



## THE PRINCIPAL IDEAL SUBGRAPH OF THE ANNIHILATING-IDEAL GRAPH OF COMMUTATIVE RINGS

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ABSTRACT. Let  $R$  be a commutative ring with identity and  $\mathbb{A}(R)$  be the set of ideals of  $R$  with non-zero annihilators. In this paper, we first introduce and investigate the principal ideal subgraph of the annihilating-ideal graph of  $R$ , denoted by  $\mathbb{AG}_P(R)$ . It is a (undirected) graph with vertices  $\mathbb{A}_P(R) = \mathbb{A}(R) \cap \mathbb{P}(R) \setminus \{(0)\}$ , where  $\mathbb{P}(R)$  is the set of proper principal ideals of  $R$  and two distinct vertices  $I$  and  $J$  are adjacent if and only if  $IJ = (0)$ . Then, we study some basic properties of  $\mathbb{AG}_P(R)$ . For instance, we characterize rings for which  $\mathbb{AG}_P(R)$  is finite graph, complete graph, bipartite graph or star graph. Also, we study diameter and girth of  $\mathbb{AG}_P(R)$ . Finally, we compare the principal ideal subgraph  $\mathbb{AG}_P(R)$  and spectrum subgraph  $\mathbb{AG}_s(R)$ .

### 1. INTRODUCTION

In recent years, assigning graphs to rings has played an important role in the study of structures of rings (see for example [7, 8, 10-14]). Let  $R$  be a commutative ring. We call an ideal  $I$  of  $R$  is an annihilating-ideal, if there exists a non-zero ideal  $J$  of  $R$  such that  $IJ = (0)$  and use the notation  $\mathbb{A}(R)$

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for the set of annihilating-ideals of  $R$ . By the annihilating-ideal graph  $\mathbb{A}\mathbb{G}(R)$  of  $R$  we mean the graph with vertices  $\mathbb{A}^*(R) = \mathbb{A}(R) \setminus \{(0)\}$  such that there is an (undirected) edge between vertices  $I$  and  $J$  if and only if  $I \neq J$  and  $IJ = (0)$ . The concept of the annihilating-ideal graph of a commutative ring was first introduced by Behboodi and Rakei in [12, 13]. Recently, this notation of the annihilating-ideal graph has been extensively studied by various authors (see for instance, [1-6, 16, 18] and many others). In [18], Taheri, Behboodi and Tehranian, introduce and investigate the spectrum graph of the annihilating-ideal graph of a commutative ring, denoted by  $\mathbb{A}\mathbb{G}_s(R)$ , that is, a graph whose vertices are all non-zero prime ideals of  $R$  with non-zero annihilators, denoted by  $\mathbb{A}_s(R)$  and two distinct vertices  $P_1, P_2$  are adjacent if and only if  $P_1P_2 = (0)$ . In this paper, we introduce and study the principal ideal graph of a commutative ring  $R$ , denoted by  $\mathbb{A}\mathbb{G}_P(R)$ , that is, the graph whose vertices are all principal ideals in  $\mathbb{A}^*(R)$  and two distinct vertices  $I, J$  are adjacent if and only if  $IJ = (0)$ . We denote by  $\mathbb{A}_P(R)$  the vertex set of  $\mathbb{A}\mathbb{G}_P(R)$ . It is clear that if  $R$  is a P.I.R. which is not integral domain, then  $\mathbb{A}\mathbb{G}_P(R) = \mathbb{A}\mathbb{G}(R)$ . First we study some basic properties of  $\mathbb{A}\mathbb{G}_P(R)$  and then we compare the principal ideal subgraph with the spectrum subgraph.

## 2. DEFINITION AND PRELIMINARIES

Throughout this paper, all rings are commutative with identity and all modules are unitary. For a ring  $R$ , we denote by  $\text{Spec}(R)$  the set of prime ideals,  $Z(R)$  the set of zero-divisors,  $\mathbb{I}(R)$  the set of non-zero proper ideals and  $\mathbb{P}(R)$  the set of proper principal ideals of  $R$ . The Jacobson radical and the set of minimal prime ideals of  $R$  are denoted by  $J(R)$  and  $\text{Min}(R)$ , respectively. Let  $X$  be an element or a subset of a ring  $R$ . The annihilator of  $X$  is the ideal  $\text{Ann}(X) = \{a \in R \mid aX = 0\}$ .

