THE PRINCIPAL IDEAL SUBGRAPH OF THE ANNIHILATING-IDEAL GRAPH OF COMMUTATIVE RINGS

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Abstract. Let $R$ be a commutative ring with identity and $A(R)$ be the set of ideals of $R$ with non-zero annihilators. In this paper, we first introduce and investigate the principal ideal subgraph of the annihilating-ideal graph of $R$, denoted by $AG_P(R)$. It is a (undirected) graph with vertices $A_P(R) = A(R) \cap P(R) \setminus \{(0)\}$, where $P(R)$ is the set of proper principal ideals of $R$ and two distinct vertices $I$ and $J$ are adjacent if and only if $IJ = (0)$. Then, we study some basic properties of $AG_P(R)$. For instance, we characterize rings for which $AG_P(R)$ is finite graph, complete graph, bipartite graph or star graph. Also, we study diameter and girth of $AG_P(R)$. Finally, we compare the principal ideal subgraph $AG_P(R)$ and spectrum subgraph $AG_s(R)$.

1. Introduction

In recent years, assigning graphs to rings has played an important role in the study of structures of rings (see for example [7, 8, 10-14]). Let $R$ be a commutative ring. We call an ideal $I$ of $R$ is an annihilating-ideal, if there exists a non-zero ideal $J$ of $R$ such that $IJ = (0)$ and use the notation $A(R)$

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for the set of annihilating-ideals of $R$. By the annihilating-ideal graph $\mathbb{A}G(R)$ of $R$ we mean the graph with vertices $\mathbb{A}^+(R) = \mathbb{A}(R) \setminus \{(0)\}$ such that there is an (undirected) edge between vertices $I$ and $J$ if and only if $I \neq J$ and $IJ = (0)$. The concept of the annihilating-ideal graph of a commutative ring was first introduced by Behroodi and Rakei in [12, 13]. Recently, this notation of the annihilating-ideal graph has been extensively studied by various authors (see for instance, [1-6, 16, 18] and many others). In [18], Taheri, Behroodi and Tehranian, introduce and investigate the spectrum graph of the annihilating-ideal graph of a commutative ring, denoted by $\mathbb{A}G_s(R)$, that is, a graph whose vertices are all non-zero prime ideals of $R$ with non-zero annihilators, denoted by $\mathbb{A}s(R)$ and two distinct vertices $P_1, P_2$ are adjacent if and only if $P_1P_2 = (0)$. In this paper, we introduce and study the principal ideal graph of a commutative ring $R$, denoted by $\mathbb{A}G_P(R)$, that is, the graph whose vertices are all principal ideals in $\mathbb{A}^+(R)$ and two distinct vertices $I, J$ are adjacent if and only if $IJ = (0)$. We denote by $\mathbb{A}_P(R)$ the vertex set of $\mathbb{A}G_P(R)$. It is clear that if $R$ is a P.I.R. which is not integral domain, then $\mathbb{A}G_P(R) = \mathbb{A}G(R)$. First we study some basic properties of $\mathbb{A}G_P(R)$ and then we compare the principal ideal subgraph with the spectrum subgraph.

2. DEFINITION AND PRELIMINARIES

Throughout this paper, all rings are commutative with identity and all modules are unitary. For a ring $R$, we denote by Spec($R$) the set of prime ideals, $Z(R)$ the set of zero-divisors, $I(R)$ the set of non-zero proper ideals and $\mathbb{P}(R)$ the set of proper principal ideals of $R$. The jacobson radical and the set of minimal prime ideals of $R$ are denoted by $J(R)$ and $\text{Min}(R)$, respectively. Let $X$ be an element or a subset of a ring $R$. The annihilator of $X$ is the ideal $\text{Ann}(X) = \{a \in R \mid aX = 0\}$.

Let $G$ be any graph. We denote the vertex set of $G$ by $V(G)$. Sometimes, two graphs $G$ and $H$ have exactly the same form, in the sense that there is a one-to-one correspondence between their vertex sets that preserves edges. In such a case, we say that the two graphs $G$ and $H$ are isomorphic and we write $G \cong H$. We say that a subgraph $H$ is an induced subgraph of $G$ if $H$ is isomorphic to a graph whose vertex set $V_1$ is a subset of the vertex set $V$ of $G$ and whose edge set $E_1$ consists of all the edges of $G$ with both end vertices in $V_1$. For every subset $A$ of $\mathbb{A}(R)$ we denote the induced subgraph of $\mathbb{A}G(R)$ with vertex set $A$ by $\mathbb{A}G_A(R)$. The graph $G$ is called connected if there is a path between every two distinct vertices. For distinct vertices $P, Q$ of $G$, let $d(P, Q)$ be the length of the shortest path from $P$ to $Q$ and, if there is no such a path, we define $d(P, Q) = \infty$. The diameter of $G$ is $\text{diam}(G) = \sup\{d(P, Q) : P \text{ and } Q \text{ are distinct vertices of } G\}$. The girth of $G$, denoted by $\text{gr}(G)$, is defined as the length of the shortest cycle in $G$ and $\text{gr}(G) = \infty$, if $G$ contains no cycles. A complete graph is a graph in which any two distinct vertices are adjacent. A complete graph with $n$ vertices denoted by $K_n$. A bipartite graph is a graph whose vertices can be divided into two disjoint sets $A$ and $B$ such that every edge connects a vertex in $A$ to one in $B$. A complete bipartite graph is a bipartite graph in which every vertex of one part is joined to every vertex of the other part. In this
case, if $|A| = n$ and $|B| = m$, we denote the graph by $K_{n,m}$. If $|A| = 1$ or $|B| = 1$, then the graph is said to be a star graph. Ultimately, we denote by $P_n$ a path of order $n$.

3. COMPARING THE PRINCIPAL IDEAL GRAPH AND THE ANNIHILATING-IDEAL GRAPH

Let $R$ be a non-domain commutative ring. In this section, we compare the features of $\mathbb{A}G_P(R)$ and $\mathbb{A}G(R)$ and express some properties of principal ideal graph. We begin with the following proposition, which characterize all rings $R$ with finite principal ideal graph.

