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# THE PRINCIPAL IDEAL SUBGRAPH OF THE ANNIHILATING-IDEAL GRAPH OF COMMUTATIVE RINGS

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Abstract. Let $R$ be a commutative ring with identity and $\mathbb{A}(R)$ be the set of ideals of $R$ with non-zero annihilators. In this paper, we first introduce and investigate the principal ideal subgraph of the annihilating-ideal graph of $R$, denoted by $\mathbb{AG}_P(R)$. It is a (undirected) graph with vertices $\mathbb{A}_P(R) = \mathbb{A}(R) \cap \mathbb{P}(R) \setminus \{(0)\}$, where $\mathbb{P}(R)$ is the set of proper principal ideals of $R$ and two distinct vertices $I$ and $J$ are adjacent if and only if $IJ = (0)$. Then, we study some basic properties of $\mathbb{AG}_P(R)$. For instance, we characterize rings for which $\mathbb{AG}_P(R)$ is finite graph, complete graph, bipartite graph or star graph. Also, we study diameter and girth of $\mathbb{AG}_P(R)$. Finally, we compare the principal ideal subgraph $\mathbb{AG}_P(R)$ and spectrum subgraph $\mathbb{AG}_s(R)$.

## 1. Introduction

In recent years, assigning graphs to rings has played an important role in the study of structures of rings (see for example [7, 8, 10-14]). Let $R$ be a commutative ring. We call an ideal $I$ of $R$ is an annihilating-ideal, if there exists a non-zero ideal $J$ of $R$ such that $IJ = (0)$ and use the notation $\mathbb{A}(R)$

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for the set of annihilating-ideals of $R$. By the annihilating-ideal graph $\mathbb{AG}(R)$ of $R$ we mean the graph with vertices $\mathbb{A}^*(R) = \mathbb{A}(R) \diagdown \{(0)\}$ such that there is an (undirected) edge between vertices $I$ and $J$ if and only if $I \neq J$ and $IJ = (0)$. The concept of the annihilating-ideal graph of a commutative ring was first introduced by Behboodi and Rakei in [12, 13]. Recently, this notation of the annihilating-ideal graph has been extensively studied by various authors (see for instance, [1-6, 16, 18] and many others). In [18], Taheri, Behboodi and Tehranian, introduce and investigate the spectrum graph of the annihilating-ideal graph of a commutative ring, denoted by $\mathbb{AG}_s(R)$, that is, a graph whose vertices are all non-zero prime ideals of $R$ with non-zero annihilators, denoted by $\mathbb{A}_s(R)$ and two distinct vertices $P_1$, $P_2$ are adjacent if and only if $P_1P_2 = (0)$. In this paper, we introduce and study the principal ideal graph of a commutative ring $R$, denoted by $\mathbb{AG}_P(R)$, that is, the graph whose vertices are all principal ideals in $\mathbb{A}^*(R)$ and two distinct vertices $I$, $J$ are adjacent if and only if $IJ = (0)$. We denote by $\mathbb{A}_P(R)$ the vertex set of $\mathbb{AG}_P(R)$. It is clear that if $R$ is a P.I.R. which is not integral domain, then $\mathbb{AG}_P(R) = \mathbb{AG}(R)$. First we study some basic properties of $\mathbb{AG}_P(R)$ and then we compare the principal ideal subgraph with the spectrum subgraph.

## 2. DEFINITION AND PRELIMINARIES

Throughout this paper, all rings are commutative with identity and all modules are unitary. For a ring $R$, we denote by $\mathrm{Spec}(R)$ the set of prime ideals, $Z(R)$ the set of zero-divisors, $\mathbb{I}(R)$ the set of non-zero proper ideals and $\mathbb{P}(R)$ the set of proper principal ideals of $R$. The jacobson radical and the set of minimal prime ideals of $R$ are denoted by $\mathrm{J}(R)$ and $\mathrm{Min}(R)$, respectively. Let $X$ be an element or a subset of a ring $R$. The annihilator of $X$ is the ideal $\mathrm{Ann}(X) = \{a \in R \mid aX = 0\}$.

let $G$ be any graph. We denote the vertex set of $G$ by $\mathrm{V}(G)$. Sometimes, two graphs $G$ and $H$ have exactly the same form, in the sense that there is a one-to-one correspondence between their vertex sets that preserves edges. In such a case, we say that the two graphs $G$ and $H$ are isomorphic and we write $G \cong H$. We say that a subgraph $H$ is an induced subgraph of $G$ if $H$ is isomorphic to a graph whose vertex set $V_1$ is a subset of the vertex set $V$ of $G$ and whose edge set $E_1$ consists of all the edges of $G$ with both end vertices in $V_1$. For every subset $A$ of $\mathbb{A}(R)$ we denote the induced subgraph of $\mathbb{AG}(R)$ with vertex set $A$ by $\mathbb{AG}_A(R)$. The graph $G$ is called connected if there is a path between every two distinct vertices. For distinct vertices $P$, $Q$ of $G$, let $\mathrm{d}(P,Q)$ be the length of the shortest path from $P$ to $Q$ and, if there is no such a path, we define $\mathrm{d}(P,Q) = \infty$. The diameter of $G$ is $\mathrm{diam}(G) = \sup\{\mathrm{d}(P,Q) : P$ and $Q$ are distinct vertices of $G\}$. The girth of $G$, denoted by $\mathrm{gr}(G)$, is defined as the length of the shortest cycle in $G$ and $\mathrm{gr}(G) = \infty$, if $G$ contains no cycles. A complete graph is a graph in which any two distinct vertices are adjacent. A complete graph with $n$ vertices denoted by $K_n$. A bipartite graph is a graph whose vertices can be divided into two disjoint sets $A$ and $B$ such that every edge connects a vertex in $A$ to one in $B$. A complete bipartite graph is a bipartite graph in which every vertex of one part is joined to every vertex of the other part. In this case, if $|A| = n$ and $|B| = m$, we denote the graph by $K_{n,m}$. If $|A| = 1$ or $|B| = 1$, then the graph is said to be a star graph. Ultimately, we denote by $P_n$ a path of order $n$.

## 3. COPARING THE PRINCIPAL IDEAL GRAPH AND THE ANNIHILATING-IDEAL GRAPH

Let $R$ be a non- domain commutative ring. In this section, we compare the features of $\mathbb{AG}_P(R)$ and $\mathbb{AG}(R)$ and express some properties of principal ideal graph. We begin with the following proposition, which characterize all rings $R$ with finite principal ideal graph.

