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THE CONCEPT OF LOGIC ENTROPY ON D-POSETS

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Abstract. In this paper, a new invariant called *logic entropy* for dynamical systems on a D-poset is introduced. Also, the *conditional logical entropy* is defined and then some of its properties are studied. The invariance of the *logic entropy* of a system under isomorphism is proved. At the end, the notion of an m-generator of a dynamical system is introduced and a version of the Kolmogorov-Sinai theorem is given.

1. Introduction

Entropy plays an important role, as a mathematical device, in a varieaty of problem areas, includying physics, information theory, biology, chemistry and others. This notion is a useful tool in studing of dynamical systems and their isomorphism. We may have different versions of the definition of the entropy, depending on the conditions of the discussed problem [5, 6, 7].To study noncompatible random events in mathematical model, in 1936, G. Birkhoff and J. Von Neumann gave a basis of quantum mathematics, called quantum logic [1]. Kopka and Chovanec, introduced D-posets as an axiomatic model for quantum logics [3]. In this paper, with the help of a state on a D-poset the notion of *logic*

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entropy and *conditional logical entropy* is introduced. Also, by using a bijective mapping between two dynamical systems, the concept of their isomorphism is defined and then it is shown that the *logic entropy* of dynamical systems is isomorphism invariant. Finally, a version of the Kolmogorov-Sinai theorem is given.

2. preliminaries

In this section we recall the basic notions on D-posets.

Definition 2.1. Effect algebra is a system $(E, +, 0, 1)$, where 0, 1 are distinguished elements of E and $+$ is a partial binary operation on E such that

1. $a + b = b + a$ if one side is defined;
2. $(a + b) + c = a + (b + c)$ if one side is defined;
3. for every $a \in E$ there exsits a unique $a^{\perp}$ with $a + a^{\perp} = 1$;
4. if $a + 1$ is defined then $a = 0$.

Every effect algebra bears a natural partial ordering given by $\leq$ if and only if $b = a + c$ for some $c \in E$. The poset $(E, \leq)$ is bounded, 0 is the bottom element and 1 is the top element. In every effect algebra, a partial subtraction can be defined as follows:
$a - b$ exists and is equal to c if and only if $a = b + c$.
The system $(E, \leq, -, 0, 1)$ so obtained is a D-poset defined by Kopka and Chovanec [3].

Definition 2.2. The structure $(D, \leq, -, 0, 1)$ is called D-poset if the relation $\leq$ is a partial ordering on D, 0 is the smallest and 1 is the largest element on D and $-$ is a partial binary operation satisfying the following conditions:

1. $b - a$ is defined if and only if $a \leq b$;
2. if $a \leq b$ then $b - a \leq b$ and $b - (b - a) = a$;
3. $a \leq b \leq c \Rightarrow c - b \leq c - a, (c - a) - (c - b) = b - a$.

For any element a in a D-poset D, the element $1 - a$ is called the orthosupplement of a and is denoted by $a^{\perp}$.

Example 2.3. Let H be a Hilbert space. A positive Hermitian operator A on H such that $O \leq A \leq I$, where O and I are operators on H defined by the formulas $Ox = 0, Ix = x$ for any $x \in H$, is said to be an effect. A system $E(H)$ of effects closed with respect to the difference $B - A$ of operators $A, B \in E(H), A \leq B$, is a D-poset.

Lemma 2.4. *Let D be a D-poset and $a, b, c \in D$. The following assertions are true:*

1. $b - a$ *is defined if and only if* $a \leq b$;
2. *if* $a \leq b$ *then* $b - a \leq b$ *and* $b - (b - a) = a$;
3. $a \leq b \leq c \Rightarrow c - b \leq c - a, (c - a) - (c - b) = b - a$;
4. $a - 0 = a$ *for all* $a \in D$;
5. $a - a = 0$ *for all* $a \in D$.

Proof. See [3]. ■

Definition 2.5. Let $(D, \leq, -, 0, 1)$ be a D-poset. Define a partial binary operation $\oplus$ and a binary operation $\odot$ as follows, for any $a, b \in D$.

$$a \oplus b = (a^{\perp} - b)^{\perp} \text{ if } a \leq b^{\perp}.$$

and

$$a \odot b == \begin{cases} a - b^{\perp} & \text{if } a^{\perp} \leq b, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.6. A state m on a D-poset D is a mapping $m : D \longrightarrow [0, 1]$ such that for all $a, b, a_n \in D$:

1. $m(1) = 1$;
2. if $a \leq b$ then $m(a) \leq m(b)$;
3. if $a \leq b$ then $m(b - a) = m(b) - m(a)$;
4. $a_n \nearrow a \Longrightarrow m(a_n) \nearrow m(a)$.