Let  $G$  be any graph. We denote the vertex set of  $G$  by  $V(G)$ . Sometimes, two graphs  $G$  and  $H$  have exactly the same form, in the sense that there is a one-to-one correspondence between their vertex sets that preserves edges. In such a case, we say that the two graphs  $G$  and  $H$  are isomorphic and we write  $G \cong H$ . We say that a subgraph  $H$  is an induced subgraph of  $G$  if  $H$  is isomorphic to a graph whose vertex set  $V_1$  is a subset of the vertex set  $V$  of  $G$  and whose edge set  $E_1$  consists of all the edges of  $G$  with both end vertices in  $V_1$ . For every subset  $A$  of  $\mathbb{A}(R)$  we denote the induced subgraph of  $\mathbb{A}\mathbb{G}(R)$  with vertex set  $A$  by  $\mathbb{A}\mathbb{G}_A(R)$ . The graph  $G$  is called connected if there is a path between every two distinct vertices. For distinct vertices  $P, Q$  of  $G$ , let  $d(P, Q)$  be the length of the shortest path from  $P$  to  $Q$  and, if there is no such a path, we define  $d(P, Q) = \infty$ . The diameter of  $G$  is  $\text{diam}(G) = \sup\{d(P, Q) : P \text{ and } Q \text{ are distinct vertices of } G\}$ . The girth of  $G$ , denoted by  $\text{gr}(G)$ , is defined as the length of the shortest cycle in  $G$  and  $\text{gr}(G) = \infty$ , if  $G$  contains no cycles. A complete graph is a graph in which any two distinct vertices are adjacent. A complete graph with  $n$  vertices denoted by  $K_n$ . A bipartite graph is a graph whose vertices can be divided into two disjoint sets  $A$  and  $B$  such that every edge connects a vertex in  $A$  to one in  $B$ . A complete bipartite graph is a bipartite graph in which every vertex of one part is joined to every vertex of the other part. In this

case, if  $|A| = n$  and  $|B| = m$ , we denote the graph by  $K_{n,m}$ . If  $|A| = 1$  or  $|B| = 1$ , then the graph is said to be a star graph. Ultimately, we denote by  $P_n$  a path of order  $n$ .

### 3. COPARING THE PRINCIPAL IDEAL GRAPH AND THE ANNIHILATING-IDEAL GRAPH

Let  $R$  be a non- domain commutative ring. In this section, we compare the features of  $\mathbb{A}\mathbb{G}_P(R)$  and  $\mathbb{A}\mathbb{G}(R)$  and express some properties of principal ideal graph. We begin with the following proposition, which characterize all rings  $R$  with finite principal ideal graph.

**Proposition 3.1.** *Let  $R$  be a ring. Then the following statements are equivalent.*

- (1)  $\mathbb{A}\mathbb{G}_P(R)$  is a finite graph.
- (2)  $\mathbb{A}\mathbb{G}(R)$  is a finite graph, moreover  $|\mathbb{A}^*(R)| \leq 2^{|\mathbb{A}_P(R)|}$ .
- (3)  $R$  has only finitely many ideals.
- (4) Every vertex of  $\mathbb{A}\mathbb{G}_P(R)$  has finite degree.
- (5) Every vertex of  $\mathbb{A}\mathbb{G}(R)$  has finite degree.

$\vec{\mathcal{E}}$  **Proof.** (1)  $\Rightarrow$  (2): Assume that  $\mathbb{A}\mathbb{G}_P(R)$  is a finite graph. Therefore,  $\mathbb{A}_P(R)$  is a finite set. For each  $I \in \mathbb{A}^*(R)$ ,  $I$  is generated by a subset of  $\mathbb{A}_P(R)$ . Since  $\mathbb{A}_P(R)$  has finitely many subsets, we can conclude that  $\mathbb{A}^*(R)$  is a finite set. Thus,  $\mathbb{A}\mathbb{G}(R)$  is a finite graph and  $|\mathbb{A}^*(R)| \leq 2^{|\mathbb{A}_P(R)|}$ .

(2)  $\Rightarrow$  (1) and (3)  $\Rightarrow$  (4) are clear.

(2)  $\Rightarrow$  (3) and (5)  $\Rightarrow$  (2) are follow from [12, Theorem 1.4].

(4)  $\Rightarrow$  (5) Suppose on the contrary that  $I \in \mathbb{A}^*(R)$  has infinite degree in  $\mathbb{A}\mathbb{G}(R)$ , so for each  $a \in I$ ,  $Ra \in \mathbb{A}_P(R)$  has infinite degree in  $\mathbb{A}\mathbb{G}_P(R)$ , a contradiction.  $\square$

In the following result, we characterize reduced rings  $R$  for which every vertex of principal ideal graph  $\mathbb{A}\mathbb{G}_P(R)$  has finite degree.