**Proposition 3.1.** Let $R$ be a ring. Then the following statements are equivalent.

1. $\mathbb{A}G_P(R)$ is a finite graph.
2. $\mathbb{A}G(R)$ is a finite graph, moreover $|\mathbb{A}^*(R)| \leq 2^{|\mathbb{A}_P(R)|}$.
3. $R$ has only finitely many ideals.
4. Every vertex of $\mathbb{A}G_P(R)$ has finite degree.
5. Every vertex of $\mathbb{A}G(R)$ has finite degree.

**Proof.** (1) $\Rightarrow$ (2): Assume that $\mathbb{A}G_P(R)$ is a finite graph. Therefore, $\mathbb{A}_P(R)$ is a finite set. For each $I \in \mathbb{A}^*(R)$, $I$ is generated by a subset of $\mathbb{A}_P(R)$. Since $\mathbb{A}_P(R)$ has finitely many subsets, we can conclude that $\mathbb{A}^*(R)$ is a finite set. Thus, $\mathbb{A}G(R)$ is a finite graph and $|\mathbb{A}^*(R)| \leq 2^{|\mathbb{A}_P(R)|}$.

(2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (4) are clear.

(2) $\Rightarrow$ (3) and (5) $\Rightarrow$ (2) are follow from [12, Theorem 1.4].

(4) $\Rightarrow$ (5) Suppose on the contrary that $I \in \mathbb{A}^*(R)$ has infinite degree in $\mathbb{A}G(R)$, so for each $a \in I$, $Ra \in \mathbb{A}_P(R)$ has infinite degree in $\mathbb{A}G_P(R)$, a contradiction. $\Box$

In the following result, we characterize reduced rings $R$ for which every vertex of principal ideal graph $\mathbb{A}G_P(R)$ has finite degree.

**Corollary 3.2.** Let $R$ be a reduced ring. Then every vertex of $\mathbb{A}G_P(R)$ has finite degree if and only if $R \cong F_1 \times F_2 \times \cdots \times F_n$, where $F_i$'s are fields for each $i$, $1 \leq i \leq n$.

**Proof.** ($\Leftarrow$) It is clear.

($\Rightarrow$) Assume that every vertex of $\mathbb{A}G_P(R)$ has finite degree. By Proposition 3.1, $R$ has only finitely many ideals and hence is an Artinian ring, so $R \cong F_1 \times F_2 \times \cdots \times F_n$, where $F_i$'s are fields for each $i$, $1 \leq i \leq n$. $\Box$

We now show that $\mathbb{A}G_P(R)$ is connected.

**Proposition 3.3.** For every ring $R$, the principal ideal subgraph $\mathbb{A}G_P(R)$ is connected and $\text{diam}(\mathbb{A}G_P(R)) \leq 3$. $\Box$
Proof. Let I and J be distinct vertices of $\mathbb{AG}_P(R)$. Then there exist principal ideals K and L such that $IK = (0)$ and $JL = (0)$. If $KL = (0)$, then $J - L - K - I$ is a path with length 3 in $\mathbb{AG}_P(R)$. If $KL \neq (0)$, then there is a principal ideal $H \subseteq KL$ and $J - H - I$ is a path with length 2 in $\mathbb{AG}_P(R)$. We can conclude that $\mathbb{AG}_P(R)$ is connected and diam$(\mathbb{AG}_P(R)) \leq 3$. \qed

In the following proposition we show that $\mathbb{AG}_P(R)$ is a complete graph if and only if $\mathbb{AG}(R)$ is complete. Also, in the next proposition we characterize rings whose principal ideal subgraphs are complete graphs.

**Proposition 3.4.** Let R be a ring. Then the following statements are equivalent.

1. $\mathbb{AG}_P(R)$ is a complete graph.
2. $\mathbb{AG}(R)$ is a complete graph.
3. Either $\mathbb{AG}(R) \cong K_2$ or $Z(R)$ is the ideal of $R$ with $Z(R)^2 = (0)$. Moreover, in the first case, either $R \cong F_1 \times F_2$, where $F_1$ and $F_2$ are fields or $(R, M)$ is a local ring with exactly two non-trivial ideals $M^2$ and $M$.

Proof. (1) $\Rightarrow$ (2) Let I and J be two distinct vertices of $\mathbb{AG}(R)$. We show that $IJ = (0)$. Without loss of generality we can assume $I \nsubseteq J$. Let $0 \neq x \in I$. We consider the following two cases:

**Case 1.** $x \in I \setminus (I \cap J)$. Then for each $y \in J$, $Rx$ and $Ry$ are two distinct vertices of $\mathbb{AG}_P(R)$ and hence $(Rx)(Ry) = (0)$. Therefore, $xJ = (0)$.

**Case 2.** $x \in (I \cap J)$. Let $a \in I \setminus (I \cap J)$, so $x + a \in I \setminus J$ and we can see for each $y \in J$, $Ry$ and $R(x + a)$ are two distinct vertices of $\mathbb{AG}_P(R)$, so $R(x + a)Ry = (0)$. Note that $y$ is an arbitrary element of $J$, thus we have $R(x + a)J = (0)$, and so $(x + a)J = (0)$. Since $aJ = (0)$, $xJ = (0)$.

Therefore, in each case, for every $x \in I$, we have $xJ = (0)$, which implies that $IJ = (0)$. We can conclude that $\mathbb{AG}(R)$ is a complete graph.

(2) $\Rightarrow$ (1) Let $\mathbb{AG}(R)$ is a complete graph. Since $\mathbb{AG}_P(R)$ is an induced subgraph of $\mathbb{AG}(R)$, we have $\mathbb{AG}_P(R)$ is a complete graph.

(2) $\Leftrightarrow$ (3) It follows from [1] Theorem 3.]\qed

We need the following lemma.