The notation "$a_n \nearrow a$" which stands for a_n is a nondecreasing sequence and $a = \oplus_{n \in \mathbb{N}} a_n$. The state m is faithful if $m(a) = 0$ implies $a = 0$ for any $a \in M$.

Definition 2.7. Two elements $a, b \in D$ are orthogonal if $a \leq b^{\perp}$, and is denoted by $a \perp b$.
A finite subset $\xi = \{a_1, a_2, ..., a_n\}$ of elements of a D-poset D is said to be $\oplus-$orthogonal if and only if

$$\oplus_{i=1}^{k} a_i \perp a_{k+1} \qquad for \quad k = 1, 2, ..., n - 1.$$

Definition 2.8. A finite collection $\xi = \{a_1, a_2, ..., a_n\}$ of elements of a D-poset D is said to be a partition of D if and only if

1. ξ is $\oplus-$orthogonal subset;
2. $m(\oplus_{i=1}^{n} a_i) = 1$.

Definition 2.9. Suppose $\xi = \{a_1, ..., a_n\}$ be any finite partition of D corresponding to a state m and $b \in D$. We say that the state m has Bayes'property if

$$m(\oplus_{i=1}^{n}(a_i \odot b)) = m(b).$$

We can easily prove that if $\xi = \{a_1, ..., a_n\}$ be a finite partition of $D, b \in D$, and the state m has Bayes'property then

$$\sum_{i=1}^{n} m(b \odot a_i) = m(b).$$

Definition 2.10. Suppose $\xi = \{a_1, ..., a_n\}$ and $\eta = \{b_1, ..., b_m\}$ are two finite partitions of a D-poset D. Then we define $\xi \prec \eta$ (i.e η is a refinement of ξ if there exists a partition $\{I(1), ..., I(n)\}$ of the set $\{1, ..., m\}$ such that

$$m(a_i) = \sum_{j \in I(i)} m(b_j) \quad i = 1, ..., n.$$

3. The Logic entropy of a partition

In this section, the concept of *logic entropy* for a finite partition on a D-poset is introduced and some of its properties are stated.

Definition 3.1. If $\xi = \{a_1, ..., a_n\}$ and $\eta = \{b_1, ..., b_m\}$ are two finite partitions of a D-poset D corresponding to a state m, then the *logic entropy* of ξ is the number

$$h_m(\xi) = \sum_{i=1}^{n} m(a_i)(1 - m(a_i)).$$

and the conditional *logical entropy* of ξ given η is defined by

$$h_m(\xi|\eta) = \sum_{i=1}^{n} \sum_{j=1}^{m} m(a_i \odot b_j)(m(b_j) - m(a_i \odot b_j)).$$

If $\eta = \{1\}$ then $h_m(\xi|\eta) = h_m(\eta)$. Also, it is clear that $h_m(\xi|\eta) \leq h_m(\eta)$.

Definition 3.2. If $\xi = \{a_1, ..., a_n\}$ and $\eta = \{b_1, ..., b_m\}$ are two finite partitions of D. Their join is

$$\xi \nabla \eta = \{a_i \odot b_j; a_i \in \xi, b_j \in \eta\}.$$

Theorem 3.3. *Let ξ, η and ζ be finite partitions of a D-poset D corresponding to a state m. Then*

$$h_m(\xi \nabla \eta|\zeta) = h_m(\xi|\zeta) + h_m(\eta|\xi \nabla \zeta).$$

Proof. By the definition we have

$$h_m(\xi \nabla \eta | \zeta) = \sum_{i,j,k} m(a_i \odot b_j \odot c_k)[(m(c_k) - m(a_i \odot b_j \odot c_k)]$$

But we may write

$$m(c_k) - m(a_i \odot b_j \odot c_k) = [m(a_i \odot c_k) - m(a_i \odot b_j \odot c_k)] + [m(c_k) - m(a_i \odot c_k)]$$

Now from Bayes'property of the state m, we have

$$\sum_k m(b_j \odot c_k \odot a_i) = m(b_j \odot a_k),$$

therefore

$$h_m(\xi \nabla \eta | \zeta) = h_m(\xi | \zeta) + h_m(\eta | \xi \nabla \zeta).$$

■

Theorem 3.4. *Let ξ, η and ζ be finite partitions of a D-poset D corresponding to a state m. Then*
1.$h_m(\xi \nabla \eta) = h_m(\xi) + h_m(\eta | \xi)$;
2.$h_m(\xi \nabla \eta) \leq h_m(\xi) + h_m(\eta)$;
3.$h_m(\xi \nabla \eta | \zeta) \leq h_m(\xi | \zeta) + h_m(\eta | \zeta)$.

