0	0	a
b	0	a	b	0	a	b
c	0	0	0	c	c	c
d	0	0	a	c	c	d
1	0	a	b	c	d	1

\oplus	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	b	b	d	1	1
b	b	b	b	1	1	1
c	c	d	1	c	d	1
d	d	1	1	d	1	1
1	1	1	1	1	1	1

$*$	0	a	b	c	d	1
	1	d	c	b	a	0

Then $(A, \oplus, \odot, *, 0, 1)$ is an MV-algebra [8]. It has only four ideals:
 $I_0 = \{0\}, I_1 = \{0, c\}, I_2 = \{0, a, b\}, I_3 = A.$

$$\begin{array}{ll} \sigma : Id(A) \longrightarrow Id(A) & \delta : Id(A) \longrightarrow Id(A) \\ I_0 \longmapsto I_0 & I_0 \longmapsto I_0 \\ I_1 \longmapsto I_1 & I_1 \longmapsto I_3 \\ I_2 \longmapsto I_3 & I_2 \longmapsto I_3 \\ I_3 \longmapsto I_3. & I_3 \longmapsto I_3. \end{array}$$

Are expansions of ideals A .

Obviously, I_1 is (σ, δ) -primary and since $a \wedge c = 0$ but $a \notin I_0$ and $c \notin I_0$, hence I_0 is not (σ, δ) -primary.

Lemma 3.5. *Let $I \in Id(A), \tau_0$ and σ_0 be the functions in Example 3.2 (1). Then I is (τ_0, σ_0) -primary if and only if it is a prime ideal of A .*

Proof. Let I be a (τ_0, σ_0) -primary, and $a \wedge b \in I$ such that $a \notin I$. So $a \notin \tau_0(I)$. Hence $b \in \sigma_0(I) = I$. Therefore I is a prime ideal of A .

Conversely, let I be a prime ideal of $A, a \wedge b \in I$ and $a \notin \tau_0(I) = I$. So $b \in I$. Hence $b \in \sigma_0(I) = I$. Therefore I is a (τ_0, σ_0) -primary. \square

Theorem 3.6. *Let σ and τ be expansions of ideals in A . Then I is a (τ, σ) -primary if and only if I is (σ, τ) -primary.*

Proof. Let I be (τ, σ) -primary, $a \wedge b \in I$ and $a \notin \sigma(I)$. We must show that $b \in \tau(I)$. Let $b \notin \tau(I)$. By the fact that I is a (τ, σ) -primary, we get $a \in \sigma(I)$, which is a contradiction. So $b \in \tau(I)$, i.e. I is a (σ, τ) -primary. The proof of the converse part is similar. \square

Theorem 3.7. *Let τ and σ be expansion of ideals in A . Then every prime ideal of A is (τ, σ) -primary.*

Proof. Let I be prime ideal of A , $a \wedge b \in I$ and $a \notin \tau(I)$. By the fact that $I \subseteq \tau(I)$, we get $a \notin I$. I is prime ideal of A , so $b \in I$. By $I \subseteq \sigma(I)$, we get $b \in \sigma(I)$. Therefore I is a (τ, σ) -primary. \square

The following example show that the converse of the above theorem may not be true, in general.

Example 3.8. Let $A = \{0, a, b, 1\}$, where $0 < a, b < 1$. Define \odot, \oplus and $*$ as follows:

\odot	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

\oplus	0	a	b	1
0	0	a	b	1
a	a	a	1	1
b	b	1	b	1
1	1	1	1	1

$*$	0	a	b	1
	1	b	a	0

Then $(A, \oplus, \odot, *, 0, 1)$ is an MV-algebra [8].

$I_0 = \{0\}, I_1 = \{0, a\}, I_2 = \{0, b\}$ and $I_3 = A$ are only ideals of A . Define

$$\sigma : Id(A) \longrightarrow Id(A)$$

$$I_0 \longmapsto I_2$$

$$I_1 \longmapsto I_3$$

$$I_2 \longmapsto I_3$$

$$I_3 \longmapsto I_3.$$

Obviously, I_0 is (τ_0, σ) -primary but I_0 is not prime ideal of A , because $a \wedge b = 0 \in I_0$ but $a \notin I_0$ and $b \notin I_0$.

Theorem 3.9. *Let τ be an expansion of ideals A . If $\tau(I)$ is a prime ideal, then I is a (τ, τ) -primary.*

Proof. Let $\tau(I)$ be a prime ideal, $a \wedge b \in I$ and $a \notin \tau(I)$. We know $I \subseteq \tau(I)$, so $a \wedge b \in \tau(I)$. Since $\tau(I)$ is a prime ideal, then $b \in \tau(I)$. Therefore I is (τ, τ) -primary. \square

Corollary 3.10. *Let τ and σ be expansions of ideals in A and $I \in Id(A)$ and $\tau(I)$ be a prime ideal. Then $\tau(I)$ is a (τ, σ) -primary.*