**Lemma 3.5.** Let $I_1, I_2$ and $I_3$ be distinct ideals of $R$. Then there exist distinct non-zero principal ideals $Ra_1, Ra_2$ and $Ra_3$ such that $Ra_i \subseteq I_i$, for $i = 1, 2, 3$.

Proof. Let $X = \{I_1, I_2, I_3\}$. Without loss of generality assume that $I_1$ is a maximal element of $X$. If $I_1 \subseteq I_2 \cup I_3$, then by Prime Avoidance Theorem [17, Theorem 3.61], $I_1 \subseteq I_2$ or $I_1 \subseteq I_3$, a contradiction to maximality of $I_1$. Therefore, there exists $0 \neq a_1 \in I_1 \setminus (I_2 \cup I_3)$. Since $I_2$ and $I_3$ are distinct, without
loss of generality assume that $I_2 \nsubseteq I_3$, so there exists $a_2 \in I_2$ such that $Ra_2 \neq Ra_3 \subseteq I_3$, for some $a_3 \in I_3$. Therefore, $Ra_1 \subseteq I_1, Ra_2 \subseteq I_2$ and $Ra_3 \subseteq I_3$ are distinct non-zero principal ideals of $R$. □

**Proposition 3.6.** Let $R$ be a ring. Then

1. $\mathbb{A}G_P(R) \cong K_1$ if and only if $R$ has only one non-zero proper ideal, moreover, either $R \cong \mathbb{Z}_p[x]/(x^2)$, or $R \cong \mathbb{Z}_p^2$.

2. Assume that $\mathbb{A}G_P(R) \cong K_2$, then $R \cong F_1 \times F_2$, where $F_1, F_2$ are two fields, or $(R, \mathcal{M})$ is a local ring with $\mathbb{A}_P(R) = \{\mathcal{M}^2, \mathcal{M}\}$.

3. Assume that $\mathbb{A}G_P(R) \cong K_3$, where $n \geq 3$, then $Z(R)$ is an ideal of $R$ with $Z(R)^2 = (0)$.

*Proof.* (1) ($\Rightarrow$) Suppose that $\mathbb{A}G_P(R) \cong K_1$ and $\mathbb{A}_P(R) = \{I\}$, where $I = Rx$ for some $0 \neq x \in R$. We claim that $|\mathbb{A}^*(R)| = 1$. By contrary suppose that $|\mathbb{A}^*(R)| \geq 2$ and $J \in \mathbb{A}^*(R) \setminus \{I\}$. Since $I \neq J$ and $I$ is a simple $R$-module, $J \nsubseteq I = Rx$ and hence there exists $y \in J \setminus Rx$ such that $Rx \neq Ry$, so $|\mathbb{A}_P(R)| \geq 2$, a contradiction. Therefore by [12, Corollary 2.9], $R$ has only one non-zero ideal. Also [1, Remark 10], implies that, either $R \cong \mathbb{Z}_p[x]/(x^2)$, or $R \cong \mathbb{Z}_p^2$.

($\Leftarrow$) Assume that $I$ is the only non-zero proper ideal of $R$. Since $R$ is an Artinian ring by [12, Proposition 1.3], $\mathbb{A}^*(R) = \{I\}$. It is clear that $I$ is a simple $R$-module and hence $I \in \mathbb{A}_P(R)$, we have $\mathbb{A}G_P(R) \cong K_1$.

(2) Suppose that $\mathbb{A}G_P(R) \cong K_2$ and $\mathbb{A}_P(R) = \{I, J\}$, where $I = Rx, J = Ry$. We claim that $\mathbb{A}^*(R) = \{I, J\}$. If $K \in \mathbb{A}^*(R) \setminus \{I, J\}$, Lemma 3.5 implies that $|\mathbb{A}_P(R)| \geq 3$, a contradiction. So $\mathbb{A}G(R) \cong K_2$ and hence by Proposition 3.4, either $R \cong F_1 \times F_2$, where $F_1, F_2$ are two fields or $(R, \mathcal{M})$ is a local ring with $\mathbb{A}_P(R) = \{\mathcal{M}^2, \mathcal{M}\}$.

(3) It is clear by Proposition 3.4. □

**Corollary 3.7.** Let $R$ be a reduced ring, then $\mathbb{A}G_P(R) \cong K_2$ if and only if $R \cong F_1 \times F_2$, where $F_1, F_2$ are two fields.

*Proof.* It is clear by Proposition 3.6. □

In [5] and [13] the authors studied diameter and girth of annihilating-ideal graph. Also, in [13], they study the interplay between the diameter of annihilating-ideal graphs and the diameter of zero-divisor graphs, and characterize rings $R$, when $gr(\mathbb{A}G(R)) \geq 4$. We now show that $diam(\mathbb{A}G_P(R)) \leq diam(\mathbb{A}G(R))$ and $gr(\mathbb{A}G(R)) = gr(\mathbb{A}G_P(R))$.

**Proposition 3.8.** Let $R$ be a ring. Then the following statements hold.

1. $diam(\mathbb{A}G_P(R)) \leq diam(\mathbb{A}G(R))$. 
(2) \(gr(\mathbb{A}G(R)) = gr(\mathbb{A}G_P(R)).\)

\[\mathcal{E}\]