Proof. 1. By Theorem 3.3 we have

$$\begin{aligned} h_m(\xi \nabla \eta) &= h_m(\xi \nabla \eta | \zeta) \\ &= h_m(\xi | \zeta) + h_m(\eta | \xi \nabla \zeta) \\ &= h_m(\xi) + h_m(\eta | \xi). \end{aligned}$$

2. By the part 1

$$h_m(\xi \nabla \eta) = h_m(\xi) + h_m(\eta | \xi) \leq h_m(\xi) + h_m(\eta).$$

3. It is clear. ■

Theorem 3.5. *Let ξ and η be finite partitions of a D-poset D corresponding to a state m. If $\xi \prec \eta$ then $h_m(\xi) \leq h_m(\eta)$.*

Proof. It is clear. ■

4. Logic entropy of dynamical systems

Definition 4.1. If D is a D-poset then by a dynamical system on D we mean a triple (D, m, φ), where $m : D \to [0, 1]$ is a state on D with the Bayes property and $\varphi : D \to D$ is a mapping satisfying the following conditions:

1. If $a \leq b^{\perp}$ then $\varphi(a) \leq \varphi(b)^{\perp}$ and $\varphi(a \oplus b) = \varphi(a) \oplus \varphi(b)$;
2. $\varphi(a \odot b) = \varphi(a) \odot \varphi(b)$;
3. $\varphi(1) = 1$;
4. $m(\varphi(a)) = m(a)$ for any $a \in D$.

Theorem 4.2. *Let (D, m, φ) be a dynamical system on D and ξ, η be two finite partitions of D. Then*

$$h_m(\varphi(\xi)|\varphi(\eta)) = h_m(\xi|\eta).$$

Proof. It is clear. ■

Definition 4.3. Let (D, m, φ) be a dynamical system on D. For any finite partition ξ we define the *logic entropy* of φ with respect to ξ as

$$h_m(\varphi, \xi) = \lim_{n\to\infty} h_m(\nabla_{i=0}^{n-1} \varphi^i(\xi)).$$

Furthermore we define the logic entropy of dynamical system by

$$h_m(\varphi) = \sup\{h_m(\varphi, \xi); \xi \text{ is a finite partition}\}.$$

Theorem 4.4. *Let (D, m, φ) be a dynamical system on D. If ξ and η be finite partitions and $\xi \prec \eta$, then*

$$h_m(\varphi, \xi) \leq h_m(\varphi, \eta).$$

Proof. Follows from Theorem 3.5. ■

Theorem 4.5. *Let (D, m, φ) be a dynamical system on D and ξ be a finite partition of D. Then*

$$h_m(\varphi, \xi) = h_m(\varphi, \nabla_{j=0}^{k} \varphi^j(\xi)).$$

Proof.

$$\begin{aligned} h_m(\varphi, \nabla_{j=0}^{k} \varphi^j(\xi)) &= \lim_{n\to\infty} h_m(\nabla_{i=0}^{n-1} \varphi^i(\nabla_{j=0}^{k} \varphi^j(\xi))) \\ &= \lim_{n\to\infty} h_m(\nabla_{t=0}^{n+k-1} \varphi^t(\xi)) \\ &= \lim_{p\to\infty} h_m(\nabla_{t=0}^{p-1} \varphi^t(\xi)) \\ &= h_m(\varphi, \xi). \end{aligned}$$

■

Theorem 4.6. *Let (D, m, φ) be a dynamical system on D and ξ be a finite partition of D. Then for every natural number k*

$$h_m(\varphi^k) = h_m(\varphi).$$

Proof. By Theorem 4.5 we have

$$\begin{aligned} h_m(\varphi^k, \xi) &= h_m(\varphi^k, \nabla_{i=0}^{k-1}\varphi^i(\xi)) \\ &= \lim_{n\to\infty} h_m(\nabla_{j=0}^{n-1}\varphi^{jk}(\nabla_{i=0}^{n-1}\varphi^i(\xi))) \\ &= \lim_{n\to\infty} h_m(\nabla_{i=0}^{nk-1}\varphi^i(\xi)) = h_m(\varphi, \xi). \end{aligned}$$