Proof. Let $\tau(I)$ be a prime ideal, $a \wedge b \in \tau(I)$ and $a \notin \tau(\tau(I))$. We have $\tau(I) \subseteq \tau(\tau(I))$, so $a \notin \tau(I)$. By the fact that $\tau(I)$ is a prime ideal of A , then $b \in \tau(I)$. Then by $\tau(I) \subseteq \sigma(\tau(I))$, we get $b \in \sigma(\tau(I))$. Therefore $\tau(I)$ is a (τ, σ) -primary. \square

Theorem 3.11. *Let τ, σ, ψ and δ be expansions of ideals in A such that $\tau \subseteq \psi$ and $\sigma \subseteq \delta$. Then every (τ, σ) -primary ideal is (ψ, δ) -primary.*

Proof. Let I be (τ, σ) -primary, $a \wedge b \in I$ and $a \notin \psi(I)$. We have $\tau \subseteq \psi$, so $a \notin \tau(I)$. Then by hypothesis, we get $b \in \sigma(I)$. Hence by the fact that $\sigma \subseteq \delta$, we get $b \in \delta(I)$, i.e. I is a (ψ, δ) -primary. \square

Corollary 3.12. *Let τ and σ be expansions of ideals in A . Then every (τ, τ_0) -primary is (τ, σ) -primary.*

Proof. Let I be a (τ, τ_0) -primary, $a \wedge b \in I$ and $a \notin \tau(I)$. By hypothesis we get $b \in \tau_0(I) = I$. We know $I \subseteq \sigma(I)$, so $b \in \sigma(I)$. Therefore I is a (τ, σ) -primary. \square

Let $y \in Id(A)$. The function $\sigma_y : Id(A) \rightarrow Id(A)$ defined by

$$(\forall I \in Id(A))(\sigma_y(I) = \{x \in A \mid x \wedge y \in I\}).$$

Lemma 3.13. *Let $I, J \in Id(A)$. Then*

- 1) $\sigma_y(I) \in Id(A)$,
- 2) $I \subseteq \sigma_y(I)$,
- 3) $I \subseteq J \implies \sigma_y(I) \subseteq \sigma_y(J)$,
- 4) $\sigma_y\left(\frac{I}{J}\right) = \frac{\sigma_y(I)}{J}$, where $J \subseteq I$,
- 5) $\sigma_y(\sigma_y(I)) = \sigma_y(I)$,
- 6) $\sigma_y(I) \rightarrow \sigma_y(J) \subseteq I \rightarrow \sigma_y(J)$,
- 7) $\sigma_y(I \rightarrow J) \subseteq \sigma_y(I \rightarrow \sigma_y(J))$,
- 8) $\sigma_y(I)$ is prime ideal if and only if $\sigma_y(I)$ is (σ_y, σ_y) -primary,
- 9) If I is $(\sigma_{x \vee y}, \sigma_0)$ -primary, then I is (σ_x, σ_0) and (σ_y, σ_0) -primary,
- 10) σ_y is an expansion of ideals in A .

Proof. 1) We have $0 \wedge y = 0 \in I$, so $0 \in \sigma_y(I)$. Now let $x \leq z$ and $z \in \sigma_y(I)$. Hence $z \wedge y \in I$. We have $x \wedge y \leq z \wedge y$. Then by ideal property, we get $x \wedge y \in I$, i.e. $x \in \sigma_y(I)$. Let $\alpha, \beta \in \sigma_y(I)$. Then $\alpha \wedge y \in I$ and $\beta \wedge y \in I$. By Lemma 2.2 (7), we have $(\alpha \oplus \beta) \wedge y \leq (\alpha \wedge y) \oplus (\beta \wedge y) \in I$. Therefore $(\alpha \oplus \beta) \wedge y \in I$. Hence $\alpha \oplus \beta \in \sigma_y(I)$. Therefore $\sigma_y(I) \in Id(A)$.

2) Let $x \in I$. We know $x \wedge y \leq x$, so $x \wedge y \in I$. Hence $x \in \sigma_y(I)$, i.e. $I \subseteq \sigma_y(I)$.

3) Let $x \in \sigma_y(I)$. Then $x \wedge y \in I$, Hence $x \wedge y \in J$, Therefore $x \in \sigma_y(J)$.

4) By Lemma 2.2, Remark 2.6, and since I, J are ideals of A such that $J \subseteq I$, we have: $\frac{x}{J} \in \sigma_y(\frac{I}{J})$ so $\frac{x}{J} \wedge \frac{y}{J} \in \frac{I}{J}$, and so $\frac{x \wedge y}{J} \in \frac{I}{J}$, $d(x \wedge y, \alpha) \in J$ thus there exist $\alpha \in I$ such that $\frac{x \wedge y}{J} = \frac{\alpha}{J}$ and so $d(x \wedge y, \alpha) \in J$. So

$$\begin{aligned} \alpha^* \odot (x \wedge y) &\leq (x \wedge y) \odot \alpha^* \oplus \alpha \odot (x \wedge y)^* \in J, \\ \implies \alpha^* \odot (x \wedge y) &\in I, \quad \alpha \in I, \\ \implies \alpha \oplus (\alpha^* \odot (x \wedge y)) &\in I, \\ \implies x \wedge y \leq \alpha \vee (x \wedge y) &\in I, \\ \implies x \wedge y &\in I, \\ \implies x &\in \sigma_y(I), \\ \implies \frac{x}{J} &\in \frac{\sigma_y(I)}{J}. \end{aligned}$$