**Corollary 3.2.** *Let  $R$  be a reduced ring. Then every vertex of  $\mathbb{A}\mathbb{G}_P(R)$  has finite degree if and only if  $R \cong F_1 \times F_2 \times \dots \times F_n$ , where  $F_i$ 's are fields for each  $i$ ,  $1 \leq i \leq n$ .*

*Proof.* ( $\Leftarrow$ ) It is clear.

( $\Rightarrow$ ) Assume that every vertex of  $\mathbb{A}\mathbb{G}_P(R)$  has finite degree. By Proposition 3.1,  $R$  has only finitely many ideals and hence is an Artinian ring, so  $R \cong F_1 \times F_2 \times \dots \times F_n$ , where  $F_i$ 's are fields for each  $i$ ,  $1 \leq i \leq n$ .  $\square$

We now show that  $\mathbb{A}\mathbb{G}_P(R)$  is connected.

**Proposition 3.3.** *For every ring  $R$ , the principal ideal subgraph  $\mathbb{A}\mathbb{G}_P(R)$  is connected and  $\text{diam}(\mathbb{A}\mathbb{G}_P(R)) \leq 3 \cdot \vec{\mathcal{E}}$*

*Proof.* Let  $I$  and  $J$  be distinct vertices of  $\mathbb{A}\mathbb{G}_P(R)$ . Then there exist principal ideals  $K$  and  $L$  such that  $IK = (0)$  and  $JL = (0)$ . If  $KL = (0)$ , then  $J - L - K - I$  is a path with length 3 in  $\mathbb{A}\mathbb{G}_P(R)$ . If  $KL \neq (0)$ , then there is a principal ideal  $H \subseteq KL$  and  $J - H - I$  is a path with length 2 in  $\mathbb{A}\mathbb{G}_P(R)$ . We can conclude that  $\mathbb{A}\mathbb{G}_P(R)$  is connected and  $\text{diam}(\mathbb{A}\mathbb{G}_P(R)) \leq 3$ .  $\square$

*In the following proposition we show that  $\mathbb{A}\mathbb{G}_P(R)$  is a complete graph if and only if  $\mathbb{A}\mathbb{G}(R)$  is complete. Also, in the next proposition we characterize rings whose principal ideal subgraphs are complete graphs.*

**Proposition 3.4.** *Let  $R$  be a ring. Then the following statements are equivalent.*

- (1)  $\mathbb{A}\mathbb{G}_P(R)$  is a complete graph.
- (2)  $\mathbb{A}\mathbb{G}(R)$  is a complete graph.
- (3) Either  $\mathbb{A}\mathbb{G}(R) \cong K_2$  or  $Z(R)$  is the ideal of  $R$  with  $Z(R)^2 = (0)$ . Moreover, in the first case, either  $R \cong F_1 \times F_2$ , where  $F_1$  and  $F_2$  are fields or  $(R, \mathcal{M})$  is a local ring with exactly two non-trivial ideals  $\mathcal{M}^2$  and  $\mathcal{M}$ .

$\vec{\mathcal{E}}$

*Proof.* (1)  $\Rightarrow$  (2) Let  $I$  and  $J$  be two distinct vertices of  $\mathbb{A}\mathbb{G}(R)$ . We show that  $IJ = (0)$ . Without loss of generality we can assume  $I \not\subseteq J$ . Let  $0 \neq x \in I$ . We consider the following two cases:

**Case 1.**  $x \in I \setminus (I \cap J)$ . Then for each  $y \in J$ ,  $Rx$  and  $Ry$  are two distinct vertices of  $\mathbb{A}\mathbb{G}_P(R)$  and hence  $(Rx)(Ry) = (0)$ . Therefore,  $xJ = (0)$ .

**Case 2.**  $x \in (I \cap J)$ . Let  $a \in I \setminus (I \cap J)$ , so  $x + a \in I \setminus J$  and we can see for each  $y \in J$ ,  $Ry$  and  $R(x + a)$  are two distinct vertices of  $\mathbb{A}\mathbb{G}_P(R)$ , so  $R(x + a)Ry = (0)$ . Note that  $y$  is an arbitrary element of  $J$ , thus we have  $R(x + a)J = (0)$ , and so  $(x + a)J = (0)$ . Since  $aJ = (0)$ ,  $xJ = (0)$ .

Therefore, in each cases, for every  $x \in I$ , we have  $xJ = (0)$ , which implies that  $IJ = (0)$ . We can conclude that  $\mathbb{A}\mathbb{G}(R)$  is a complete graph.

(2)  $\Rightarrow$  (1) Let  $\mathbb{A}\mathbb{G}(R)$  is a complete graph. Since  $\mathbb{A}\mathbb{G}_P(R)$  is an induced subgraph of  $\mathbb{A}\mathbb{G}(R)$ , we have  $\mathbb{A}\mathbb{G}_P(R)$  is a complete graph.