**Proof.** (1) For a ring \(R, \text{[12, Theorem 2.1]}\) implies that \(0 \leq \text{diam}(\mathbb{A}G(R)) \leq 3.\) Note that \(\text{diam}(\mathbb{A}G(R)) = 0,\) if and only if, \(\mathbb{A}G(R) \cong K_1,\) if and only if, \(\mathbb{A}G_P(R) \cong K_1,\) if and only if, \(\text{diam}(\mathbb{A}G_P(R)) = 0.\) Moreover, by Proposition 3.4, we can conclude that \(\text{diam}(\mathbb{A}G(R)) = 1\) if and only if \(\mathbb{A}G_P(R) \cong K_1.\) Now assume that \(\text{diam}(\mathbb{A}G(R)) = 2.\) It is clear that \(\text{diam}(\mathbb{A}G_P(R)) \geq 2.\) Let \(Ra \) and \(Rb \in \mathbb{A}_P(R)\) such that \((Ra)(Rb) \neq (0).\) Since \(Ra, Rb \in \mathbb{A}^*(R),\) there exists an ideal \(I\) such that

\[
Ra \quad I \quad Rb
\]

is a path in \(\mathbb{A}G(R).\) Let \(0 \neq c \in I.\) Then

\[
Ra \quad Rc \quad Rb
\]

is a path in \(\mathbb{A}G_P(R).\) Therefore, \(\text{diam}(\mathbb{A}G(R)) = 2 = \text{diam}(\mathbb{A}G_P(R)).\) Now assume that \(\text{diam}(\mathbb{A}G(R)) = 3,\) since \(\text{diam}(\mathbb{A}G_P(R)) \leq 3\) (see Proposition 3.3), we can conclude that in every cases \(\text{diam}(\mathbb{A}G_P(R)) \leq \text{diam}(\mathbb{A}G(R)).\)

(2) First note that \(\mathbb{A}G_P(R)\) is an induced subgraph of \(\mathbb{A}G(R),\) thus \(gr(\mathbb{A}G(R)) \leq gr(\mathbb{A}G_P(R)).\) Also, by \(\text{[12, Theorem 2.1]},\) we have \(gr(\mathbb{A}G(R)) = 3, 4\) or \(\infty.\)

Assume that \(gr(\mathbb{A}G(R)) = 3.\) Then there exist distinct ideals \(I_1, I_2\) and \(I_3\) such that

\[
I_3 \quad I_1 \quad I_2
\]

is a cycle in \(\mathbb{A}G(R).\) By Lemma 3.5, there exist distinct non-zero principal ideals \(Ra_1 \subseteq I_1, Ra_2 \subseteq I_2\) and \(Ra_3 \subseteq I_3.\) Then

\[
Ra_2 \quad Ra_3 \quad Ra_1
\]

is a cycle in \(\mathbb{A}G_P(R).\) Therefore, \(gr(\mathbb{A}G(R)) = gr(\mathbb{A}G_P(R)) = 3.\)

Assume that \(gr(\mathbb{A}G(R)) = 4.\) Then there exist distinct ideals \(I_1, I_2, I_3\) and \(I_4\) such that
is a cycle in $AG(R)$. Since $I_1$ and $I_4$ are distinct ideals, without loss of generality, assume that there exists $a_4 \in I_4 \setminus I_1$. By Lemma 3.5, we have three distinct principal ideals $Ra_1, Ra_2$ and $Ra_3$ such that $Ra_i \subseteq I_i$, for $i = 1, 2, 3$. If $Ra_4 \in \{Ra_1, Ra_2, Ra_3\}$, then

is a cycle in $AG_P(R)$, which is a contradiction since $gr(AG_P(R)) \geq gr(AG(R)) = 4$. So we may assume that $Ra_4 \notin \{Ra_1, Ra_2, Ra_3\}$, then we have the following cycle in $AG_P(R)$

and so $gr(AG_P(R)) = gr(AG(R)) = 4$. Also, if $gr(AG(R)) = \infty$, then since $gr(AG_P(R)) \geq gr(AG(R)) = \infty$, we must have $gr(AG_P(R)) = gr(AG(R)) = \infty$.\[\]

Note that if $AG(R)$ is a bipartite graph, then $gr(AG(R)) \geq 4$. Also, In [5], there is a characterization of rings $R$, when $gr(AG(R)) \geq 4$. We will show that $AG(R)$ is a bipartite graph if and only if $AG_P(R)$ is bipartite. First we need the following lemma.

**Lemma 3.9.** Let $G$ be a graph. Then $G$ is a bipartite graph if and only if it contains no odd cycles.

**Proof.** See [9, Theorem 3.4].\]

**Proposition 3.10.** Let $R$ be a ring. Then $AG_P(R)$ is a bipartite graph if and only if $AG(R)$ is a bipartite graph.\]

**Proof.** If $AG(R)$ is a bipartite graph, then since $AG_P(R)$ is an induced subgraph of $AG(R)$, we can conclude that $AG_P(R)$ is a bipartite graph. Now assume that $AG_P(R)$ is a bipartite graph. Suppose on the contrary that $AG(R)$ is not bipartite, so by [2, Corollary 25], $AG(R)$ contains a triangle and
thus \( gr(\mathcal{A}\mathcal{G}(R)) = 3 \). Theorem 3.8 implies that \( gr(\mathcal{A}\mathcal{G}_P(R)) = 3 \), so \( \mathcal{A}\mathcal{G}_P(R) \) contains an odd cycle. Therefore, it is not bipartite graph (see Lemma 3.9), a contradiction.\( \square \)

**Corollary 3.11.** Let \( R \) be an Artinian ring. If \( \mathcal{A}\mathcal{G}_P(R) \) is a bipartite graph, then \( R \) is one of the following three types of rings:

(a) \( R \cong F_1 \times F_2 \), where \( F_1 \) and \( F_2 \) are two fields.
(b) \( R \) is a local ring with non-zero maximal ideal \( M \) with \( M^4 = (0) \).
(c) \( R \cong F \times S \), where \( F \) is a field and \( S \) is a ring with a unique non-trivial ideal.

The converse is also true when in case (b), \( M^3 \neq (0) \) and \( \mathcal{A}^*(R) = \{M, M^2, M^3\} \).

**Proof.** \((\Rightarrow)\) Assume that \( \mathcal{A}\mathcal{G}_P(R) \) is a bipartite graph. By Proposition 3.10, \( \mathcal{A}\mathcal{G}(R) \) is a bipartite graph, so by \([3, \text{Theorem 3}]\), either \( \mathcal{A}\mathcal{G}(R) \) is a star graph or \( \mathcal{A}\mathcal{G}(R) \cong P_4 \). Moreover \( \mathcal{A}\mathcal{G}(R) \cong P_4 \), if and only if \( R \cong F \times S \), where \( F \) is a field and \( S \) is a ring with a unique non-trivial ideal. If \( \mathcal{A}\mathcal{G}(R) \) is a star graph, \([12, \text{Lemma 2.5}]\) implies that \( R \cong F_1 \times F_2 \), where \( F_1 \) and \( F_2 \) are two fields or \( R \) is a local ring with non-zero maximal ideal \( M \) with \( M^4 = (0) \).