Therefore $\sigma_y(\frac{I}{J}) \subseteq \frac{\sigma_y(I)}{J}$. Let $\frac{x}{J} \in \frac{\sigma_y(I)}{J}$, so there exist $t \in \sigma_y(I)$ such that $\frac{x}{J} = \frac{t}{J}$ and so $d(x, t) \in J$. Therefore

$$\begin{aligned} x \odot t^* &\leq (x \odot t^*) \oplus (t \odot x^*) \in J \subseteq I \subseteq \sigma_y(I), \\ \implies x \odot t^* &\in \sigma_y(I), \quad t \in \sigma_y(I), \\ \implies t \oplus (t^* \odot x) &\in \sigma_y(I), \\ \implies x \leq t \vee x &\in \sigma_y(I), \\ \implies x &\in \sigma_y(I), \\ \implies x \wedge y &\in I, \\ \implies \frac{x}{J} \wedge \frac{y}{J} &\in \frac{I}{J}, \\ \implies \frac{x}{J} &\in \sigma_y(\frac{I}{J}). \end{aligned}$$

Hence $\frac{\sigma_y(I)}{J} \subseteq \sigma_y\left(\frac{I}{J}\right)$. Therefore $\sigma_y\left(\frac{I}{J}\right) = \frac{\sigma_y(I)}{J}$.

5) We have $\sigma_y(I) \subseteq \sigma_y(\sigma_y(I))$. Now let $x \in \sigma_y(\sigma_y(I))$. Then $x \wedge y \in \sigma_y(I)$, hence $x \wedge y \wedge y \in I$, and so $x \wedge y \in I$, i.e. $x \in \sigma_y(I)$.

6) Let $x \in \sigma_y(I) \rightarrow \sigma_y(J)$. Hence $(x] \cap I \subseteq (x] \cap \sigma_y(I) \subseteq \sigma_y(J)$. Thus $x \in I \rightarrow \sigma_y(J)$.

7) Let $x \in \sigma_y(I \rightarrow J)$. Then $x \wedge y \in I \rightarrow J$. It follows that $(x \wedge y] \cap I \subseteq J \subseteq \sigma_y(J)$. Hence $x \wedge y \in I \rightarrow \sigma_y(J)$. Thus $x \in \sigma_y(I \rightarrow \sigma_y(J))$.

8) Let $\sigma_y(I)$ be a prime ideal, $a \wedge b \in \sigma_y(I)$ and $a \notin \sigma_y(\sigma_y(I))$. By part (5), we have $\sigma_y(\sigma_y(I)) = \sigma_y(I)$. So, $a \wedge b \in \sigma_y(I)$ and $a \notin \sigma_y(I)$. Hence by hypothesis we get $b \in \sigma_y(I)$. Thus $b \in \sigma_y(\sigma_y(I))$, i.e. $b \in \sigma_y(I)$. Therefore $\sigma_y(I)$ is a (σ_y, σ_y) -primary.

Conversely, let $\sigma_y(I)$ be a (σ_y, σ_y) -primary, $a \wedge b \in \sigma_y(I)$ and $a \notin \sigma_y(I)$. Hence by part (5), we have $a \notin \sigma_y(\sigma_y(I))$. Then by hypothesis we get $b \in \sigma_y(\sigma_y(I))$, i.e. $b \in \sigma_y(I)$. Therefore $\sigma_y(I)$ is a prime ideal of A .

9) Let I be $(\sigma_{x \vee y}, \sigma_0)$ -primary, $a \wedge b \in I$ and $a \notin \sigma_y(I)$. Then $a \wedge y \notin I$. We know $a \wedge y \leq a \wedge (x \vee y)$, so $a \wedge (x \vee y) \notin I$. Hence $a \notin \sigma_{x \vee y}$. So by hypothesis we get $b \in \sigma_0(I) = I$, i.e. I is a (σ_y, σ_0) -primary. Similarly, we can prove that I is a (σ_x, σ_0) -primary.

10) By parts (1), (2), (3), the proof is clear. \square

Proposition 3.14. *Let $\{I_j\}_{j \in J}$ be a family of ideals of A . Then*

$$1) \bigcap_{j \in J} \sigma_y(I_j) = \sigma_y\left(\bigcap_{j \in J} I_j\right).$$

$$2) \text{ If } \{I_j\}_{j \in J} \text{ is a chain of ideals, then } \bigcup_{j \in J} \sigma_y(I_j) = \sigma_y\left(\bigcup_{j \in J} I_j\right).$$

Proof. 1) We have

$$\begin{aligned} a \in \bigcap_{j \in J} \sigma_y(I_j) &\iff a \in \sigma_y(I_j), \forall j \in J, \\ &\iff a \wedge y \in I_j, \forall j \in J, \\ &\iff a \wedge y \in \bigcap_{j \in J} I_j, \\ &\iff a \in \sigma_y\left(\bigcap_{j \in J} I_j\right). \end{aligned}$$

2) Let $\{I_j\}_{j \in J}$ be a chain of ideals. It is easy to see that $\bigcup_{j \in J} I_j$ is an ideal. Let $x \in \bigcup_{j \in J} \sigma_y(I_j)$.