(2)  $\Leftrightarrow$  (3) It follows from [1, Theorem 3].  $\square$

*We need the following lemma.*

**Lemma 3.5.** *Let  $I_1, I_2$  and  $I_3$  be distinct ideals of  $R$ . Then there exist distinct non-zero principal ideals  $Ra_1, Ra_2$  and  $Ra_3$  such that  $Ra_i \subseteq I_i$ , for  $i = 1, 2, 3$ .*

*Proof.* Let  $\mathbf{X} = \{I_1, I_2, I_3\}$ . Without loss of generality assume that  $I_1$  is a maximal element of  $\mathbf{X}$ . If  $I_1 \subseteq I_2 \cup I_3$ , then by Prime Avoidance Theorem [17, Theorem 3.61],  $I_1 \subseteq I_2$  or  $I_1 \subseteq I_3$ , a contradiction to maximality of  $I_1$ . Therefore, there exists  $0 \neq a_1 \in I_1 \setminus (I_2 \cup I_3)$ . Since  $I_2$  and  $I_3$  are distinct, without

loss of generality assume that  $I_2 \not\subseteq I_3$ , so there exists  $a_2 \in I_2$  such that  $Ra_2 \neq Ra_3 \subseteq I_3$ , for some  $a_3 \in I_3$ . Therefore,  $Ra_1 \subseteq I_1, Ra_2 \subseteq I_2$  and  $Ra_3 \subseteq I_3$  are distinct non-zero principal ideals of  $R$ .  $\square$

**Proposition 3.6.** *Let  $R$  be a ring. Then*

- (1)  $\mathbb{A}\mathbb{G}_P(R) \cong K_1$  if and only if  $R$  has only one non-zero proper ideal, moreover, either  $R \cong \frac{\mathbb{Z}_p[x]}{(x^2)}$ , or  $R \cong \mathbb{Z}_{p^2}$ .
- (2) Assume that  $\mathbb{A}\mathbb{G}_P(R) \cong K_2$ , then  $R \cong F_1 \times F_2$ , where  $F_1, F_2$  are two fields, or  $(R, \mathcal{M})$  is a local ring with  $\mathbb{A}_P(R) = \{\mathcal{M}^2, \mathcal{M}\}$ .
- (3) Assume that  $\mathbb{A}\mathbb{G}_P(R) \cong K_n$ , where  $n \geq 3$ , then  $Z(R)$  is an ideal of  $R$  with  $Z(R)^2 = (0)$ .

$\vec{\mathcal{E}}$

*Proof.* (1)  $(\Rightarrow)$  Suppose that  $\mathbb{A}\mathbb{G}_P(R) \cong K_1$  and  $\mathbb{A}_P(R) = \{I\}$ , where  $I = Rx$  for some  $0 \neq x \in R$ . We claim that  $|\mathbb{A}^*(R)| = 1$ . By contrary suppose that  $|\mathbb{A}^*(R)| \geq 2$  and  $J \in \mathbb{A}^*(R) \setminus \{I\}$ . Since  $I \neq J$  and  $I$  is a simple  $R$ -module,  $J \not\subseteq I = Rx$  and hence there exists  $y \in J \setminus Rx$  such that  $Rx \neq Ry$ , so  $|\mathbb{A}_P(R)| \geq 2$ , a contradiction. Therefore by [12, Corollary 2.9],  $R$  has only one non-zero ideal. Also [1, Remark 10], implies that, either  $R \cong \frac{\mathbb{Z}_p[x]}{(x^2)}$ , or  $R \cong \mathbb{Z}_{p^2}$ .

$(\Leftarrow)$  Assume that  $I$  is the only non-zero proper ideal of  $R$ . Since  $R$  is an Artinian ring by [12, Proposition 1.3],  $\mathbb{A}^*(R) = \{I\}$ . It is clear that  $I$  is a simple  $R$ -module and hence  $I \in \mathbb{A}_P(R)$ , we have  $\mathbb{A}\mathbb{G}_P(R) \cong K_1$ .

(2) Suppose that  $\mathbb{A}\mathbb{G}_P(R) \cong K_2$  and  $\mathbb{A}_P(R) = \{I, J\}$ , where  $I = Rx, J = Ry$ . We claim that  $\mathbb{A}^*(R) = \{I, J\}$ . If  $K \in \mathbb{A}^*(R) \setminus \{I, J\}$ , Lemma 3.5 implies that  $|\mathbb{A}_P(R)| \geq 3$ , a contradiction. So  $\mathbb{A}\mathbb{G}(R) \cong K_2$  and hence by Proposition 3.4, either  $R \cong F_1 \times F_2$ , where  $F_1, F_2$  are two fields or  $(R, \mathcal{M})$  is a local ring with  $\mathbb{A}_P(R) = \{\mathcal{M}^2, \mathcal{M}\}$ .