**Proposition 3.1.** *Let $R$ be a ring. Then the following statements are equivalent.*

(1) *$\mathbb{AG}_P(R)$ is a finite graph.*
(2) *$\mathbb{AG}(R)$ is a finite graph, moreover $|\mathbb{A}^*(R)| \leqslant 2^{|\mathbb{A}_P(R)|}$.*
(3) *$R$ has only finitely many ideals.*
(4) *Every vertex of $\mathbb{AG}_P(R)$ has finite degree.*
(5) *Every vertex of $\mathbb{AG}(R)$ has finite degree.*

**Proof.** (1) $\Rightarrow$ (2)*: Assume that $\mathbb{AG}_P(R)$ is a finite graph. Therefore, $\mathbb{A}_P(R)$ is a finite set. For each $I \in \mathbb{A}^*(R)$, $I$ is generated by a subset of $\mathbb{A}_P(R)$. Since $\mathbb{A}_P(R)$ has finitely many subsets, we can conclude that $\mathbb{A}^*(R)$ is a finite set. Thus, $\mathbb{AG}(R)$ is a finite graph and $|\mathbb{A}^*(R)| \leqslant 2^{|\mathbb{A}_P(R)|}$.*

(2) $\Rightarrow$ (1) *and* (3) $\Rightarrow$ (4) *are clear.*

(2) $\Rightarrow$ (3) *and* (5) $\Rightarrow$ (2) *are follow from* [12, Theorem 1.4]*.*

(4) $\Rightarrow$ (5) *Suppose on the contrary that $I \in \mathbb{A}^*(R)$ has infinite degree in $\mathbb{AG}(R)$, so for each $a \in I$, $Ra \in \mathbb{A}_P(R)$ has infinite degree in $\mathbb{AG}_P(R)$, a contradiction.* □

*In the following result, we characterize reduced rings $R$ for which every vertex of principal ideal graph $\mathbb{AG}_P(R)$ has finite degree.*

**Corollary 3.2.** *Let $R$ be a reduced ring. Then every vertex of $\mathbb{AG}_P(R)$ has finite degree if and only if $R \cong F_1 \times F_2 \times \cdots \times F_n$ , where $F_i$'s are fields for each $i$, $1 \leqslant i \leqslant n$.*

*Proof.* ($\Leftarrow$) It is clear.

($\Rightarrow$) Assume that every vertex of $\mathbb{AG}_P(R)$ has finite degree. By Proposition 3.1, $R$ has only finitely many ideals and hence is an Artinian ring, so $R \cong F_1 \times F_2 \times \cdots \times F_n$ , where $F_i$'s are fields for each $i$, $1 \leqslant i \leqslant n$. □

*We now show that $\mathbb{AG}_P(R)$ is connected.*

**Proposition 3.3.** *For every ring $R$, the principal ideal subgraph $\mathbb{AG}_P(R)$ is connected and* $\operatorname{diam}(\mathbb{AG}_P(R)) \leqslant 3$.

*Proof.* Let $I$ and $J$ be distinct vertices of $\mathbb{AG}_P(R)$. Then there exist principal ideals $K$ and $L$ such that $IK = (0)$ and $JL = (0)$. If $KL = (0)$, then $J — L — K — I$ is a path with length 3 in $\mathbb{AG}_P(R)$. If $KL \neq (0)$, then there is a principal ideal $H \subseteq KL$ and $J — H — I$ is a path with length 2 in $\mathbb{AG}_P(R)$. We can conclude that $\mathbb{AG}_P(R)$ is connected and $\mathrm{diam}(\mathbb{AG}_P(R)) \leq 3$. □

*In the following proposition we show that $\mathbb{AG}_P(R)$ is a complete graph if and only if $\mathbb{AG}(R)$ is complete. Also, in the next proposition we characterize rings whose principal ideal subgraphs are complete graphs.*

**Proposition 3.4.** *Let $R$ be a ring. Then the following statements are equivalent.*

(1) $\mathbb{AG}_P(R)$ *is a complete graph.*
(2) $\mathbb{AG}(R)$ *is a complete graph.*
(3) *Either $\mathbb{AG}(R) \cong K_2$ or $Z(R)$ is the ideal of $R$ with $Z(R)^2 = (0)$. Moreover, in the first case, either $R \cong F_1 \times F_2$, where $F_1$ and $F_2$ are fields or $(R, \mathcal{M})$ is a local ring with exactly two non-trivial ideals $\mathcal{M}^2$ and $\mathcal{M}$.*

$\vec{\mathcal{E}}$

*Proof.* (1) $\Rightarrow$ (2) Let $I$ and $J$ be two distinct vertices of $\mathbb{AG}(R)$. We show that $IJ = (0)$. Without loss of generality we can assume $I \nsubseteq J$. Let $0 \neq x \in I$. We consider the following two cases:

**Case 1.** $x \in I \setminus (I \cap J)$. Then for each $y \in J$, $Rx$ and $Ry$ are two distinct vertices of $\mathbb{AG}_P(R)$ and hence $(Rx)(Ry) = (0)$. Therefore, $xJ = (0)$.

**Case 2.** $x \in (I \cap J)$. Let $a \in I \setminus (I \cap J)$, so $x + a \in I \setminus J$ and we can see for each $y \in J$, $Ry$ and $R(x + a)$ are two distinct vertices of $\mathbb{AG}_P(R)$, so $R(x + a)Ry = (0)$. Note that $y$ is an arbitrary element of $J$, thus we have $R(x + a)J = (0)$, and so $(x + a)J = (0)$. Since $aJ = (0)$, $xJ = (0)$.
Therefore, in each cases, for every $x \in I$, we have $xJ = (0)$, which implies that $IJ = (0)$. We can conclude that $\mathbb{AG}(R)$ is a complete graph.

(2) $\Rightarrow$ (1) Let $\mathbb{AG}(R)$ is a complete graph. Since $\mathbb{AG}_P(R)$ is an induced subgraph of $\mathbb{AG}(R)$, we have $\mathbb{AG}_P(R)$ is a complete graph.