Then there exists $j \in J$ such that $x \in \sigma_y(I_j)$, i.e. there exists $j \in J$ such that $x \wedge y \in I_j$. Then $x \wedge y \in \bigcup_{j \in J} I_j$ and so $x \in \sigma_y\left(\bigcup_{j \in J} I_j\right)$ therefore $\bigcup_{j \in J} \sigma_y(I_j) \subseteq \sigma_y\left(\bigcup_{j \in J} I_j\right)$.

Conversely, let $x \in \sigma_y\left(\bigcup_{j \in J} I_j\right)$. Then $x \wedge y \in \bigcup_{j \in J} I_j$. since $\{I_j\}_{j \in J}$ is a chain of ideals, there

exists $j \in J$ such that $x \wedge y \in I_j$. Hence $x \in \sigma_y(I_j)$, for some $j \in J$ and so $x \in \bigcup_{j \in J} \sigma_y(I_j)$.

Therefore $\sigma_y(\bigcup_{j \in J} I_j) \subseteq \bigcup_{j \in J} \sigma_y(I_j)$. Hence $\bigcup_{j \in J} \sigma_y(I_j) = \sigma_y(\bigcup_{j \in J} I_j)$. \square

Remark 3.15. If I is a prime ideal, then so is $\sigma_y(I)$.

Proof. Let I be a prime ideal and $a \wedge b \in \sigma_y(I)$. Hence $(a \wedge b) \wedge y \in I$. Since I is a prime ideal, $a \in I$ or $b \wedge y \in I$. Then $a \wedge y \leq a \in I$ or $b \wedge y \in I$. Thus $a \in \sigma_y(I)$ or $b \in \sigma_y(I)$. Therefore $\sigma_y(I)$ is a prime ideal. \square

Lemma 3.16. *Let I be an ideal. Then we have $\sigma_y(I) = (y] \rightarrow I$ in the complete Heyting algebra $(Id(A), \wedge, \vee, \rightarrow, \{0\}, A)$.*

Proof. Let $x \in \sigma_y(I)$. We prove that $x \in (y] \rightarrow I$. It is sufficient to show that $(x] \cap (y] \subseteq I$. By Lemma 2.2 (8), we have $(x \wedge y] = (x] \cap (y] \subseteq I$. Suppose that $t \in (x \wedge y]$, then there exists $n \in \mathbb{N}$ such that $t \leq n(x \wedge y)$. Since $x \in \sigma_y(I)$, so $x \wedge y \in I$. Hence we have $t \leq n(x \wedge y) \in I$. Thus $t \in I$. Therefore $\sigma_y(I) \subseteq (y] \rightarrow I$.

Conversely, let $x \in (y] \rightarrow I$. Hence $(x \wedge y] = (x] \cap (y] \subseteq I$. We obtain that $x \wedge y \in I$. Thus $x \in \sigma_y(I)$. We get $\sigma_y(I) = (y] \rightarrow I$ in the Heyting algebra $Id(A)$. \square

Definition 3.17. An ideal I is called constant relative to expansion σ_y if $I = \sigma_y(I)$.

Lemma 3.18. *Let I be an ideal of A . Then $\sigma_y(I)$ is the smallest constant ideal relative to expansion σ_y such that $I \subseteq \sigma_y(I)$.*

Proof. By Lemma 3.13 (5), we have $\sigma_y(\sigma_y(I)) = \sigma_y(I)$. Thus $\sigma_y(I)$ is constant ideal relative to expansion σ_y . Let J be constant ideal relative to expansion σ_y such that $I \subseteq J$. It follows from Lemma 3.13 (3) that $\sigma_y(I) \subseteq \sigma_y(J) = J$. \square

Corollary 3.19. *By Lemma 3.18, we get $\sigma_y(I)$ is constant ideal relative to σ_y . Hence every constant ideal relative to σ_y is in $C(\sigma_y)$, where $C(\sigma_y) = \{\sigma_y(I) \mid I \in Id(A)\}$. For all elements $\sigma_y(I), \sigma_y(J) \in C(\sigma_y)$, we define two operations \sqcap and \sqcup as follows:*

$$\sigma_y(I) \sqcap \sigma_y(J) = \sigma_y(I \wedge J) \text{ and } \sigma_y(I) \sqcup \sigma_y(J) = \sigma_y(I \vee J).$$

Where $\sigma_y(I \wedge J)$ (or $\sigma_y(I \vee J)$) is infimum (supremum) of $\{\sigma_y(I), \sigma_y(J)\}$ in $C(\sigma_y)$. It is easy to show that $\sigma_y(I) \sqcap \sigma_y(J) = \sigma_y(I \wedge J)$. We show that $\sigma_y(I \vee J)$ is a supremum of $\sigma_y(I), \sigma_y(J)$ in $C(\sigma_y)$. By Lemma 3.13 (3), we get $\sigma_y(I), \sigma_y(J) \subseteq \sigma_y(I \vee J)$. For any constant ideal relative to $\sigma_y, \sigma_y(K)$ such that $\sigma_y(I), \sigma_y(J) \subseteq \sigma_y(K)$, we prove that $\sigma_y(I \vee J) \subseteq \sigma_y(K)$. Let $x \in \sigma_y(I \vee J)$.