(3) It is clear by Proposition 3.4.  $\square$

**Corollary 3.7.** *Let  $R$  be a reduced ring, then  $\mathbb{A}\mathbb{G}_P(R) \cong K_2$  if and only if  $R \cong F_1 \times F_2$ , where  $F_1, F_2$  are two fields.*

*Proof.* It is clear by Proposition 3.6.  $\square$

In [5] and [13] the authors studied diameter and girth of annihilating-ideal graph. Also, in [13], they study the interplay between the diameter of annihilating-ideal graphs and the diameter of zero-divisor graphs, and characterize rings  $R$ , when  $gr(\mathbb{A}\mathbb{G}(R)) \geq 4$ . We now show that  $diam(\mathbb{A}\mathbb{G}_P(R)) \leq diam(\mathbb{A}\mathbb{G}(R))$  and  $gr(\mathbb{A}\mathbb{G}(R)) = gr(\mathbb{A}\mathbb{G}_P(R))$ .

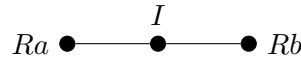
**Proposition 3.8.** *Let  $R$  be a ring. Then the following statements hold.*

- (1)  $diam(\mathbb{A}\mathbb{G}_P(R)) \leq diam(\mathbb{A}\mathbb{G}(R))$ .

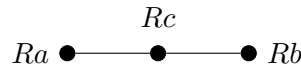
$$(2) \text{ gr}(\mathbb{A}\mathbb{G}(R)) = \text{gr}(\mathbb{A}\mathbb{G}_P(R)).$$

$\vec{\mathcal{E}}$

*Proof.* (1) For a ring  $R$ , [12, Theorem 2.1] implies that  $0 \leq \text{diam}(\mathbb{A}\mathbb{G}(R)) \leq 3$ . Note that  $\text{diam}(\mathbb{A}\mathbb{G}(R)) = 0$ , if and only if,  $\mathbb{A}\mathbb{G}(R) \cong K_1$ , if and only if,  $\mathbb{A}\mathbb{G}_P(R) \cong K_1$ , if and only if,  $\text{diam}(\mathbb{A}\mathbb{G}_P(R)) = 0$ . Moreover, by Proposition 3.4, we can conclude that  $\text{diam}(\mathbb{A}\mathbb{G}(R)) = 1$  if and only if  $\text{diam}(\mathbb{A}\mathbb{G}_P(R)) = 1$ . Now assume that  $\text{diam}(\mathbb{A}\mathbb{G}(R)) = 2$ . It is clear that  $\text{diam}(\mathbb{A}\mathbb{G}_P(R)) \geq 2$ . Let  $Ra$  and  $Rb \in \mathbb{A}_P(R)$  such that  $(Ra)(Rb) \neq (0)$ . Since  $Ra, Rb \in \mathbb{A}^*(R)$ , there exists an ideal  $I$  such that



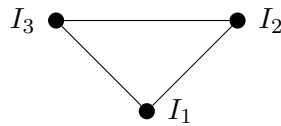
is a path in  $\mathbb{A}\mathbb{G}(R)$ . Let  $0 \neq c \in I$ . Then



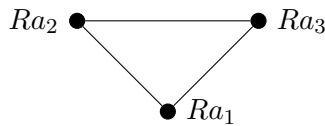
is a path in  $\mathbb{A}\mathbb{G}_P(R)$ . Therefore,  $\text{diam}(\mathbb{A}\mathbb{G}(R)) = 2 = \text{diam}(\mathbb{A}\mathbb{G}_P(R))$ . Now assume that  $\text{diam}(\mathbb{A}\mathbb{G}(R)) = 3$ , since  $\text{diam}(\mathbb{A}\mathbb{G}_P(R)) \leq 3$  (see Proposition 3.3), we can conclude that in every cases  $\text{diam}(\mathbb{A}\mathbb{G}_P(R)) \leq \text{diam}(\mathbb{A}\mathbb{G}(R))$ .

(2) First note that  $\mathbb{A}\mathbb{G}_P(R)$  is an induced subgraph of  $\mathbb{A}\mathbb{G}(R)$ , thus  $\text{gr}(\mathbb{A}\mathbb{G}(R)) \leq \text{gr}(\mathbb{A}\mathbb{G}_P(R))$ . Also, by [12, Theorem 2.1], we have  $\text{gr}(\mathbb{A}\mathbb{G}(R)) = 3, 4$  or  $\infty$ .