(2) $\Leftrightarrow$ (3) It follows from [1, Theorem 3].□

*We need the following lemma.*

**Lemma 3.5.** *Let $I_1, I_2$ and $I_3$ be distinct ideals of $R$. Then there exist distinct non-zero principal ideals $Ra_1, Ra_2$ and $Ra_3$ such that $Ra_i \subseteq I_i$, for $i = 1, 2, 3$.*

*Proof.* Let $\mathbf{X} = \{I_1, I_2, I_3\}$. Without loss of generality assume that $I_1$ is a maximal element of $\mathbf{X}$. If $I_1 \subseteq I_2 \cup I_3$, then by Prime Avoidance Theorem [17, Theorem 3.61], $I_1 \subseteq I_2$ or $I_1 \subseteq I_3$, a contradiction to maximality of $I_1$. Therefore, there exists $0 \neq a_1 \in I_1 \setminus (I_2 \cup I_3)$. Since $I_2$ and $I_3$ are distinct, without loss of generality assume that $I_2 \nsubseteq I_3$, so there exists $a_2 \in I_2$ such that $Ra_2 \neq Ra_3 \subseteq I_3$, for some $a_3 \in I_3$. Therefore, $Ra_1 \subseteq I_1, Ra_2 \subseteq I_2$ and $Ra_3 \subseteq I_3$ are distinct non-zero principal ideals of $R$. □

**Proposition 3.6.** *Let $R$ be a ring. Then*

(1) *$\mathbb{AG}_P(R) \cong K_1$ if and only if $R$ has only one non-zero proper ideal, moreover, either $R \cong \dfrac{\mathbb{Z}_p[x]}{(x^2)}$, or $R \cong \mathbb{Z}_{p^2}$.*

(2) *Assume that $\mathbb{AG}_P(R) \cong K_2$, then $R \cong F_1 \times F_2$, where $F_1$, $F_2$ are two fields, or $(R, \mathcal{M})$ is a local ring with $\mathbb{A}_P(R) = \{\mathcal{M}^2, \mathcal{M}\}$.*

(3) *Assume that $\mathbb{AG}_P(R) \cong K_n$, where $n \geq 3$, then $Z(R)$ is an ideal of $R$ with $Z(R)^2 = (0)$.*

$\vec{\mathcal{E}}$

*Proof.* (1) ($\Rightarrow$) Suppose that $\mathbb{AG}_P(R) \cong K_1$ and $\mathbb{A}_P(R) = \{I\}$, where $I = Rx$ for some $0 \neq x \in R$. We claim that $|\mathbb{A}^*(R)| = 1$. By contrary suppose that $|\mathbb{A}^*(R)| \geq 2$ and $J \in \mathbb{A}^*(R) \setminus \{I\}$. Since $I \neq J$ and $I$ is a simple $R$-module, $J \nsubseteq I = Rx$ and hence there exists $y \in J \setminus Rx$ such that $Rx \neq Ry$, so $|\mathbb{A}_P(R)| \geq 2$, a contradiction. Therefore by [12, Corollary 2.9], $R$ has only one non-zero ideal. Also [1, Remark 10], implies that, either $R \cong \dfrac{\mathbb{Z}_p[x]}{(x^2)}$, or $R \cong \mathbb{Z}_{p^2}$.

($\Leftarrow$) Assume that $I$ is the only non-zero proper ideal of $R$. Since $R$ is an Artinian ring by [12, Proposition 1.3], $\mathbb{A}^*(R) = \{I\}$. It is clear that $I$ is a simple $R$-module and hence $I \in \mathbb{A}_P(R)$, we have $\mathbb{AG}_P(R) \cong K_1$.

(2) Suppose that $\mathbb{AG}_P(R) \cong K_2$ and $\mathbb{A}_P(R) = \{I, J\}$, where $I = Rx$, $J = Ry$. We claim that $\mathbb{A}^*(R) = \{I, J\}$. If $K \in \mathbb{A}^*(R) \setminus \{I, J\}$, Lemma 3.5 implies that $|\mathbb{A}_P(R)| \geqslant 3$, a contradiction. So $\mathbb{AG}(R) \cong K_2$ and hence by Proposition 3.4, either $R \cong F_1 \times F_2$, where $F_1$, $F_2$ are two fields or $(R, \mathcal{M})$ is a local ring with $\mathbb{A}_P(R) = \{\mathcal{M}^2, \mathcal{M}\}$.

(3) It is clear by Proposition 3.4.□

**Corollary 3.7.** *Let $R$ be a reduced ring, then $\mathbb{AG}_P(R) \cong K_2$ if and only if $R \cong F_1 \times F_2$ , where $F_1$, $F_2$ are two fields.*

*Proof.* It is clear by Proposition 3.6.□

*In* [5] *and* [13] *the authors studied diameter and girth of annihilating-ideal graph. Also, in* [13], *they study the interplay between the diameter of annihilating-ideal graphs and the diameter of zero-divisor graphs, and characterize rings $R$, when* $\mathrm{gr}(\mathbb{AG}(R)) \geqslant 4$. *We now show that* $\mathrm{diam}(\mathbb{AG}_P(R)) \leq \mathrm{diam}(\mathbb{AG}(R))$ *and* $gr(\mathbb{AG}(R)) = gr(\mathbb{AG}_P(R))$.

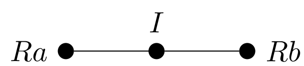
**Proposition 3.8.** *Let $R$ be a ring. Then the following statements hold.*

(1) $\mathrm{diam}(\mathbb{AG}_P(R)) \leq \mathrm{diam}(\mathbb{AG}(R))$.

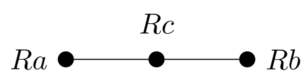
(2) $gr(\mathbb{AG}(R)) = gr(\mathbb{AG}_P(R))$.

