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A NOTE ON ARTINIAN PRIMES AND SECOND MODULES

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Abstract. Prime submodules and artinian prime modules are characterized. Furthermore, some previous results on prime modules and second modules are generalized.

1. Introduction

Throughout this note the ring R is commutative with a non-zero identity. An R-module M is said to be prime if whenever $rm = 0$ either $m = 0$ or $rM = 0$. Note that if M is a prime R-module, then any submodule N of M is prime. A proper submodule N of an R-module M is said to be a prime submodule of M if factor module M/N is prime. Thus N is a prime submodule of M if and only if $P = Ann(M/N) = (N : M)$ is a prime ideal of R and M/N is a torsion free R/P -module. This notion of prime submodule was first introduced and systematically studied in [4, 5] and recently from several authors, see for example [2, 7, 8, 9, 13]. Note that there is no need an R-module contains a prime submodule, for example take $\mathbb{Z}_{P^\infty}$ as Z-module . The purpose of this paper is to introduce some relation between dual of prime

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submodules, prime submodules, prime modules and artinian modules. Let M be a prime R-module. The non-zero submodule N of an R- module M is said to be second submodule of M if for each $x \in M$, $xN = 0$ or $xN = N$. This implies that $Ann(N) = P$ is a prime ideal of R and N is said to be P-second. Also recall that R (domain)-module M is said to be divisible if $xM = M$ for all $0 \neq x \in R$. In this paper we will characterize second submodules of prime artinian modules. It is proved that N is a prime submodule of an artinian module M, if and only if $(N : M)$ is a maximal ideal of R (Proposition 2.2) moreover, the second submodule of artinian modules are studied. Recall that a submodule N of M is called semiprime if $N \neq M$, and $r^2m \in N$ implies $rm \in N$ for all $r \in R$, $m \in M$. We will prove that if M is a finitely generated faithfully prime R-module such that each cyclic submodule of M is semiprime, then M is a vector space (Theorem 2.11 and Corollary 2.12). In Theorem 2.13 and Corollary 2.14 we will give a simple proof for [14 Theorem 1.3].

Lemma 1.1. *Let M be a prime R-module. Then we have the following:*

(1) *$P = (0 : M) = Ann(M)$ is a prime ideal.*
(2) *M is a torsion free $R/Ann(M)$-module.*
(3) *$Ann(N) = Ann(M)$ for every non-zero submodule N of M.*
(4) *If N is a minimal (simple) submodule of M then $Ann(M)$ is a maximal ideal of R. Because $N = Rx \cong R/Ann(x) = R/Ann(M)$ for every $0 \neq x \in N$, and hence M is vector space over $R/Ann(M)$.*
(5) *Every submodule N of M is prime module.*
(6) *For any non-zero x in M, (0:x)=(0:M)=Ann(M).*

Lemma 1.2. *Every simple R-module is prime.*

Proof. Let M be a simple R-module and $0 \neq m \in M$. We have $M = Rm$. If $rm = 0$, then $rM = rRm = 0$. Note that if $M = \bigoplus_{i \in I} M_i$ is prime R-module then $M_i (i \in I)$ is prime. But the converse is not true in general as the following example illustrates. □

Example 1.3. *Let $Z_6 = \{\bar{0}, \bar{3}\} \bigoplus \{\bar{0}, \bar{2}, \bar{4}\}$ (as a Z-module). Although $\{\bar{0}, \bar{3}\}$ and $\{\bar{0}, \bar{2}, \bar{4}\}$ are prime and Z_6 is semisimple, Z_6 itself is not a prime Z-module. Now see the following proposition.*

Proposition 1.1. *Let $M = \bigoplus_{i \in I} M_i$ be a semisimple R-module and $Ann(M_i) = Ann(M)(i \in I)$. Then M is a prime module.*

2. Artinian Modules and Second Submodules

Recall that a non-zero submodule N of an R-module M is said to be second submodule if for each $x \in R$, $xN = 0$ or $xN = N$. This implies that $Ann(N) = P$ is a prime ideal of R, and N is said to be P-second. Also recall that R (domain)-module M is said to be divisible if $rM = M$ for all $0 \neq r \in R$.

The following theorem is easy to obtain.