Then $x \wedge y \in I \vee J$. Hence $x \wedge y \leq a \oplus c$, for some $a \in I \subseteq \sigma_y(I)$ and $c \in J \subseteq \sigma_y(J)$. We obtain $x \wedge y \in \sigma_y(I) \vee \sigma_y(J) \subseteq \sigma_y(K)$. Thus $x \in \sigma_y(\sigma_y(K)) = \sigma_y(K)$ (by Lemma 3.13 (5)). This means that $\sigma_y(I \vee J)$ is the supremum of $\{\sigma_y(I), \sigma_y(J)\}$ in $C(\sigma_y)$. Thus $(C(\sigma_y), \sqcap, \sqcup)$ is a lattice.

Theorem 3.20. $(C(\sigma_y), \sqcap, \sqcup, \rightarrow, \sigma_y(\{0\}), A)$ is a complete Heyting algebra.

Proof. By Remark 3.19 and Proposition 3.14, we have $\bigwedge_{j \in J} \sigma_y(I_j) = \sigma_y(\bigwedge_{j \in J} I_j)$ and so $C(\sigma_y)$ is complete. We only show that for $\sigma_y(I), \sigma_y(J), \sigma_y(K) \in C(\sigma_y)$,

- 1) $\sigma_y(I) \rightarrow \sigma_y(J) \in C(\sigma_y)$,
- 2) $\sigma_y(I) \sqcap \sigma_y(J) \subseteq \sigma_y(K) \Leftrightarrow \sigma_y(I) \leq \sigma_y(J) \rightarrow \sigma_y(K)$.

By Lemma 3.16 and Lemma 2.10, we obtain

$$\begin{aligned}
 \sigma_y(I) \rightarrow \sigma_y(J) &= ((y] \rightarrow I) \rightarrow ((y] \rightarrow J) \\
 &= (y] \rightarrow [((y] \rightarrow I) \rightarrow J) \\
 &= [(y] \odot ((y] \rightarrow I)] \rightarrow J \\
 &= ((y] \wedge I) \rightarrow J \\
 &= ((y] \odot I) \rightarrow J \\
 &= (y] \rightarrow (I \rightarrow J) \in C(\sigma_y).
 \end{aligned}$$

For the case (2), it follows from Lemma 3.16 and Lemma 2.10 that

$$\begin{aligned}
 \sigma_y(I) \sqcap \sigma_y(J) \subseteq \sigma_y(K) &\Leftrightarrow \sigma_y(I \wedge J) \subseteq \sigma_y(K), \\
 &\Leftrightarrow (y] \rightarrow I \wedge J \subseteq (y] \rightarrow K, \\
 &\Leftrightarrow (y] \wedge ((y] \rightarrow I \wedge J) \subseteq K, \\
 &\Leftrightarrow (y] \wedge I \wedge J \subseteq K, \\
 &\Leftrightarrow (y] \wedge I \subseteq J \rightarrow K, \\
 &\Leftrightarrow (y] \rightarrow ((y] \wedge I) \subseteq (y] \rightarrow (J \rightarrow K), \\
 &\Leftrightarrow ((y] \rightarrow (y]) \wedge ((y] \rightarrow I) \subseteq (y] \rightarrow (J \rightarrow K), \\
 &\Leftrightarrow (y] \rightarrow I \subseteq (y] \rightarrow (J \rightarrow K), \\
 &\Leftrightarrow \sigma_y(I) \subseteq \sigma_y(J) \rightarrow \sigma_y(K).
 \end{aligned}$$

Thus, $C(\sigma_y)$ of all constant ideals relative to expansion σ_y is the Heyting algebra. \square

Lemma 3.21. Let $I, J \in Id(A)$ such that $J \subseteq I$. If I is prime, then $\sigma_y(J) \subseteq I$, for all $y \in A - I$.

Proof. Let $x \in \sigma_y(J)$. Then $x \wedge y \in J \subseteq I$. By the fact that I is a prime ideal, we get $x \in I$ or $y \in I$. Since $y \in A - I$, we have $x \in I$, i.e. $\sigma_y(J) \subseteq I$. \square

Theorem 3.22. *Let $I \in Id(A)$. Then I is prime if and only if $\sigma_y(I) = I$, for all $y \in A - I$.*

Proof. Let I be a prime ideal of A and $x \in \sigma_y(I)$. Then $x \wedge y \in I$. Hence $x \in I$ or $y \in I$. We have $y \in A - I$, so $x \in I$. Therefore $\sigma_y(I) \subseteq I$. Thus $\sigma_y(I) = I$, for all $y \in A - I$.