Assume that  $\text{gr}(\mathbb{A}\mathbb{G}(R)) = 3$ . Then there exist distinct ideals  $I_1, I_2$  and  $I_3$  such that

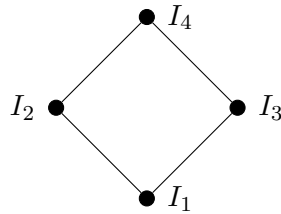


is a cycle in  $\mathbb{A}\mathbb{G}(R)$ . By Lemma 3.5, there exist distinct non-zero principal ideals  $Ra_1 \subseteq I_1, Ra_2 \subseteq I_2$  and  $Ra_3 \subseteq I_3$ . Then

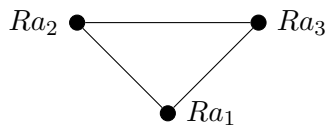


is a cycle in  $\mathbb{A}\mathbb{G}_P(R)$ . Therefore,  $\text{gr}(\mathbb{A}\mathbb{G}(R)) = \text{gr}(\mathbb{A}\mathbb{G}_P(R)) = 3$ .

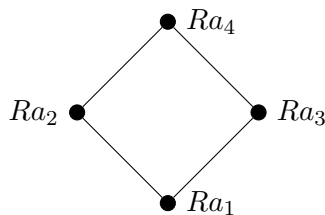
Assume that  $\text{gr}(\mathbb{A}\mathbb{G}(R)) = 4$ . Then there exist distinct ideals  $I_1, I_2, I_3$  and  $I_4$  such that



is a cycle in  $\mathbb{A}\mathbb{G}(R)$ . Since  $I_1$  and  $I_4$  are distinct ideals, without loss of generality, assume that there exists  $a_4 \in I_4 \setminus I_1$ . By Lemma 3.5, we have three distinct principal ideals  $Ra_1, Ra_2$  and  $Ra_3$  such that  $Ra_i \subseteq I_i$ , for  $i = 1, 2, 3$ . If  $Ra_4 \in \{Ra_1, Ra_2, Ra_3\}$ , then



is a cycle in  $\mathbb{A}\mathbb{G}_P(R)$ , which is a contradiction since  $gr(\mathbb{A}\mathbb{G}_P(R)) \geq gr(\mathbb{A}\mathbb{G}(R)) = 4$ . So we may assume that  $Ra_4 \notin \{Ra_1, Ra_2, Ra_3\}$ , then we have the following cycle in  $\mathbb{A}\mathbb{G}_P(R)$



and so  $gr(\mathbb{A}\mathbb{G}_P(R)) = gr(\mathbb{A}\mathbb{G}(R)) = 4$ . Also, if  $gr(\mathbb{A}\mathbb{G}(R)) = \infty$ , then since  $gr(\mathbb{A}\mathbb{G}_P(R)) \geq gr(\mathbb{A}\mathbb{G}(R)) = \infty$ , we must have  $gr(\mathbb{A}\mathbb{G}_P(R)) = gr(\mathbb{A}\mathbb{G}(R)) = \infty$ .  $\square$

*Note that if  $\mathbb{A}\mathbb{G}(R)$  is a bipartite graph, then  $gr(\mathbb{A}\mathbb{G}(R)) \geq 4$ . Also, In [5], there is a characterization of rings  $R$ , when  $gr(\mathbb{A}\mathbb{G}(R)) \geq 4$ . We will show that  $\mathbb{A}\mathbb{G}(R)$  is a bipartite graph if and only if  $\mathbb{A}\mathbb{G}_P(R)$  is bipartite. First we need the following lemma.*

**Lemma 3.9.** *Let  $G$  be a graph. Then  $G$  is a bipartite graph if and only if it contains no odd cycles.*

*Proof.* See [9, Theorem 3.4].  $\square$

**Proposition 3.10.** *Let  $R$  be a ring. Then  $\mathbb{A}\mathbb{G}_P(R)$  is a bipartite graph if and only if  $\mathbb{A}\mathbb{G}(R)$  is a bipartite graph.  $\vec{\mathcal{E}}$*