Theorem 2.1. *Let N be a submodule of M such that N is prime module, then the following are equivalent:*

(1) *N is a $(0 : N)$-second submodule of M;*
(2) *N is a divisible $R/(0 : N)$-module;*
(3) *rN=N for all $r \in R \backslash (0 : N)$;*
(4) *$IN = N$ for all ideal $I \nsubseteq (0 : N)$;*
(5) *$\{x \in R \mid xN \subset N\} = (0 : N) = Ann(N)$.*

Note: Since every submodule of a prime module is prime module, therefore if M is prime and $N \leq M$, than Theorem 2.1 holds for submodule N.

Proposition 2.1. *If N is a simple submodule of the prime module M. Then N is an $Ann(M)$-second submodule.*

Proof. There exists an element $0 \neq x \in M$ such that $N = Rx \cong R/Ann(x)$, where $Ann(x)$ is a maximal ideal of R. Since M is a prime module, $Ann(x) = Ann(M)$. Hence N is a vector space over $R/Ann(M)$, and hence, N is a divisible $R/Ann(M)$-module. Thus N is an $Ann(M)$-second submodule by Theorem 2.1. □

Corollary 2.2. *An Artinian prime R-module M is:*

(1) *$Ann(M)$-second module.*
(2) *An injective $R/Ann(M)$-module.*
(3) *A flat $R/Ann(M)$-module.*

Proof. (1) Let $T = \{S \mid S$ is a non-trivial submodule of $M\}$. Suppose that N is a minimal element of T. Obviously N is a non-zero simple module. Now the proof is clear by Lemma 1.1(4) and Proposition 2.1.

(2) By Part (1) and Lemma 1.1(4), M is a torsion-free $R/Ann(M)$-module and vector space over filed $R/Ann(M)$. Also by (1), M is a second module and therefore M is a divisible $R/Ann(M)$-module, by Theorem 2.1.

(3) By the proof of Part (1) and Lemma 1.1 (4), M is a vector space over the filed $R/Ann(M)$. Hence M is flat $R/Ann(M)$-module.

So the proof is complete. □

Lemma 2.3. *Let M be an R-module and N be a submodule of M. Then N is a prime submodule of M if and only if M/N is a prime R-module.*

It is easy to see that a submodule N of an R-module M is prime submodule if and only if $(N : M)$ is a prime ideal of R and M/N is a torsion-free $R/(N : M)$- module.

Proposition 2.2. *Let N be a submodule of an artinian R-module M. Then N is a prime submodule of M if and only if $(N : M)$ is a maximal ideal of R.*

Proof. Suppose that N is a prime submodule of M. Then M/N is an artinian prime R-module. Let $T = \{S \mid S$ is a non-trivial submodule of $M/N\}$. Suppose that K is a minimal element of T. Obviously K is a non-zero simple submodule of M/N. Hence there exists an element $0 \neq x+N \in M/N$ such that $K = R(x+N) \cong R/Ann(x+N)$, where $Ann(x+N)$ is a maximal ideal of R. Since M/N is a prime module, $Ann(x + N) = Ann(M/N) = (N : M)$. Hence $(N : M)$ is a maximal ideal of R.

Conversely if $(N : M)$ is a maximal ideal of R, then M/N is a vector space over the filed $R/(N : M) = R/Ann(M/N)$. Thus it is torsion-free. Hence N is a prime submodule of M. □

Corollary 2.4. *Let N be a prime submodule of an artinian R-module M. Then M/N is:*

(1) $(N : M)$*-second module.*
(2) *An injective* $R/(N : M)$*-module.*
(3) *A flat* $R/(N : M)$*-module.*

Proof. Evident by Corollary 2.2 and Proposition 2.2. □

Note that in example 1.3 we saw that although $\{\bar{0}, \bar{3}\}$ and $\{\bar{0}, \bar{2}, \bar{4}\}$ were prime but $Z_6 = \{\bar{0}, \bar{3}\} \bigoplus \{\bar{0}, \bar{2}, \bar{4}\}$ was not a prime Z-module. Now see the following Theorem.

The following theorem is an immediate result from ([10], 2.4).

Theorem 2.5. *Let M be a finitely generated R-module. If M is a second module, then M is a prime module.*

Proposition 2.3. *Let P be a prime ideal of R, Then the sum of P-second modules is a P-second module.*

Proof. Let $M_1, M_2, \cdots, M_n$ be P-second module. Then for any $1 \leq i \leq n$ we have $Ann(M_i) = P$ and M_i is a divisible R/P-module. Therefore $Ann(\sum M_i) = P$ and $\sum M_i$ is a divisible R/P-module. Thus $\sum M_i$ is a P-second module. □

We use this results in the next Theorem.