\((\Leftarrow)\) It is clear that \( \mathcal{A}\mathcal{G}(R) \cong P_2, P_3 \) or \( P_4 \). Since \( \mathcal{A}\mathcal{G}_P(R) \) is an induced subgraph of \( \mathcal{A}\mathcal{G}(R) \), \( \mathcal{A}\mathcal{G}_P(R) \) is a bipartite graph.\( \square \)

In the following proposition we characterize all rings \( R \) for which the principal ideal graph \( \mathcal{A}\mathcal{G}_P(R) \) has a vertex which is adjacent to every other vertex of \( \mathcal{A}\mathcal{G}_P(R) \).

**Proposition 3.12.** Let \( R \) be a ring. Then the following statements are equivalent.

1. There is a vertex of \( \mathcal{A}\mathcal{G}_P(R) \) which is adjacent to every other vertex of \( \mathcal{A}\mathcal{G}_P(R) \).
2. There is a vertex of \( \mathcal{A}\mathcal{G}(R) \) which is adjacent to every other vertex of \( \mathcal{A}\mathcal{G}(R) \).
3. Either \( R = F \times D \), where \( F \) is a field and \( D \) is an integral domain, or \( Z(R) = \text{Ann}(x) \) for some \( 0 \neq x \in R \).

**Proof.** \((1) \Rightarrow (2)\) Assume that \( I \in \mathcal{A}P(R) \) is a vertex which is adjacent to every other vertex in \( \mathcal{A}\mathcal{G}_P(R) \). Let \( J \in \mathcal{A}^*(R) \setminus \mathcal{A}P(R) \), we claim that \( IJ = (0) \). Since \( I \) and \( J \) are distinct vertex of \( \mathcal{A}\mathcal{G}(R) \), \( I \not\subset J \) or \( J \not\subset I \). First assume that \( I \not\subset J \). Thus for each \( y \in J \), \( I \neq Ry \). Therefore, \( I(Ry) = (0) \), which implies that \( IJ = (0) \). Now assume that \( J \not\subset I \). Let \( 0 \neq t \in J \). We consider the following two cases:

1. **Case 1.** \( t \in J \setminus (J \cap I) \). So for each \( x \in I \), \( Rx \neq Rt \). Therefore, \( I \neq Rt \). Then we have \( I(Rt) = (0) \).
2. **Case 2.** \( t \in (J \cap I) \). Let \( z \in J \setminus (J \cap I) \). Then \( t + z \in J \setminus I \) and so \( I \neq R(t + z) \). Therefore, \( IR(t + z) = (0) \) and we have \( (t + z)I = (0) \). Since \( zI = (0) \), \( tI = (0) \). Thus \( (Rt)I = (0) \).

Since in both cases we have \( (Rt)I = (0) \), we can conclude that \( IJ = (0) \).
(2) ⇒ (3) It follows from [12, Theorem 2.2].

(3) ⇒ (1) If \( R = F \times D \), where \( F \) is a field and \( D \) is an integral domain, then \( F \times (0) \in \mathcal{A}_P(R) \) is adjacent to every other vertex. If \( Z(R) = \text{Ann}(x) \) for some non-zero \( x \in R \), then \( Rx \in \mathcal{A}_P(R) \) is adjacent to every other vertex. □

**Proposition 3.13.** Let \( R \) be a ring. Then \( \mathcal{A}_G_P(R) \) is a star graph if and only if \( \mathcal{A}_G(R) \) is star. □

**Proof.** (⇐) If \( \mathcal{A}_G(R) \) is a star graph, then since \( \mathcal{A}_G_P(R) \) is an induced subgraph of \( \mathcal{A}_G(R) \), we can conclude that \( \mathcal{A}_G_P(R) \) is a star graph.

(⇒) Assume that \( \mathcal{A}_G_P(R) \) is a star graph and \( Rx \in \mathcal{A}_P(R) \) is adjacent to every other vertex of \( \mathcal{A}_G_P(R) \). Let \( I \in \mathcal{A}^*(R) \setminus \mathcal{A}_P(R) \). We claim that \( I \) is only adjacent to \( Rx \). By the same argument in Theorem 3.12, \( I(Rx) = (0) \). If \( J \in \mathcal{A}^*(R) \setminus \{Rx\} \) such that \( IJ = (0) \), then \( J \bullet I \bullet Rx \)

is a triangle in \( \mathcal{A}_G(R) \) and so \( gr(\mathcal{A}_G(R)) = 3 \). Therefore, By Proposition 3.8, \( gr(\mathcal{A}_G_P(R)) = 3 \), it is impossible. Then \( \mathcal{A}_G(R) \) is a star graph. □

The following corollary classifies all rings with star principal ideal graph.

**Corollary 3.14.** Let \( R \) be a ring. Then

1. If \( R \) is reduced, then \( \mathcal{A}_G_P(R) \) is a star graph if and only if \( R \cong F \times D \), where \( F \) is a field and \( D \) is an integral domain.
2. If \( R \) is non-reduced, then \( \mathcal{A}_G(R) \) is a star graph if and only if \( \mathcal{A}_G_P(R) \) is a bipartite graph and \( Z(R) = \text{Ann}(x) \), for some \( x \in R \).