■

Definition 4.7. We say that two dynamical systems (D_1, m_1, φ_1) and (D_2, m_2, φ_2) are isomorphic if there exists a bijective map $\Psi : D_1 \to D_2$ satisfying the following conditions for each $a, b \in D_1$:

1. If $a \le b^\perp$ then $\Psi(a) \le \Psi(b)^\perp$ and $\Psi(a \oplus b) = \Psi(a) \oplus \Psi(b)$;
2. $\Psi(a \odot b) = \Psi(a) \odot \Psi(b)$;
3. $\Psi(\varphi_1(a)) = \varphi_2(\Psi(a))$;
4. $m_1(a) = m_2(\Psi(a))$.

Theorem 4.8. *Let (D_1, m_1, φ_1) and (D_2, m_2, φ_2) are isomorphic dynamical systems, then $h_m(\varphi_1) = h(\varphi_2)$, i.e., logic entropy of their dynamical systems is an isomorphism invariant.*

Proof. Let (D_1, m_1, φ_1) and (D_2, m_2, φ_2) be isomorphic dynamical systems and $\Psi : D_1 \to D_2$ be the mapping representing the isomorphism of dynamical systems. Let $\xi = \{a_1, ..., a_n\}$ be a finite partition of D_1, then $\Psi(\xi)$ is the finite partition of D_2. Now

$$\begin{aligned} h_{m_2}(\Psi(\xi)) &= \sum_{i=1}^{n} m_2(\Psi(a_i))(1 - m_2(\Psi(a_i)) \\ &= \sum_{i=1}^{n} m_1(a_i)(1 - m_1(a_i)) = h_{m_1}(\xi). \end{aligned}$$

Thus

$$\begin{aligned} h_{m_2}(\varphi_2, \Psi(\xi)) &= \lim_{n\to\infty} h_{m_2}(\nabla_{i=0}^{n-1} {\varphi_2}^i(\Psi(\xi))) \\ &= \lim_{n\to\infty} h_{m_2}(\nabla_{i=0}^{n-1} \Psi({\varphi_1}^i(\xi))) \\ &= \lim_{n\to\infty} h_{m_2}(\Psi(\nabla_{i=0}^{n-1} {\varphi_1}^i(\xi))) \\ &= \lim_{n\to\infty} h_{m_1}(\nabla_{i=0}^{n-1} {\varphi_1}^i(\xi)) \\ &= h_{m_1}(\varphi_1, \xi). \end{aligned}$$

■

5. m-Generators of dynamical systems

Definition 5.1. Let (D, m, φ) be a dynamical system on D. Then a finite partition ξ of D is called an m-generator if there exsits an integer $k > 0$ such that

$$\eta \prec \nabla_{i=0}^{k} \varphi^i \xi,$$

for every finite partition η of D.

Theorem 5.2. *Let (D, m, φ) be a dynamical system on D and ξ be an m-generator. Then*

$$h_m(\varphi, \eta) \leq h_m(\varphi, \xi),$$

for every finite partition η of D.

Proof. Since ξ is an m-generator, then for partition η there exsists an integer $k > 0$ such that

$$\eta \prec \nabla_{i=0}^{k} \varphi^i \xi.$$

Hence

$$h_m(\varphi, \eta) \leq h_m(\varphi, \nabla_{i=0}^{k} \varphi^i \xi) = h_m(\varphi, \xi).$$

■

Now we can deduce the following version of Kolmogorov-Sinai theorem [7].

Theorem 5.3. *Let (D, m, φ) be a dynamical system on D and ξ be an m-generator. Then*

$$h_m(\varphi) = h_m(\varphi, \xi).$$

Proof. Obvious. ■

6. Concluding Remarks

In this paper, the notion of *logic entropy* of dynamical systems on a D-poset is introduced and its properties are investigated. Also, it is shown that *logic entropy* is a useful object for the classification of the dynamical systems based on isomorphism. Because, it is an invariant object under isomorphism relation. Finally, the notion of an m-generator of a dynamical system is introduced and a version of the Kolmogorov-Sinai theorem is stated.

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