$\vec{\mathcal{E}}$

*Proof.* (1) For a ring $R$, [12, Theorem 2.1] implies that $0 \leqslant \operatorname{diam}(\mathbb{AG}(R)) \leqslant 3$. Note that $\operatorname{diam}(\mathbb{AG}(R)) = 0$, if and only if, $\mathbb{AG}(R) \cong K_1$, if and only if, $\mathbb{AG}_P(R) \cong K_1$, if and only if, $\operatorname{diam}(\mathbb{AG}_P(R)) = 0$. Moreover, by Proposition 3.4, we can conclude that $\operatorname{diam}(\mathbb{AG}(R)) = 1$ if and only if $\operatorname{diam}(\mathbb{AG}_P(R)) = 1$. Now assume that $\operatorname{diam}(\mathbb{AG}(R)) = 2$. It is clear that $\operatorname{diam}(\mathbb{AG}_P(R)) \geqslant 2$. Let $Ra$ and $Rb \in \mathbb{A}_P(R)$ such that $(Ra)(Rb) \neq (0)$. Since $Ra, Rb \in \mathbb{A}^*(R)$, there exists an ideal $I$ such that



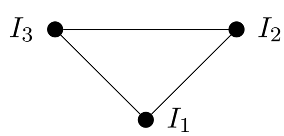

is a path in $\mathbb{AG}(R)$. Let $0 \neq c \in I$. Then



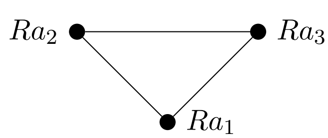

is a path in $\mathbb{AG}_P(R)$. Therefore, $\operatorname{diam}(\mathbb{AG}(R)) = 2 = \operatorname{diam}(\mathbb{AG}_P(R))$. Now assume that $\operatorname{diam}(\mathbb{AG}(R)) = 3$, since $\operatorname{diam}(\mathbb{AG}_P(R)) \leqslant 3$ (see Proposition 3.3), we can conclude that in every cases $\operatorname{diam}(\mathbb{AG}_P(R)) \leq \operatorname{diam}(\mathbb{AG}(R))$.

(2) First note that $\mathbb{AG}_P(R)$ is an induced subgraph of $\mathbb{AG}(R)$, thus $gr(\mathbb{AG}(R)) \leqslant gr(\mathbb{AG}_P(R))$. Also, by [12, Theorem 2.1], we have $gr(\mathbb{AG}(R)) = 3, 4$ or $\infty$.

Assume that $gr(\mathbb{AG}(R)) = 3$. Then there exist distinct ideals $I_1, I_2$ and $I_3$ such that

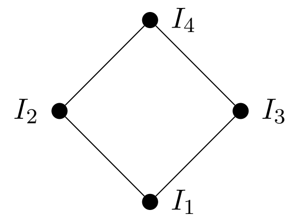



is a cycle in $\mathbb{AG}(R)$. By Lemma 3.5, there exist distinct non-zero principal ideals $Ra_1 \subseteq I_1, Ra_2 \subseteq I_2$ and $Ra_3 \subseteq I_3$. Then

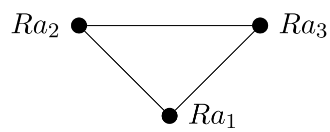



is a cycle in $\mathbb{AG}_P(R))$. Therefore, $gr(\mathbb{AG}(R)) = gr(\mathbb{AG}_P(R)) = 3$.

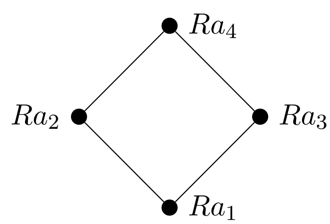
Assume that $gr(\mathbb{AG}(R)) = 4$. Then there exist distinct ideals $I_1, I_2, I_3$ and $I_4$ such that




is a cycle in $\mathbb{AG}(R)$. Since $I_1$ and $I_4$ are distinct ideals, without loss of generality, assume that there exists $a_4 \in I_4 \setminus I_1$. By Lemma 3.5, we have three distinct principal ideals $Ra_1, Ra_2$ and $Ra_3$ such that $Ra_i \subseteq I_i$, for $i = 1, 2, 3$. If $Ra_4 \in \{Ra_1, Ra_2, Ra_3\}$, then




is a cycle in $\mathbb{AG}_P(R)$, which is a contradiction since $gr(\mathbb{AG}_P(R)) \geqslant gr(\mathbb{AG}(R)) = 4$. So we may assume that $Ra_4 \notin \{Ra_1, Ra_2, Ra_3\}$, then we have the following cycle in $\mathbb{AG}_P(R)$




and so $gr(\mathbb{AG}_P(R)) = gr(\mathbb{AG}(R)) = 4$. Also, if $gr(\mathbb{AG}(R)) = \infty$, then since $gr(\mathbb{AG}_P(R)) \geqslant gr(\mathbb{AG}(R)) = \infty$, we must have $gr(\mathbb{AG}_P(R)) = gr(\mathbb{AG}(R)) = \infty$.□

*Note that if* $\mathbb{AG}(R)$ *is a bipartite graph, then* $\mathrm{gr}(\mathbb{AG}(R)) \geq 4$. *Also, In* [5], *there is a characterization of rings* $R$, *when* $\mathrm{gr}(\mathbb{AG}(R)) \geq 4$. *We will show that* $\mathbb{AG}(R)$ *is a bipartite graph if and only if* $\mathbb{AG}_P(R)$ *is bipartite. First we need the following lemma.*

**Lemma 3.9.** *Let* $G$ *be a graph. Then* $G$ *is a bipartite graph if and only if it contains no odd cycles.*

*Proof.* See [9, Theorem 3.4]. □

**Proposition 3.10.** *Let* $R$ *be a ring. Then* $\mathbb{AG}_P(R)$ *is a bipartite graph if and only if* $\mathbb{AG}(R)$ *is a bipartite graph.* $\vec{\mathcal{E}}$

*Proof.* If $\mathbb{AG}(R)$ is a bipartite graph, then since $\mathbb{AG}_P(R)$ is an induced subgraph of $\mathbb{AG}(R)$, we can conclude that $\mathbb{AG}_P(R)$ is a bipartite graph. Now assume that $\mathbb{AG}_P(R)$ is a bipartite graph. Suppose on the contrary that $\mathbb{AG}(R)$ is not bipartite, so by [2, Corollary 25], $\mathbb{AG}(R)$ contains a triangle and thus $gr(\mathbb{AG}(R)) = 3$. Theorem 3.8 implies that $gr(\mathbb{AG}_P(R)) = 3$, so $\mathbb{AG}_P(R)$ contains an odd cycle. Therefore, it is not bipartite graph (see Lemma 3.9), a contradiction.□

**Corollary 3.11.** *Let $R$ be an Artinian ring. If $\mathbb{AG}_P(R)$ is a bipartite graph, then $R$ is one of the following three types of rings:*