*Proof.* If  $\mathbb{A}\mathbb{G}(R)$  is a bipartite graph, then since  $\mathbb{A}\mathbb{G}_P(R)$  is an induced subgraph of  $\mathbb{A}\mathbb{G}(R)$ , we can conclude that  $\mathbb{A}\mathbb{G}_P(R)$  is a bipartite graph. Now assume that  $\mathbb{A}\mathbb{G}_P(R)$  is a bipartite graph. Suppose on the contrary that  $\mathbb{A}\mathbb{G}(R)$  is not bipartite, so by [2, Corollary 25],  $\mathbb{A}\mathbb{G}(R)$  contains a triangle and

thus  $gr(\mathbb{A}\mathbb{G}(R)) = 3$ . Theorem 3.8 implies that  $gr(\mathbb{A}\mathbb{G}_P(R)) = 3$ , so  $\mathbb{A}\mathbb{G}_P(R)$  contains an odd cycle. Therefore, it is not bipartite graph (see Lemma 3.9), a contradiction.  $\square$

**Corollary 3.11.** *Let  $R$  be an Artinian ring. If  $\mathbb{A}\mathbb{G}_P(R)$  is a bipartite graph, then  $R$  is one of the following three types of rings:*

- (a)  $R \cong F_1 \times F_2$ , where  $F_1$  and  $F_2$  are two fields.
- (b)  $R$  is a local ring with non-zero maximal ideal  $\mathcal{M}$  with  $\mathcal{M}^4 = (0)$ .
- (c)  $R \cong F \times S$ , where  $F$  is a field and  $S$  is a ring with a unique non-trivial ideal.

The converse is also true when in case (b),  $\mathcal{M}^3 \neq (0)$  and  $\mathbb{A}^*(R) = \{\mathcal{M}, \mathcal{M}^2, \mathcal{M}^3\}$ .

*Proof.* ( $\Rightarrow$ ) Assume that  $\mathbb{A}\mathbb{G}_P(R)$  is a bipartite graph. By Proposition 3.10,  $\mathbb{A}\mathbb{G}(R)$  is a bipartite graph, so by [3, Theorem 3], either  $\mathbb{A}\mathbb{G}(R)$  is a star graph or  $\mathbb{A}\mathbb{G}(R) \cong P_4$ . Moreover  $\mathbb{A}\mathbb{G}(R) \cong P_4$ , if and only if  $R \cong F \times S$ , where  $F$  is a field and  $S$  is a ring with a unique non-trivial ideal. If  $\mathbb{A}\mathbb{G}(R)$  is a star graph, [12, Lemma 2.5] implies that  $R \cong F_1 \times F_2$ , where  $F_1$  and  $F_2$  are two fields or  $R$  is a local ring with non-zero maximal ideal  $\mathcal{M}$  with  $\mathcal{M}^4 = (0)$ .

( $\Leftarrow$ ) It is clear that  $\mathbb{A}\mathbb{G}(R) \cong P_2, P_3$  or  $P_4$ . Since  $\mathbb{A}\mathbb{G}_P(R)$  is an induced subgraph of  $\mathbb{A}\mathbb{G}(R)$ ,  $\mathbb{A}\mathbb{G}_P(R)$  is a bipartite graph.  $\square$

*In the following proposition we characterize all rings  $R$  for which the principal ideal graph  $\mathbb{A}\mathbb{G}_P(R)$  has a vertex which is adjacent to every other vertex of  $\mathbb{A}\mathbb{G}_P(R)$ .*

**Proposition 3.12.** *Let  $R$  be a ring. Then the following statements are equivalent.*

- (1) *There is a vertex of  $\mathbb{A}\mathbb{G}_P(R)$  which is adjacent to every other vertex of  $\mathbb{A}\mathbb{G}_P(R)$ .*
- (2) *There is a vertex of  $\mathbb{A}\mathbb{G}(R)$  which is adjacent to every other vertex of  $\mathbb{A}\mathbb{G}(R)$ .*
- (3) *Either  $R = F \times D$ , where  $F$  is a field and  $D$  is an integral domain, or  $Z(R) = \text{Ann}(x)$  for some  $0 \neq x \in R$ .*

$\vec{\mathcal{E}}$

*Proof.* (1)  $\Rightarrow$  (2) Assume that  $I \in \mathbb{A}_P(R)$  is a vertex which is adjacent to every other vertex in  $\mathbb{A}\mathbb{G}_P(R)$ . Let  $J \in \mathbb{A}^*(R) \setminus \mathbb{A}_P(R)$ , we claim that  $IJ = (0)$ . Since  $I$  and  $J$  are distinct vertex of  $\mathbb{A}\mathbb{G}(R)$ ,  $I \not\subseteq J$  or  $J \not\subseteq I$ . First assume that  $I \not\subseteq J$ . Thus for each  $y \in J$ ,  $I \neq Ry$ . Therefore,  $I(Ry) = (0)$ , which implies that  $IJ = (0)$ . Now assume that  $J \not\subseteq I$ . Let  $0 \neq t \in J$ . We consider the following two cases:

**Case 1.**  $t \in J \setminus (J \cap I)$ . So for each  $x \in I$ ,  $Rx \neq Rt$ . Therefore,  $I \neq Rt$ . Then we have  $I(Rt) = (0)$ .