We say that the annihilating-ideal graph \( \mathcal{A}_G(R) \) has ACC (respectively, DCC) on its vertices if \( R \) has ACC (respectively, DCC) on \( \mathcal{A}^*(R) \). Also, we say that the principal ideal graph \( \mathcal{A}_G_P(R) \) has ACC (respectively, DCC) on its vertices if \( R \) has ACC (respectively, DCC) on \( \mathcal{A}_P(R) \). In [12, Theorem 1.1], it is shown that \( \mathcal{A}_G(R) \) has ACC (respectively, DCC) on its vertices if and only if \( R \) is a Noetherian (respectively, an Artinian) ring. One may naturally ask, is \( R \) Noetherian when \( \mathcal{A}_G_P(R) \) has ACC (respectively, DCC) on its vertices? The answer is no! The following example gives a non-Noetherian ring \( R \) for which \( \mathcal{A}_G_P(R) \) has ACC (respectively, DCC) on vertices.
Example 3.15. Let \( R = \mathbb{Z}_2[\{X_i \mid i \in \mathbb{N}\}]/\langle \{X_iX_j \mid i, j \in \mathbb{N}\} \rangle \). Then \( R \) is a non-Noetherian local ring with \( \mathcal{M} = \bigoplus_{i=1}^{\infty} Rx_i \) (where \( x_i = X_i + \langle \{X_iX_j \mid i, j \in \mathbb{N}\} \rangle \)). It is clear that \( \mathcal{M}^2 = \{0\} \) and \( \mathcal{M} \) is a \( R \)-module. Since every principal ideal of \( R \) is a simple \( R \)-module, we can conclude that \( R \) has ACC (respectively, DCC) on \( \mathcal{A}_P(R) \) but \( R \) is not a Noetherian ring. E

Let \( R \) be a ring. Then every non-zero proper principal ideal of \( R \) is a vertex of \( \mathcal{A}_G(R) \), i.e., \( \mathcal{A}_P(R) = \mathbb{P}(R) \) if and only if every element in \( R \) is a unit or zero-divisor (see [12, Proposition 1.13]). The following example gives a Noetherian ring such that \( \mathbb{A}_G(R) \) is an induced subgraph of \( \mathcal{A}_G(R) \), but if \( R \) is a non-domain ring, then by Proposition 3.3, \( \mathcal{A}_G(R) \) is a connected graph.

Example 3.16. Let \( R = F \times \mathbb{Z} \) where \( F \) is a field. Assume that \( X = \{(0) \times n\mathbb{Z}\} \), where \( n \in \mathbb{N} \). Then \( F \times (0) \) is adjacent to each vertex in \( X \), so \( \mathbb{A}_G(R) \) is an infinity star graph and \( F \times 2\mathbb{Z} \) is a principal ideal in \( \mathbb{P}(R) \) which is not belong to \( \mathcal{A}_P(R) \). E

4. THE PRINCIPAL IDEAL GRAPH AND THE SPECTRUM GRAPH

In this section, we have a comparison between the principal ideal graph and the spectrum graph. We begin with the following remark.

Remark 4.1. By [12, Example 1.9], there exists a local zero-dimensional ring \( R \) such that \( \mathcal{A}^*(R) \neq \emptyset \) and \( \mathcal{A}_G_s(R) \) is an empty graph, but \( \mathcal{A}_G(R) \) is non empty. Also, [18, Example 2.7] gives a non-connected spectrum graph of a local ring, but if \( R \) is a non-domain ring, then by Proposition 3.3, \( \mathbb{A}_G(R) \) is a connected graph.

Lemma 4.2. ([3, Theorem 10]) Let \( R \) be a ring such that \( |\text{Min}(R)| = 1 \). If \( \mathcal{A}_G(R) \) is a triangle-free graph, then \( \mathcal{A}_G(R) \) is a star graph.

Proposition 4.3. Let \( R \) be a non-domain ring. Then

1. If \( \mathcal{A}_G(P(R)) \) is a finite graph, then \( \mathcal{A}_G_s(R) \) is a finite graph.
2. If \( \mathcal{A}_G(P(R)) \) is a complete graph, then \( \mathcal{A}_G_s(R) \cong K_1 \) or \( \mathcal{A}_G_s(R) \cong K_2 \).
3. Assume that \( R \) is an Artinian ring such that \( \mathcal{A}_G(R) \) is a triangle-free graph, then \( \mathcal{A}_G(P(R)) \) is a star graph if and only if \( \mathcal{A}_G_s(R) \) is a star graph.

Proof. (1) It is clear by Proposition 3.1.

(2) Assume that \( \mathcal{A}_G(P(R)) \) is a complete graph. Then by Proposition 3.4, \( \mathcal{A}_G(R) \) is a complete graph. Since \( \mathcal{A}_G_s(R) \) is an induced subgraph of \( \mathcal{A}_G(R) \), \( \mathcal{A}_G_s(R) \) is a complete graph, too. Therefore, by [18, Proposition 2.5], \( \mathcal{A}_G_s(R) \cong K_1 \) or \( \mathcal{A}_G_s(R) \cong K_2 \).

(3) \( (\Rightarrow) \) Suppose that \( \mathcal{A}_G(P(R)) \) is a star graph. Thus, \( \mathcal{A}_G(R) \) is a star graph (see Proposition 3.13). Since \( R \) is an Artinian ring, by [12, Lemma 2.5], either \( R = F_1 \times F_2 \), where \( F_1, F_2 \) are fields, or \( R \) is a local ring. Therefore, either \( \mathcal{A}_G_s(R) \cong K_1 \) or \( \mathcal{A}_G_s(R) \cong K_2 \).
(\Leftrightarrow) Assume that \(\mathcal{A}\mathcal{G}_s(R)\) is a star graph. Since \(R\) is an Artinian ring, by [18, Theorem 3.10], either \(R \cong R_1 \times R_2\) or \(R\) is a local ring. If \(R \cong R_1 \times R_2\), \(\mathcal{A}\mathcal{G}_P(R) \cong K_2\). Let \(R\) be a local ring. Since \(R\) is an Artinian ring, \(|\text{Min}(R)| = 1\). So by Lemma 4.2, \(\mathcal{A}\mathcal{G}(R)\) is a star graph and hence \(\mathcal{A}\mathcal{G}_P(R)\) is star, too. \(\square\)

The following example shows that the converse of Proposition 4.3 (1) and (2) does not hold.