(a) *$R \cong F_1 \times F_2$, where $F_1$ and $F_2$ are two fields.*
(b) *$R$ is a local ring with non-zero maximal ideal $\mathcal{M}$ with $\mathcal{M}^4 = (0)$.*
(c) *$R \cong F \times S$, where $F$ is a field and $S$ is a ring with a unique non-trivial ideal.*

*The converse is also true when in case (b), $\mathcal{M}^3 \neq (0)$ and $\mathbb{A}^*(R) = \{\mathcal{M}, \mathcal{M}^2, \mathcal{M}^3\}$.*

*Proof.* ($\Rightarrow$) Assume that $\mathbb{AG}_P(R)$ is a bipartite graph. By Proposition 3.10, $\mathbb{AG}(R)$ is a bipartite graph, so by [3, Theorem 3], either $\mathbb{AG}(R)$ is a star graph or $\mathbb{AG}(R) \cong P_4$. Moreover $\mathbb{AG}(R) \cong P_4$, if and only if $R \cong F \times S$, where $F$ is a field and $S$ is a ring with a unique non-trivial ideal. If $\mathbb{AG}(R)$ is a star graph, [12, Lemma 2.5] implies that $R \cong F_1 \times F_2$, where $F_1$ and $F_2$ are two fields or $R$ is a local ring with non-zero maximal ideal $\mathcal{M}$ with $\mathcal{M}^4 = (0)$.

($\Leftarrow$) It is clear that $\mathbb{AG}(R) \cong P_2, P_3$ or $P_4$. Since $\mathbb{AG}_P(R)$ is an induced subgraph of $\mathbb{AG}(R)$, $\mathbb{AG}_P(R)$ is a bipartite graph.□

*In the following proposition we characterize all rings $R$ for which the principal ideal graph $\mathbb{AG}_P(R)$ has a vertex which is adjacent to every other vertex of $\mathbb{AG}_P(R)$.*

**Proposition 3.12.** *Let $R$ be a ring. Then the following statements are equivalent.*

(1) *There is a vertex of $\mathbb{AG}_P(R)$ which is adjacent to every other vertex of $\mathbb{AG}_P(R)$.*
(2) *There is a vertex of $\mathbb{AG}(R)$ which is adjacent to every other vertex of $\mathbb{AG}(R)$.*
(3) *Either $R = F \times D$, where $F$ is a field and $D$ is an integral domain, or $Z(R) = \mathrm{Ann}(x)$ for some $0 \neq x \in R$.*

$\vec{\mathcal{E}}$

*Proof.* (1) $\Rightarrow$ (2) Assume that $I \in \mathbb{A}_P(R)$ is a vertex which is adjacent to every other vertex in $\mathbb{AG}_P(R)$. Let $J \in \mathbb{A}^*(R) \setminus \mathbb{A}_P(R)$, we claim that $IJ = (0)$. Since $I$ and $J$ are distinct vertex of $\mathbb{AG}(R)$, $I \nsubseteq J$ or $J \nsubseteq I$. First assume that $I \nsubseteq J$. Thus for each $y \in J$, $I \neq Ry$. Therefore, $I(Ry) = (0)$, which implies that $IJ = (0)$. Now assume that $J \nsubseteq I$. Let $0 \neq t \in J$. We consider the following two cases:

**Case 1.** $t \in J \setminus (J \cap I)$. So for each $x \in I$, $Rx \neq Rt$. Therefore, $I \neq Rt$. Then we have $I(Rt) = (0)$.

**Case 2.** $t \in (J \cap I)$. Let $z \in J \setminus (J \cap I)$. Then $t + z \in J \setminus I$ and so $I \neq R(t + z)$. Therefore, $IR(t + z) = (0)$ and we have $(t + z)I = (0)$. Since $zI = (0)$, $tI = (0)$. Thus $(Rt)I = (0)$.

Since in both cases we have $(Rt)I = (0)$, we can conclude that $IJ = (0)$.

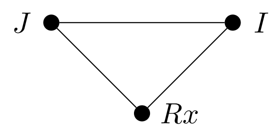
(2) $\Rightarrow$ (3) It follows from [12, Theorem 2.2].

(3) $\Rightarrow$ (1) If $R = F \times D$, where $F$ is a field and $D$ is an integral domain, then $F \times (0) \in \mathbb{A}_P(R)$ is adjacent to every other vertex. If $Z(R) = \mathrm{Ann}(x)$ for some non-zero $x \in R$, then $Rx \in \mathbb{A}_P(R)$ is adjacent to every other vertex.□

**Proposition 3.13.** *Let $R$ be a ring. Then $\mathbb{AG}_P(R)$ is a star graph if and only if $\mathbb{AG}(R)$ is star.* $\vec{\mathcal{E}}$

*Proof.* ($\Leftarrow$) If $\mathbb{AG}(R)$ is a star graph, then since $\mathbb{AG}_P(R)$ is an induced subgraph of $\mathbb{AG}(R)$, we can conclude that $\mathbb{AG}_P(R)$ is a star graph.

($\Rightarrow$) Assume that $\mathbb{AG}_P(R)$ is a star graph and $Rx \in \mathbb{A}_P(R)$ is adjacent to every other vertex of $\mathbb{AG}_P(R)$. Let $I \in \mathbb{A}^*(R) \setminus \mathbb{A}_P(R)$. We claim that $I$ is only adjacent to $Rx$. By the same argument in Theorem 3.12, $I(Rx) = (0)$. If $J \in \mathbb{A}^*(R) \setminus \{Rx\}$ such that $IJ = (0)$, then




is a triangle in $\mathbb{AG}(R)$ and so $gr(\mathbb{AG}(R)) = 3$. Therefore, By Proposition 3.8, $gr(\mathbb{AG}_P(R)) = 3$, it is impossible. Then $\mathbb{AG}(R)$ is a star graph.□

*The following corollary classifies all rings with star principal ideal graph.*

**Corollary 3.14.** *Let $R$ be a ring. Then*

(1) *If $R$ is reduced, then $\mathbb{AG}_P(R)$ is a star graph if and only if $R \cong F \times D$, where $F$ is a field and $D$ is an integral domain.*
(2) *If $R$ is non-reduced, then $\mathbb{AG}(R)$ is a star graph if and only if $\mathbb{AG}_P(R)$ is a bipartite graph and $Z(R) = \mathrm{Ann}(x)$, for some $x \in R$.*