Conversely, let $\sigma_y(I) = I$, for all $y \in A - I$. Also $x \wedge y \in I$ and $y \notin I$. Then $x \in \sigma_y(I)$. So by hypothesis, we get $x \in I$, i.e. I is a prime ideal of A . \square

Corollary 3.23. *An ideal I of A is prime if and only if I is (σ_y, σ_0) -primary for all $y \in A - I$.*

Proof. Let I be a prime ideal of A , $a \wedge b \in I$ and $a \notin \sigma_y(I)$. By Theorem 3.22, $a \notin I$. Since I is a prime ideal, hence $b \in I$. Therefore $b \in \sigma_0(I)$.

Conversely, let I be a (σ_y, σ_0) -primary for all $y \in A - I$. By Theorem 3.22, it is enough to show that $\sigma_y(I) \subseteq I$, for all $y \in A - I$. Let $z \in \sigma_y(I)$. Then $z \wedge y \in I$. We have $y \wedge y = y \notin I$, so $y \notin \sigma_y(I)$. Hence by hypothesis we get $z \in \sigma_0(I) = I$. \square

Definition 3.24. Let $I, J \in Id(A)$. Then (I, J) is defined to be

$$(I, J) = \bigcap_{y \in J} \sigma_y(I) = \{x \in A \mid x \wedge y \in I, \forall y \in J\}.$$

Theorem 3.25. *Let $I, J \in Id(A)$. Then $(I, J) \in Id(A)$.*

Proof. Let $z, t \in (I, J)$. We must show that $z \oplus t \in (I, J)$. We get $z \wedge y, t \wedge y \in I$, for all $y \in J$. So $(z \wedge y) \oplus (t \wedge y) \in I$. By Lemma 2.2 (7), we have $y \wedge (z \oplus t) \leq (y \wedge z) \oplus (y \wedge t) \in I$, for all $y \in J$. So $z \oplus t \in (I, J)$.

Now let $z \in (I, J)$ and $t \leq z$. Then $z \wedge y \in I$, for all $y \in J$, and also $t \wedge y \leq z \wedge y$. Hence $t \wedge y \in I$, for all $y \in J$. Therefore $t \in (I, J)$, and so $(I, J) \in Id(A)$. \square

Theorem 3.26. *Let $I, J \in Id(A)$ and I be (σ, σ_0) -primary. Then*

- 1) *If J is not contained in $\sigma(I)$, then $(I, J) = I$,*
- 2) *(I, J) is (σ, σ_0) -primary.*

Proof. 1) Let $x \in I$. Then $x \wedge y \in I$, for all $y \in J$. So $x \in (I, J)$, i.e. $I \subseteq (I, J)$. Now let $z \in (I, J)$. Then $z \wedge y \in I$, for all $y \in J$. We have $J \not\subseteq \sigma_y(I)$. So there exists $y_0 \in J$ such that $y_0 \notin \sigma(I)$. By the fact that $y_0 \in J$, we get $z \wedge y_0 \in I$. Then by $y_0 \notin \sigma(I)$. Since I is (σ, σ_0) -primary, we get $z \in \sigma_0(I) = I$. Therefore $(I, J) \subseteq I$. Hence $(I, J) = I$.

2) Let $a \wedge b \in (I, J)$ and $a \notin \sigma((I, J))$. We have $I \subseteq (I, J)$, so $\sigma(I) \subseteq \sigma((I, J))$. Hence $a \notin \sigma(I)$.

Also by $a \wedge b \in (I, J)$, we get $a \wedge b \wedge y \in I$, for all $y \in J$. By the fact that $a \wedge b \wedge y \in I$ and $a \notin \sigma(I)$, we get $b \wedge y \in I$, for all $y \in J$, because I is (σ, σ_0) -primary. Therefore $b \in (I, J)$. So the proof is complete. \square

4. (τ, σ) -OBSTINATE AND (τ, σ) -BOOLEAN IDEALS IN MV -ALGEBRAS

Definition 4.1. Let τ and σ be expansions of ideals. Then an ideal I of A is said to be (τ, σ) -obstinate if

$$(\forall a \in A)(a \in \tau(I) \text{ or } a^* \in \sigma(I)).$$

Example 4.2. Consider MV -algebra A of Example 3.4, Let:

$$\tau : Id(A) \longrightarrow Id(A)$$

$$I_0 \longmapsto I_1$$

$$I_1 \longmapsto I_1$$

$$I_2 \longmapsto I_3$$

$$I_3 \longmapsto I_3.$$

Obviously, I_2 is (τ, σ_0) -obstinate and I_1 is not (τ, σ_0) -obstinate ideal of A , because $a \notin \tau(I_1) = I_1$ and $a^* = d \notin \sigma_0(I_1) = I_1$.