**Example 4.4.** Let \(R = F \times \mathbb{Z}\), where \(F\) is a field. Then \(\mathcal{A}_s(R) = \{(0) \times \mathbb{Z}, F \times (0)\}\) and hence \(\mathcal{A}\mathcal{G}_s(R) \cong K_2\). But \(\mathcal{A}\mathcal{G}_P(R)\) is an infinite star graph. \(\mathcal{E}\)

**Corollary 4.5.** Let \(R\) be an Artinian ring such that \(\mathcal{A}\mathcal{G}_P(R)\) is a complete graph. Then, either \(R\) is a local ring or \(R \cong R_1 \times R_2\), where \(R_1, R_2\) are fields.

**Proof.** It is clear with Proposition 4.3 and [18, Theorem 3.10]. \(\square\)

In the next proposition, we have a compression between diameter and girth of spectrum graph and principal ideal graph.

**Proposition 4.6.** Let \(R\) be a non-domain ring. Then

1. \(\text{gr}(\mathcal{A}\mathcal{G}_P(R)) \leq \text{gr}(\mathcal{A}\mathcal{G}_s(R))\).
2. If \(\mathcal{A}\mathcal{G}_s(R)\) is a connected graph, then \(\text{diam}(\mathcal{A}\mathcal{G}_s(R)) \leq \text{diam}(\mathcal{A}\mathcal{G}_P(R))\).
3. Assume that \(R\) is a Noetherian ring and \(\text{diam}(\mathcal{A}\mathcal{G}_s(R)) = 2\), then \(\text{diam}(\mathcal{A}\mathcal{G}_P(R)) = 2\).
4. Assume that \(R\) is a Noetherian ring such that \(R\) is not Artinian and \(\text{diam}(\mathcal{A}\mathcal{G}_s(R)) = 1\), then \(\text{diam}(\mathcal{A}\mathcal{G}_P(R)) = 2\).

**Proof.** (1) It is clear since for every ring \(R\), \(\text{gr}(\mathcal{A}\mathcal{G}_s(R)) = \infty\) (see [18, Corollary 2.4]).

(2) Assume that \(\mathcal{A}\mathcal{G}_s(R)\) is a connected graph. So by [18, Proposition 2.5], \(\text{diam}(\mathcal{A}\mathcal{G}_s(R)) \in \{0, 1, 2\}\). If \(\text{diam}(\mathcal{A}\mathcal{G}_s(R)) = 0\) or 1, then there is nothing to prove. Let \(\text{diam}(\mathcal{A}\mathcal{G}_s(R)) = 2\), so there are \(P_1, P_2 \in \mathcal{A}_s(R)\) such that \(P_1P_2 \neq (0)\). Since \(P_1, P_2 \in \mathcal{A}\mathcal{G}(R)\), \(\text{diam}(\mathcal{A}\mathcal{G}(R)) \geq 2\) and hence \(\text{diam}(\mathcal{A}\mathcal{G}_P(R)) \geq 2 = \text{diam}(\mathcal{A}\mathcal{G}_s(R))\).

(3) Suppose that \(\text{diam}(\mathcal{A}\mathcal{G}_s(R)) = 2\), by [18, Proposition 2.3], \(\mathcal{A}\mathcal{G}_s(R) \cong K_{1,\infty}\). Thus, there is a vertex in \(\mathcal{A}\mathcal{G}_s(R)\) which is adjacent to every other vertex in \(\mathcal{A}\mathcal{G}_s(R)\). Since \(|\mathcal{A}_s(R)| > 2\), by [18, Proposition 3.2], there is a vertex in \(\mathcal{A}\mathcal{G}(R)\) which is adjacent to every other vertex in \(\mathcal{A}\mathcal{G}(R)\), so \(\text{diam}(\mathcal{A}\mathcal{G}(R)) \leq 2\). Therefore, \(\text{diam}(\mathcal{A}\mathcal{G}_P(R)) \leq 2\). By (2), \(2 = \text{diam}(\mathcal{A}\mathcal{G}_s(R)) \leq \text{diam}(\mathcal{A}\mathcal{G}_P(R))\), so \(\text{diam}(\mathcal{A}\mathcal{G}_P(R)) = 2\).

(4) Since \(\text{diam}(\mathcal{A}\mathcal{G}_s(R)) = 1\), \(\mathcal{A}\mathcal{G}_s(R) \cong K_2\) (see [18, Proposition 2.3]). Therefore, by [18, Proposition 3.6], \(\mathcal{A}\mathcal{G}(R)\) is a complete bipartite graph, so \(\text{diam}(\mathcal{A}\mathcal{G}_P(R)) \leq 2\), by Proposition 3.8 (1). Note that \(R\) is not Artinian. So by [12, Theorem 2.10], \(\mathcal{A}\mathcal{G}_s(R)\) is a proper subgraph of \(\mathcal{A}\mathcal{G}(R)\) and hence
|\mathbb{A}^*(R)| \geq 3. If \text{diam}(\mathbb{A}_P(R)) = 0, then by Proposition 3.6 (1), |\mathbb{A}^*(R)| = 1, a contradiction. Now assume that \text{diam}(\mathbb{A}_P(R)) = 1, so \mathbb{A}_P(R) \cong K_n , where n \geq 2. If n = 2, then Proposition 3.6 (2) implies that |\mathbb{A}^*(R)| = 2, a contradiction. For case n > 2, we have \mathbb{A}_G(R) \cong K_n, where n > 2 (see Proposition 3.4). Thus \mathbb{A}_G(R) is not a complete bipartite graph, a contradiction. Therefore \text{diam}(\mathbb{A}_P(R)) \neq 0,1 and hence \text{diam}(\mathbb{A}_P(R)) = 2. \qed

**Proposition 4.7.** Let R be an Artinian local ring such that \mathbb{A}_s(R) is a subgraph of \mathbb{A}_P(R). Then \mathbb{A}_P(R) is a finite graph.