*Proof.* It follows from [2, Corollary 26] and Proposition 3.13.□

*We say that the annihilating-ideal graph $\mathbb{AG}(R)$ has ACC (respectively, DCC) on its vertices if $R$ has ACC (respectively, DCC) on $\mathbb{A}^*(R)$. Also, we say that the principal ideal graph $\mathbb{AG}_P(R)$ has ACC (respectively, DCC) on its vertices if $R$ has ACC (respectively, DCC) on $\mathbb{A}_P(R)$. In* [12, Theorem 1.1], *it is shown that $\mathbb{AG}(R)$ has ACC (respectively, DCC) on its vertices if and only if $R$ is a Noetherian (respectively, an Artinian) ring. One may naturally ask, is $R$ Noetherian when $\mathbb{AG}_P(R)$ has ACC (respectively, DCC) on its vertices? The answer is no! The following example gives a non-Noetherian ring $R$ for which $\mathbb{AG}_P(R)$ has ACC (respectively, DCC) on vertices.*

**Example 3.15.** Let $R = \mathbb{Z}_2[\{X_i|\ i \in \mathbb{N}\}]/\langle\{X_iX_j|\ i, j \in \mathbb{N}\}\rangle$. Then $R$ is a non-Noetherian local ring with $\mathcal{M} = \bigoplus_{i=1}^{\infty} Rx_i$ (where $x_i = X_i + \langle\{X_iX_j|\ i, j \in \mathbb{N}\}\rangle$). It is clear that $\mathcal{M}^2 = (0)$ and $\mathcal{M}$ is a $\frac{R}{\mathcal{M}}$-vector space. For every $I \in \mathbb{P}(R)$, $I$ is a $\frac{R}{\mathcal{M}}$-vector space of one dimension. Thus $I$ is a minimal ideal of $R$, and so $I$ is a simple $R$-module. Since every principal ideal of $R$ is a simple $R$-module, we can conclude that $R$ has ACC (respectively, DCC) on $\mathbb{A}_P(R)$ but $R$ is not a Noetherian ring. E

Let $R$ be a ring. Then every non-zero proper principal ideal of $R$ is a vertex of $\mathbb{AG}(R)$, i.e., $\mathbb{A}_P(R) = \mathbb{P}(R)$ if and only if every element in $R$ is a unit or zero-divisor (see [12, Proposition 1.13] ). The following example gives a Noetherian ring such that $\mathbb{AG}_P(R) \cong K_{1,\infty}$ and $\mathbb{A}_P(R) \subsetneqq \mathbb{P}(R)$.

**Example 3.16.** Let $R = F \times \mathbb{Z}$ where $F$ is a field. Assume that $X = \{(0) \times n\mathbb{Z}\}$, where $n \in \mathbb{N}$. Then $F \times (0)$ is adjacent to each vertex in $X$, so $\mathbb{AG}_P(R)$ is an infinity star graph and $F \times 2\mathbb{Z}$ is a principal ideal in $\mathbb{P}(R)$ which is not belong to $\mathbb{A}_P(R)$. E

## 4. THE PRINCIPAL IDEAL GRAPH AND THE SPECTRUM GRAPH

In this section, we have a comparison between the principal ideal graph and the spectrum graph. We begin with the following remark.

**Remark 4.1.** By [12, Example 1.9], there exists a local zero-dimensional ring $R$ such that $\mathbb{A}^*(R) \neq \emptyset$ and $\mathbb{AG}_s(R)$ is an empty graph, but $\mathbb{AG}_P(R)$ is non empty. Also, [18, Example 2.7] gives a non-connected spectrum graph of a local ring, but if $R$ is a non-domain ring, then by Proposition 3.3, $\mathbb{AG}_P(R)$ is a connected graph.

**Lemma 4.2.** ([3, Theorem 10]) *Let $R$ be a ring such that $|\mathrm{Min}(R)| = 1$. If $\mathbb{AG}(R)$ is a triangle-free graph, then $\mathbb{AG}(R)$ is a star graph.*

**Proposition 4.3.** *Let $R$ be a non-domain ring. Then*

(1) *If $\mathbb{AG}_P(R)$ is a finite graph, then $\mathbb{AG}_s(R)$ is a finite graph.*
(2) *If $\mathbb{AG}_P(R)$ is a complete graph, then $\mathbb{AG}_s(R) \cong K_1$ or $\mathbb{AG}_s(R) \cong K_2$.*
(3) *Assume that $R$ is an Artinian ring such that $\mathbb{AG}(R)$ is a triangle-free graph, then $\mathbb{AG}_P(R)$ is a star graph if and only if $\mathbb{AG}_s(R)$ is a star graph.*

$\vec{\mathcal{E}}$

*Proof.* (1) It is clear by Proposition 3.1 .

(2) Assume that $\mathbb{AG}_P(R)$ is a complete graph. Then by Proposition 3.4, $\mathbb{AG}(R)$ is a complete graph. Since $\mathbb{AG}_s(R)$ is an induced subgraph of $\mathbb{AG}(R)$, $\mathbb{AG}_s(R)$ is a complete graph, too. Therefore, by [18, Proposition 2.5], $\mathbb{AG}_s(R) \cong K_1$ or $\mathbb{AG}_s(R) \cong K_2$.

(3) ($\Rightarrow$) Suppose that $\mathbb{AG}_P(R)$ is a star graph. Thus, $\mathbb{AG}(R)$ is a star graph (see Proposition 3.13). Since $R$ is an Artinian ring, by [12, Lemma 2.5], either $R = F_1 \times F_2$ , where $F_1$, $F_2$ are fields, or $R$ is a local ring. Therefore, either $\mathbb{AG}_s(R) \cong K_1$ or $\mathbb{AG}_s(R) \cong K_2$.