Theorem 2.6. *If $M_i(1 \leq i \leq n)$ are finitely generated P-second modules, then $M_1 \oplus M_2 \oplus \cdots \oplus M_n$ is a prime module.*

Corollary 2.7. *Let M be a semisimple R-module and second module. Then M is a prime R-module.*

Corollary 2.8. *Let $M = \bigoplus_{i \in I} M_i$ be a R-module such that $Ann(M) = Ann(M_i)(i \in I)$ and $M_i(i \in I)$ are prime module. Then M is a prime module.*

Corollary 2.9. *Let $M = \bigoplus_{i \in I} M_i$ be semisimple R-module and $Ann(M) = Ann(M_i)(i \in I)$. Then M is a prime module.*

Corollary 2.10. *Let M be a semisimple artinian prime module. Then $M = \bigoplus_{i=1}^{n} M_i$ with M_i simple for some $n \in N$*

Proof. Immediate follows from ([1], Theorem 10.6) and Corollary 2.8. □

Recall that a submodule N of M is called semiprime if $N \neq M$ and $r^2m \in N$ implies $rm \in N$ for all $r \in R$, $m \in M$ and non-zero module M is called semiprime R-module if its zero submodule is a semiprime submodule.

Theorem 2.11. *Let M be a finitely generated faithfully prime R-module such that each cyclic submodule of M is semiprime. Then M is a vector space.*

Proof. Since M is prime and faithful, $0 = Ann(M) = Ann(y)$, for all $0 \neq y \in M$. Hence $R \cong Ry$. Now suppose that $0 \neq a \in R$ and $a^2y \in Ra^2y$, then $ay \in Ra^2y$ because Ra^2y is a cyclic semiprime. So, there exists $b \in R$ such that $ay = ba^2y$. Because M is faithfull,$ba = 1$.Hence the element a is invertible and consequently M is a vector space. □

Corollary 2.12. *Let M be a artinian prime R-module such that each cyclic submodule of M is semiprime. Then M is a vector space over $R/Ann(M)$.*

Proof. By Corollary 2.8, M is finitely generated. Hence it follows by Theorem 2.11. □

Recall that if N and M are R-module, then $\pi(N, M) = \sum_{\varphi} \varphi(N)$, where $\varphi \in Hom(N, M)$.

Definition 2.1. Let M be an R-module. Then M is said to be a π_{-}module if $\pi(N, M) = M$ for each non-zero submodule N of M. A submodule N of M is said to be dense in M if $\pi(N, M) = M$.

Theorem 2.13. *Let M be a injective prime R-module. Then M is a π-module.*

Proof. Let N be a non-zero submodule of M and x a non-zero element of N. We have $Ann(x) = Ann(M) = Ann(y)$ for each y in M. Define f from Rx to M by $f(rx) = ry$ for all $r \in R$. So that f is homomorphism. Since M is N-injective, f lifts to a homomorphism $\varphi : N \to M$. Also $\varphi(x) = f(x) = y$. Thus $y \in \varphi(N) \subseteq \pi(N, M)$. Hence M is a π-module. □

Corollary 2.14. *Let M be an artinian prime R-module. Then M is a π-module.*

Proof. It follows from Corollary 2.2 and Theorem 2.13. □

Definition 2.2. Let M be an R-module. Then M is said to be a fully π-module if every submodule of M is π-module.

Corollary 2.15. *Let M be an artinian prime R-module, then M is fully π-module.*

Proof. If N is a submodule, then N is artinian prime module. This completes the proof. □

Note that the class of modules which have secondary representation is larger that of artinian modules. In ([10], page 43) Matsumura introduced the notion of a minimal secondary representation module and it is shown that if $M = M_1 + M_2 + \cdots + M_n$ is a minimal secondary representation for M with $\sqrt{(0 : M_i)} = P_i$ for $i = 1, \cdots, n$, then the set $\{P_1, P_2, \cdots, P_n\}$ of prime ideals of R is independent of the choice of a minimal secondary representation for M. We denote this set by $Att(M)$ and refer to its members as the attached prime ideals of M.