**Proof.** Assume that (R,M) is an Artinian local ring. It is clear that \mathbb{A}_s(R) \cong K_1, where \mathbb{A}_s(R) = \{M\}. Since \mathbb{A}_s(R) is a subgraph of \mathbb{A}_P(R), J(R) = M = Rx, where 0 \neq x \in R. Since R is an Artinian ring, there is k \in \mathbb{N}, such that x^k = 0 and x^{k-1} \neq 0. Now, by [15, Theorem 9], (0) = Rx^k \subset Rx^{k-1} \subset \cdots \subset Rx \subset R are the only ideals of R. Since R is an Artinian ring, \mathbb{A}_P(R) = \{J(R), Rx^2, \ldots , Rx^{k-1}\}. Therefore, \mathbb{A}_P(R) is a finite graph. \qed

The following proposition characterizes all Artinian rings R for which \mathbb{A}_s(R) is a subgraph of \mathbb{A}_P(R).

**Proposition 4.8.** Let R be an Artinian ring. Then \mathbb{A}_s(R) is a subgraph of \mathbb{A}_P(R) if and only if R is a principal ideal ring.

**Proof.** (\Rightarrow) It is clear.

(\Leftarrow) Assume that \mathbb{A}_s(R) is a subgraph of \mathbb{A}_P(R). Since R is an Artinian ring and \mathbb{A}_s(R) is a subgraph of \mathbb{A}_P(R), by [12, Proposition 1.3], Spec(R) = \mathbb{A}_s(R) \subseteq \mathbb{A}_P(R). Let X = \{I : I \in \mathbb{I}(R) \setminus \mathbb{P}(R)\}. We claim that X = \emptyset. Suppose on the contrary that X \neq \emptyset. Since R is a Noetherian ring, we may assume that J is a Maximal element of X. It is clear that J \notin \text{Spec}(R), so there are x, y \in R such that xy \in J but x, y \notin J. Therefore, J \subsetneq J + Rx. Since J is a maximal element of X, J + Rx \in \mathbb{P}(R). So there is z \in J + Rx such that J + Rx = Rz. Let Y = \{r \in R : rz \in J\}. It is clear that J \subseteq Y and z \in J + Rx. Since yz \in yJ + R(xy) \subseteq J, y \in Y and hence J \subsetneq Y. Therefore Y \in \mathbb{P}(R). Let Y = Rt, where t \in Y. Now we claim that J = Rzt. By definition, zY = zR(t) = R(zt) \subseteq J. On the other hand, if s \in J, then s \in J + Rx = Rz, so s = kz for some k, where clearly k \in Y. Thus s \in zY = Rzt. Therefore J = Rzt \in \mathbb{P}(R), a contradiction. We can conclude that X = \emptyset as we claimed and we have R is a principal ideal ring. \qed

In view of the above proposition, one may naturally ask, is R a principal ideal ring, when \mathbb{A}_s(R) is a principal ideal graph? The answer is no! The following example shows that the Artinian hypothesis is necessary in Proposition 4.8.
Example 4.9. Let $F$ be a field and

$$R = F[[X,Y]]/(XY).$$

Then $R$ is a local ring with maximal ideal $M = Rx + Ry$, where $x = X + (XY)$ and $y = Y + (XY)$. Note that $\mathbb{A}_s(R) = \{Rx, Ry\} \subseteq \mathbb{A}_P(R)$, but $R$ is not a principal ideal ring. E

Proposition 4.10. Let $R$ be a non-domain reduced ring. Then every vertex of $\mathbb{A}_P(R)$ has finite degree and $\mathbb{A}_s(R)$ is a connected graph if and only if $R \cong F_1 \times F_2$, where $F_1$, $F_2$ are fields.

Proof. $(\Leftarrow)$ It is clear.

$(\Rightarrow)$ Since every vertex of $\mathbb{A}_P(R)$ has finite degree, by Corollary 3.2, $R \cong F_1 \times F_2 \times \cdots \times F_n$, where $n \geq 2$ and $F_i$’s are field. If $n > 2$. Then $\mathbb{A}_s(R)$ is not a connected graph, thus $n = 2$ and $R \cong F_1 \times F_2$. □

We conclude this paper with the following proposition.

Proposition 4.11. Let $R$ be a non-local Artinian ring. Then $\mathbb{A}_P(R)$ is a complete bipartite graph if and only if $\mathbb{A}_s(R) \cong K_2$. □$

Proof. $(\Leftarrow)$ Assume that $\mathbb{A}_s(R) \cong K_2$. Since $R$ is an Artinian ring, [18, Theorem 3.10] implies that $R \cong F_1 \times F_2$, so $\mathbb{A}_P(R)$ is a complete bipartite graph.

$(\Rightarrow)$ Suppose that $\mathbb{A}_P(R)$ is a complete bipartite graph with two parts $X$ and $Y$. Since $R$ is an Artinian ring which is not local, $R \cong R_1 \times R_2$, where $R_1$, $R_2$ are two non-zero rings. We claim that $R_1$ is a field. Suppose on the contrary that $R_1$ is not a field, so there exists non-zero proper ideal $J$ of $R_1$. Let $a \in J$ and $I = R_1a$. Since $R_1$ ia an Artinian ring, [12, Proposition 1.3] implies that $J \in \mathbb{A}^*(R_1)$ and hence $I \in \mathbb{A}_P(R_1)$. Therefore $I \times R_2 = R(a,1)$ and $I \times R_2, (0) \times R_2 \in \mathbb{A}_P(R)$. Since $(I \times R_2)((0) \times R_2) \neq (0) \times (0)$, both $I \times R_2$ and $(0) \times R_2$ are in $X$ or $Y$. But $R_1 \times (0)$ is adjacent to $(0) \times R_2$ and is not adjacent to $I \times R_2$, a contradiction. Thus, $R_1$ is a field and similarly, $R_2$ is a field, too. Consequently $\mathbb{A}_s(R) \cong K_2$. □

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