Lemma 4.3. Let $I \in Id(A)$. I is (τ_0, σ_0) -obstinate ideal if and only if it is an obstinate ideal of A .

Proof. By Lemma 2.5, it is clear. \square

Theorem 4.4. Let σ and τ be expansions of ideals in A . Then I is (τ, σ) -obstinate ideal of A if and only if I is (σ, τ) -obstinate ideal of A .

Proof. Let I be an (τ, σ) -obstinate and $\forall a \in A, a \notin \sigma(I)$. We have $(a^*)^* = a \notin \sigma(I)$ and I is (τ, σ) -obstinate. Therefore $a^* \in \tau(I)$.

Conversely, it is similar. \square

Theorem 4.5. Let σ and τ be expansions of ideals in A . Then every obstinate ideal of A is (τ, σ) -obstinate.

Proof. Let I be obstinate ideal and for all $a \in A, a \notin \tau(I)$. By the fact that $I \subseteq \tau(I)$, we get $a \notin I$. Since I is an obstinate ideal of A , so $a^* \in I$. By $I \subseteq \sigma(I)$, we get $a^* \in \sigma(I)$. Therefore I is an (τ, σ) -obstinate. \square

The following example shows that the converse of the above theorem may not be true, in general.

Example 4.6. 1) Let $\Omega = \{1, 2\}$ and $A = P(\Omega) = \{\{1\}, \{2\}, \{1, 2\}, \emptyset\}$, which is an MV -algebra with $\oplus = \cup, \odot = \cdot = \cap$ and $D^* = \Omega - D$ for any $D \in A$. It is clear that $I_0 = \{\emptyset\}, I_1 = \{\{1\}, \emptyset\}, I_2 = \{\{2\}, \emptyset\}$ and $I_3 = A$ are ideals of A . Let $\tau(I) = A$ for all $I \in Id(A)$. Obviously I_0 is (τ, σ_0) -obstinate ideal but I_0 is not obstinate ideal of A , because $\{1\} \notin I_0$ and $\{1\}^* = \{2\} \notin I_0$.

2) In Example 3.4, Let $\tau(I) = A$ for all $I \in Id(A)$. Obviously I_0 is (τ, σ_0) -obstinate ideal but I_0 is not obstinate ideal of A , because $a \notin I_0$ and $a^* = d \notin I_0$.

Theorem 4.7. *Let τ be an expansion of ideals in A . If $\tau(I)$ is an obstinate ideal, then I is an (τ, τ) -obstinate ideal of A .*

Proof. Let $\tau(I)$ be an obstinate ideal and for all $a \in A, a \notin \tau(I)$. $\tau(I)$ is an obstinate ideal, Then $a^* \in \tau(I)$. Therefore I is an (τ, τ) -obstinate. \square

Corollary 4.8. *Let τ and σ be expansions of ideals in A and $I \in Id(A)$. If $\tau(I)$ is an obstinate ideal, then $\tau(I)$ is an (τ, σ) -obstinate ideal of A .*

Proof. Let $\tau(I)$ be an obstinate ideal and for all $a \in A, a \notin \tau(\tau(I))$. We have $\tau(I) \subseteq \tau(\tau(I))$, so $a \notin \tau(I)$. By the fact that $\tau(I)$ is an obstinate ideal, then $a^* \in \tau(I)$. Hence by $\tau(I) \subseteq \sigma(\tau(I))$, we get $a^* \in \sigma(\tau(I))$. Therefore $\tau(I)$ is an (τ, σ) -obstinate ideal of A . \square

Theorem 4.9. *Let τ, σ, ψ and δ be expansions of ideals in A such that $\tau \subseteq \psi$ and $\sigma \subseteq \delta$. Then every (τ, σ) -obstinate ideal is (ψ, δ) -obstinate.*

Proof. Let I be an (τ, σ) -obstinate and for all $a \in A, a \notin \psi(I)$. We have $\tau \subseteq \psi$, so $a \notin \tau(I)$. Then by hypothesis, we get $a^* \in \sigma(I)$. Hence by the fact that $\sigma \subseteq \delta$, we get $a^* \in \delta(I)$, i.e. I is an (ψ, δ) -obstinate ideal of A . \square

Corollary 4.10. *Let τ and σ be expansions of ideals in A . Then every (τ, τ_0) -obstinate ideal is an (τ, σ) -obstinate ideal of A .*

Proof. Let I be an (τ, τ_0) -obstinate and for all $a \in A, a \notin \tau(I)$. By hypothesis, we get $a^* \in \tau_0(I) = I$. We know $I \subseteq \sigma(I)$, so $a^* \in \sigma(I)$. Therefore I is an (τ, σ) -obstinate ideal of A . \square

Lemma 4.11. *Let $I \in Id(A)$. Then $\sigma_y(I)$ is an obstinate ideal if and only if $\sigma_y(I)$ is an (σ_y, σ_y) -obstinate ideal of A .*

Proof. Let $\sigma_y(I)$ be an obstinate ideal and for all $a \in A, a \notin \sigma_y(\sigma_y(I))$. By part (5) of Lemma 3.13, we have $\sigma_y(\sigma_y(I)) = \sigma_y(I)$. So $a \notin \sigma_y(I)$. Hence by hypothesis, we get $a^* \in \sigma_y(I)$. Thus $a^* \in \sigma_y(\sigma_y(I))$, i.e. $\sigma_y(I)$ is an (σ_y, σ_y) -obstinate ideal of A .