Theorem 2.16. *([13],Theorem 7.30) Let M be a module over the commutative ring R, and assume that M is annihilated by the product of finitely many (not necessarily distinct) maximal ideals of R, that is, there exists $n \in N$ and maximal ideals $m_1, \cdots, m_n$ of R such that $m_1 m_2 \cdots m_n M = 0$. Then M is a noetherian R-module if and only if M is an artinian R-module.*

Proposition 2.4. *Let (R, m) be a local ring and M be an artinian R-module. Then M is a noetherian R-module if and only if $Att(M) \subseteq \{m\}$.*

Proof. If $Att(M) \subseteq \{m\}$, then there exists $n \in N$ such that $m^n M = 0$. Hence, M is a noetherian R-module by Theorem 2.16. Conversely, if M is noetherian module, then it is either zero or m-secondary. □

Theorem 2.17. *Let M be a non-zero prime R-module. Then M is artinian R-module if and only if M is noetherian R-module with $Soc(M) \neq 0$*

Proof. Since $Soc(M) \neq 0$ and M is prime, we have that $N \cong R/(0 : x) = R/Ann(M)$ where N is a simple submodule and $Ann(M)$ is maximal ideal. Hence M is a vector space over the field $R/Ann(M)$. Thus the claim follows. □

Recall that a prime submodule K of M has height$n$, if exits a chain $K = K_0 \supset K_1 \supset \cdots \supset K_n$ of prime submodules $K_i(1 \leq i \leq n)$ of M, but no such longer chain exits. Otherwise, we say that it has infinite height. We shall denote the height of K by htK . We define the $h - dim(M)$ to be the supremum of the heights of all prime submodules of M. If M has no prime submodule, we set $h - dim(M) = -1$.

A module M is called a catenary module if for any prime submodules N and $N^{'}$ of M with $N \subset N^{'}$ all the saturated chains of prime submodules of M starting from N and ending at $N^{'}$ have the seme length (see [11]).

Recall that an R-module M is said to be finitely cogenereted (the dual notion of finitely generated) if $E(M)$ (the injective envelope of M) is isomorphic to a direct sum of finitely many injective envelopes of simple modules. It is well known that a module is finitely cogenerated if and only if its socle is a finitely generated and essential submodule.

Example 2.18. (1) *The only prime submodule of Q (rational number) is the submodule (0) but Q is not finitely generated. Hence if every prime submodule of an R-module M is finitely generated, then M is not necessarily finitely generated.*

(2) *$\bigoplus_{i=1}^{k} Z_p$ is an artinian prime Z-module which is finitely generated and finitely cogenerated.*

(3) *Z_p^{∞} is artinian $\mathbb{Z}$-module which is not prime and finitely generated but it is finitely cogenerated.*

(4) *Z_p^{∞} as Z-module has not any prime submodule.*

(5) *Let K be a field, then $K[x, y]$ is a noetherian prime module but it is not a artinian module.*

(6) *Any vector space is catenary if and only if it is finite dimensional.*

Corollary 2.19. *Let M be a non-zero artinian prime R-module. Then*

(1) *M is finitely generated.*

(2) *M is a noetherian R-module.*

(3) *There exits $N \leq M$ such that $E(N) \cong \bigoplus_{i=1}^{k} E(N_i)$ for some $k \in N$, where $E(N_i)(1 \leq i \leq k)$ are injective envelopes of simple modules.*

(4) *There exists $N \leq M$ such that every non-zero submodule of N contains a simple prime module, hence second submodule.*

(5) *M is a catenary module.*

(6) *M/N is catenary $R/(N : M)$-module for every prime submodule N.*

(7) $h - dim(M) < \infty$

(8) *M is a second module.*

Proof. (1) It follows from Theorem 2.17.

(2) It follows from Theorem 2.17.

(3) By Theorem 2.17 we have $Soc(M) \neq 0$. It suffices to put $N = Soc(M)$.

(4) Proof is the same as Part (3).

(5) M is a vector space over $R/Ann(M)$. Hence M is catenary.

(6) Let N be a prime submodule of R-module M. Then by Proposition 2.2, M/N is a vector space over $R/Ann(M/N)$. Hence M/N is a catenary module.

(7) M is a finite dimensional vector space over $R/Ann(M)$.

(8) It follows from Corollary 2.2.

So the proof is complete. $\square$

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