($\Leftarrow$) Assume that $\mathbb{AG}_s(R)$ is a star graph. Since $R$ is an Artinian ring, by [18, Theorem 3.10], either $R \cong F_1 \times F_2$ or $R$ is a local ring. If $R \cong F_1 \times F_2$, $\mathbb{AG}_P(R) \cong K_2$. Let $R$ be a local ring. Since $R$ is an Artinian ring, $|\mathrm{Min}(R)| = 1$. So by Lemma 4.2, $\mathbb{AG}(R)$ is a star graph and hence $\mathbb{AG}_P(R)$ is star, too.□

*The following example shows that the converse of Proposition 4.3 (1) and (2) does not hold.*

**Example 4.4.** Let $R = F \times \mathbb{Z}$, where $F$ is a field. Then $\mathbb{A}_s(R) = \{(0) \times \mathbb{Z}, F \times (0)\}$ and hence $\mathbb{AG}_s(R) \cong K_2$. But $\mathbb{AG}_P(R)$ is an infinite star graph. E

**Corollary 4.5.** *Let $R$ be an Artinian ring such that $\mathbb{AG}_P(R)$ is a complete graph. Then, either $R$ is a local ring or $R \cong F_1 \times F_2$, where $F_1$, $F_2$ are fields.*

*Proof.* It is clear with Proposition 4.3 and [18, Theorem 3.10].□

In the next proposition, we have a compression between diameter and girth of spectrum graph and principal ideal graph.

**Proposition 4.6.** *Let $R$ be a non-domain ring. Then*

(1) $\mathrm{gr}(\mathbb{AG}_P(R)) \leqslant \mathrm{gr}(\mathbb{AG}_s(R))$.

(2) *If $\mathbb{AG}_s(R)$ is a connected graph, then* $\mathrm{diam}(\mathbb{AG}_s(R)) \leqslant \mathrm{diam}(\mathbb{AG}_P(R))$.

(3) *Assume that $R$ is a Noetherian ring and* $\mathrm{diam}(\mathbb{AG}_s(R)) = 2$, *then* $\mathrm{diam}(\mathbb{AG}_P(R)) = 2$.

(4) *Assume that $R$ is a Noetherian ring such that $R$ is not Artinian and* $\mathrm{diam}(\mathbb{AG}_s(R)) = 1$, *then* $\mathrm{diam}(\mathbb{AG}_P(R)) = 2$.

$\vec{\mathcal{E}}$

*Proof.* (1) It is clear since for every ring $R$, $\mathrm{gr}(\mathbb{AG}_s(R)) = \infty$ (see [18, Corollary 2.4]).

(2) Assume that $\mathbb{AG}_s(R)$ is a connected graph. So by [18, Proposition 2.5], $\mathrm{diam}(\mathbb{AG}_s(R)) \in \{0, 1, 2\}$. If $\mathrm{diam}(\mathbb{AG}_s(R)) = 0$ or $1$, then there is nothing to prove. Let $\mathrm{diam}(\mathbb{AG}_s(R)) = 2$, so there are $P_1, P_2 \in \mathbb{A}_s(R)$ such that $P_1P_2 \neq (0)$. Since $P_1, P_2 \in \mathbb{AG}(R)$, $\mathrm{diam}(\mathbb{AG}(R)) \geqslant 2$ and hence $\mathrm{diam}(\mathbb{AG}_P(R)) \geqslant 2 = \mathrm{diam}(\mathbb{AG}_s(R))$.

(3) Suppose that $\mathrm{diam}(\mathbb{AG}_s(R)) = 2$, by [18, Proposition 2.3], $\mathbb{AG}_s(R) \cong K_{1,\infty}$. Thus, there is a vertex in $\mathbb{AG}_s(R)$ which is adjacent to every other vertex in $\mathbb{AG}_s(R)$. Since $|\mathbb{A}_s(R)| > 2$, by [18, Proposition 3.2], there is a vertex in $\mathbb{AG}(R)$ which is adjacent to every other vertex in $\mathbb{AG}(R)$, so $\mathrm{diam}(\mathbb{AG}(R)) \leqslant 2$. Therefore, $\mathrm{diam}(\mathbb{AG}_P(R)) \leqslant 2$. By (2), $2 = \mathrm{diam}(\mathbb{AG}_s(R)) \leqslant \mathrm{diam}(\mathbb{AG}_P(R))$, so $\mathrm{diam}(\mathbb{AG}_P(R)) = 2$.

(4) Since $\mathrm{diam}(\mathbb{AG}_s(R)) = 1$, $\mathbb{AG}_s(R) \cong K_2$ (see [18, Proposition 2.3]). Therefore, by [18, Proposition 3.6], $\mathbb{AG}(R)$ is a complete bipartite graph, so $\mathrm{diam}(\mathbb{AG}_P(R)) \leqslant 2$, by Proposition 3.8 (1). Note that $R$ is not Artinian. So by [12, Theorem 2.10], $\mathbb{AG}_s(R)$ is a proper subgraph of $\mathbb{AG}(R)$ and hence

$|\mathbb{A}^*(R)| \geqslant 3$. If $\text{diam}(\mathbb{AG}_P(R)) = 0$, then by Proposition 3.6 (1), $|\mathbb{A}^*(R)| = 1$, a contradiction. Now assume that $\text{diam}(\mathbb{AG}_P(R)) = 1$, so $\mathbb{AG}_P(R) \cong K_n$ , where $n \geq 2$. If $n = 2$, then Proposition 3.6 (2) implies that $|\mathbb{A}^*(R)| = 2$, a contradiction. For case $n > 2$, we have $\mathbb{AG}(R) \cong K_n$, where $n > 2$ (see Proposition 3.4). Thus $\mathbb{AG}(R)$ is not a complete bipartite graph, a contradiction. Therefore $\text{diam}(\mathbb{AG}_P(R)) \neq 0, 1$ and hence $\text{diam}(\mathbb{AG}_P(R)) = 2$. □

**Proposition 4.7.** *Let $R$ be an Artinian local ring such that $\mathbb{AG}_s(R)$ is a subgraph of $\mathbb{AG}_P(R)$. Then $\mathbb{AG}_P(R)$ is a finite graph.* $\vec{\mathcal{E}}$

*Proof.* Assume that $(R, \mathcal{M})$ is an Artinian local ring. It is clear that $\mathbb{AG}_s(R) \cong K_1$, where $\mathbb{A}_s(R) = \{\mathcal{M}\}$. Since $\mathbb{AG}_s(R)$ is a subgraph of $\mathbb{AG}_P(R)$, $\text{J}(R) = \mathcal{M} = Rx$, where $0 \neq x \in R$. Since $R$ is an Artinian ring, there is $k \in \mathbb{N}$, such that $x^k = 0$ and $x^{k-1} \neq 0$. Now, by [15, Theorem 9], $(0) = Rx^k \subset Rx^{k-1} \subset \cdots \subset Rx \subset R$ are the only ideals of $R$. Since $R$ is an Atinian ring, $\mathbb{A}_P(R) = \{J(R), Rx^2, \ldots, Rx^{k-1}\}$. Therefore, $\mathbb{AG}_P(R)$ is a finite graph.□

*The following proposition characterizes all Artinian rings $R$ for which $\mathbb{AG}_s(R)$ is a subgraph of $\mathbb{AG}_P(R)$.*

**Proposition 4.8.** *Let $R$ be an Artinian ring. Then $\mathbb{AG}_s(R)$ is a subgraph of $\mathbb{AG}_P(R)$ if and only if $R$ is a principal ideal ring.* $\vec{\mathcal{E}}$

*Proof.* ($\Leftarrow$) It is clear.