Conversely, let $\sigma_y(I)$ be an (σ_y, σ_y) -obstinate ideal and for all $a \in A, a \notin \sigma_y(I)$. Hence by part (5) of Lemma 3.13, we have $a \notin \sigma_y(\sigma_y(I))$. Then by hypothesis, we get $a^* \in \sigma_y(\sigma_y(I))$, i.e. $a^* \in \sigma_y(I)$. Therefore $\sigma_y(I)$ is an obstinate ideal of A . \square

Theorem 4.12. *Let τ and σ be expansions of ideals A . Then every (τ, σ) -obstinate is (τ, σ) -primary ideal of A .*

Proof. Let $a \wedge b \in I$ and $a \notin \tau(I)$. Since I is an (τ, σ) -obstinate and $\sigma(I)$ is an ideal of A , if $a^* \in \sigma(I)$, then $a^* \oplus (a \wedge b) \in \sigma(I)$, so $a^* \oplus (a \odot (a^* \oplus b)) \in \sigma(I)$, and so , $a^* \oplus b \leq a^* \vee (a^* \oplus b) \in \sigma(I)$. Therefore $b \leq a^* \oplus b \in \sigma(I)$ and so $b \in \sigma(I)$. \square

The following example shows that the converse of the above theorem may not be true in general.

Example 4.13. Consider MV -algebra A in Example 3.4. Define expansion τ of ideals as follows:

$$\tau : Id(A) \longrightarrow Id(A)$$

$$I_0 \longmapsto I_0$$

$$I_1 \longmapsto I_1$$

$$I_2 \longmapsto I_3$$

$$I_3 \longmapsto I_3.$$

Obviously, $I_1 = \{0, c\}$ is a prime ideal of A . By Theorem 3.7, I_1 is (τ, σ_0) -primary but I_1 is not (τ, σ_0) -obstinate because $d \notin \tau(I_1)$ and $d^* = a \notin \sigma_0(I_1) = I_1$.

Theorem 4.14. *Let I be Boolean ideal of A and (τ, σ) -primary ideal of A . Then I is (τ, σ) -obstinate ideal of A .*

Proof. Let for all $a \in A, a \notin \tau(I)$. Since I is Boolean ideal, hence $a \wedge a^* \in I$ and I is (τ, σ) -primary ideal, therefore $a^* \in \sigma(I)$. Then I is (τ, σ) -obstinate ideal of A . \square

Definition 4.15. Let τ and σ be expansions of ideals A . Then ideal I of A is said to be (τ, σ) -Boolean if

$$(\forall a \in A)(a \wedge a^* \in \tau(I) \text{ or } a \wedge a^* \in \sigma(I)).$$

Example 4.16. Consider MV -algebra A in Example 3.4. Define expansion σ of ideals as follows:

$$\sigma : Id(A) \longrightarrow Id(A)$$

$$I_0 \longmapsto I_1$$

$$I_1 \longmapsto I_1$$

$$I_2 \longmapsto I_3$$

$$I_3 \longmapsto I_3.$$

I_2 is (τ_0, σ) -Boolean and I_0 is not (τ_0, σ) -Boolean ideal of A , because $a \wedge a^* = a \wedge d = a, a \notin \tau_0(I_0) = I_0$ and $a \notin \sigma(I_0) = I_1$.

Theorem 4.17. *Let τ and σ be expansions of ideals. Every (τ, σ) -obstinate ideal is (τ, σ) -Boolean ideal of A .*

Proof. Let I be an (τ, σ) -obstinate and for all $x \in A, x \wedge x^* \notin \tau(I)$ and $x \wedge x^* \notin \sigma(I)$. We have $x \wedge x^* \leq x$ and $x \wedge x^* \leq x^*$, so $x \notin \tau(I)$ and $x^* \notin \sigma(I)$, that is a contradiction. Therefore for all $x \in A, x \wedge x^* \in \tau(I)$ or $x \wedge x^* \in \sigma(I)$. Hence I is a (τ, σ) -Boolean ideal of A . \square

The following example shows that the converse of the above theorem may not be true, in general.

Example 4.18. Consider MV -algebra A in Example 3.8. Define expansion σ of ideals as follows:

$$\sigma : Id(A) \longrightarrow Id(A)$$

$$I_0 \longmapsto I_0$$

$$I_1 \longmapsto I_1$$

$$I_2 \longmapsto I_3$$

$$I_3 \longmapsto I_3.$$

Obviously, I_0 is a (τ_0, σ) -Boolean but I_0 is not an (τ_0, σ) -obstinate ideal of A , because $a \notin \tau_0(I_0) = I_0$ and $a^* = b \notin \sigma(I_0) = I_0$.

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