($\Rightarrow$) Assume that $\mathbb{AG}_s(R)$ is a subgraph of $\mathbb{AG}_P(R)$. Since $R$ is an Artinian ring and $\mathbb{AG}_s(R)$ is a subgraph of $\mathbb{AG}_P(R)$, by [12, Proposition 1.3], $\text{Spec}(R) = \mathbb{A}_s(R) \subseteq \mathbb{A}_P(R)$. Let $\mathbf{X} = \{I : I \in \mathbb{I}(R) \setminus \mathbb{P}(R)\}$. We claim that $\mathbf{X} = \emptyset$. Suppose on the contrary that $\mathbf{X} \neq \emptyset$. Since $R$ is a Noetherian ring, we may assume that $J$ is a Maximal element of $\mathbf{X}$. It is clear that $J \notin \text{Spec}(R)$, so there are $x, y \in R$ such that $xy \in J$ but $x, y \notin J$. Therefore, $J \subsetneqq J + Rx$. Since $J$ is a maximal element of $\mathbf{X}$, $J + Rx \in \mathbb{P}(R)$. So there is $z \in J + Rx$ such that $J + Rx = Rz$. Let $\mathbf{Y} = \{r \in R : rz \in J\}$. It is clear that $J \subseteq \mathbf{Y}$ and $z \in J + Rx$. Since $yz \in yJ + R(xy) \subseteq J$, $y \in \mathbf{Y}$ and hence $J \subsetneqq \mathbf{Y}$. Therefore $\mathbf{Y} \in \mathbb{P}(R)$. Let $\mathbf{Y} = Rt$, where $t \in \mathbf{Y}$. Now we claim that $J = R(zt)$. By definition, $z\mathbf{Y} = zR(t) = R(zt) \subseteq J$. On the other hand, if $s \in J$, then $s \in J + Rx = Rz$, so $s = kz$ for some $k$, where clearly $k \in \mathbf{Y}$. Thus $s \in z\mathbf{Y} = R(zt)$. Therefore $J = R(zt) \in \mathbb{P}(R)$, a contradiction. We can conclude that $\mathbf{X} = \emptyset$ as we claimed and we have $R$ is a principal ideal ring.□

*In view of the above proposition, one may naturally ask, is $R$ a principal ideal ring, when $\mathbb{AG}_s(R)$ is a principal ideal graph? The answer is no! The following example shows that the Artinian hypothesis is necessary in Proposition 4.8.*

**Example 4.9.** Let $F$ be a field and

$$R = F[[X, Y]]/\langle XY \rangle.$$

Then $R$ is a local ring with maximal ideal $\mathcal{M} = Rx + Ry$, where $x = X + \langle XY \rangle$ and $y = Y + \langle XY \rangle)$. Note that $\mathbb{A}_s(R) = \{Rx, Ry\} \subseteq \mathbb{A}_P(R)$, but $R$ is not a principal ideal ring. E

**Proposition 4.10.** *Let $R$ be a non-domain reduced ring. Then every vertex of $\mathbb{AG}_P(R)$ has finite degree and $\mathbb{AG}_s(R)$ is a connected graph if and only if $R \cong F_1 \times F_2$ , where $F_1$, $F_2$ are fields.* $\vec{\mathcal{E}}$

*Proof.* ($\Leftarrow$) It is clear.

($\Rightarrow$) Since every vertex of $\mathbb{AG}_P(R)$ has finite degree, by Corollary 3.2, $R \cong F_1 \times F_2 \times \cdots \times F_n$ , where $n \geqslant 2$ and $F_i$'s are field. If $n > 2$. Then $\mathbb{AG}_s(R)$ is not a connected graph, thus $n = 2$ and $R \cong F_1 \times F_2$.□

*We conclude this paper with the following proposition.*

**Proposition 4.11.** *Let $R$ be a non-local Artinian ring. Then $\mathbb{AG}_P(R)$ is a complete bipartite graph if and only if $\mathbb{AG}_s(R) \cong K_2$.* $\vec{\mathcal{E}}$

*Proof.* ($\Leftarrow$) Assume that $\mathbb{AG}_s(R) \cong K_2$. Since $R$ is an Artinian ring, [18, Theorem 3.10] implies that $R \cong F_1 \times F_2$, so $\mathbb{AG}_P(R)$ is a complete bipartite graph.

($\Rightarrow$) Suppose that $\mathbb{AG}_P(R)$ is a complete bipartite graph with two parts $X$ and $Y$. Since $R$ is an Artinian ring which is not local, $R \cong R_1 \times R_2$, where $R_1$, $R_2$ are two non-zero rings. We claim that $R_1$ is a field. Suppose on the contrary that $R_1$ is not a field, so there exists non-zero proper ideal $J$ of $R_1$. Let $a \in J$ and $I = R_1 a$. Since $R_1$ ia an Artinian ring, [12, Proposition 1.3] implies that $J \in \mathbb{A}^*(R_1)$ and hence $I \in \mathbb{A}_P(R_1)$. Therefore $I \times R_2 = R(a, 1)$ and $I \times R_2, (0) \times R_2 \in \mathbb{A}_P(R)$. Since $(I \times R_2)((0) \times R_2) \neq (0) \times (0)$ , both $I \times R_2$ and $(0) \times R_2$ are in $X$ or $Y$. But $R_1 \times (0)$ is adjacent to $(0) \times R_2$ and is not adjacent to $I \times R_2$, a contradiction. Thus, $R_1$ is a field and similarly, $R_2$ is a field, too. Consequently $\mathbb{AG}_s(R) \cong K_2$.□

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