**Case 2.**  $t \in (J \cap I)$ . Let  $z \in J \setminus (J \cap I)$ . Then  $t + z \in J \setminus I$  and so  $I \neq R(t + z)$ . Therefore,  $IR(t + z) = (0)$  and we have  $(t + z)I = (0)$ . Since  $zI = (0)$ ,  $tI = (0)$ . Thus  $(Rt)I = (0)$ .

Since in both cases we have  $(Rt)I = (0)$ , we can conclude that  $IJ = (0)$ .



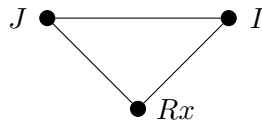
(2)  $\Rightarrow$  (3) It follows from [12, Theorem 2.2].

(3)  $\Rightarrow$  (1) If  $R = F \times D$ , where  $F$  is a field and  $D$  is an integral domain, then  $F \times (0) \in \mathbb{A}_P(R)$  is adjacent to every other vertex. If  $Z(R) = \text{Ann}(x)$  for some non-zero  $x \in R$ , then  $Rx \in \mathbb{A}_P(R)$  is adjacent to every other vertex.  $\square$

**Proposition 3.13.** *Let  $R$  be a ring. Then  $\mathbb{A}\mathbb{G}_P(R)$  is a star graph if and only if  $\mathbb{A}\mathbb{G}(R)$  is star.  $\vec{\mathcal{E}}$*

*Proof.* ( $\Leftarrow$ ) If  $\mathbb{A}\mathbb{G}(R)$  is a star graph, then since  $\mathbb{A}\mathbb{G}_P(R)$  is an induced subgraph of  $\mathbb{A}\mathbb{G}(R)$ , we can conclude that  $\mathbb{A}\mathbb{G}_P(R)$  is a star graph.

( $\Rightarrow$ ) Assume that  $\mathbb{A}\mathbb{G}_P(R)$  is a star graph and  $Rx \in \mathbb{A}_P(R)$  is adjacent to every other vertex of  $\mathbb{A}\mathbb{G}_P(R)$ . Let  $I \in \mathbb{A}^*(R) \setminus \mathbb{A}_P(R)$ . We claim that  $I$  is only adjacent to  $Rx$ . By the same argument in Theorem 3.12,  $I(Rx) = (0)$ . If  $J \in \mathbb{A}^*(R) \setminus \{Rx\}$  such that  $IJ = (0)$ , then



is a triangle in  $\mathbb{A}\mathbb{G}(R)$  and so  $gr(\mathbb{A}\mathbb{G}(R)) = 3$ . Therefore, By Proposition 3.8,  $gr(\mathbb{A}\mathbb{G}_P(R)) = 3$ , it is impossible. Then  $\mathbb{A}\mathbb{G}(R)$  is a star graph.  $\square$

*The following corollary classifies all rings with star principal ideal graph.*

**Corollary 3.14.** *Let  $R$  be a ring. Then*

- (1) *If  $R$  is reduced, then  $\mathbb{A}\mathbb{G}_P(R)$  is a star graph if and only if  $R \cong F \times D$ , where  $F$  is a field and  $D$  is an integral domain.*
- (2) *If  $R$  is non-reduced, then  $\mathbb{A}\mathbb{G}(R)$  is a star graph if and only if  $\mathbb{A}\mathbb{G}_P(R)$  is a bipartite graph and  $Z(R) = \text{Ann}(x)$ , for some  $x \in R$ .*

*Proof.* It follows from [2, Corollary 26] and Proposition 3.13.  $\square$

*We say that the annihilating-ideal graph  $\mathbb{A}\mathbb{G}(R)$  has ACC (respectively, DCC) on its vertices if  $R$  has ACC (respectively, DCC) on  $\mathbb{A}^*(R)$ . Also, we say that the principal ideal graph  $\mathbb{A}\mathbb{G}_P(R)$  has ACC (respectively, DCC) on its vertices if  $R$  has ACC (respectively, DCC) on  $\mathbb{A}_P(R)$ . In [12, Theorem 1.1], it is shown that  $\mathbb{A}\mathbb{G}(R)$  has ACC (respectively, DCC) on its vertices if and only if  $R$  is a Noetherian (respectively, an Artinian) ring. One may naturally ask, is  $R$  Noetherian when  $\mathbb{A}\mathbb{G}_P(R)$  has ACC (respectively, DCC) on its vertices? The answer is no! The following example gives a non-Noetherian ring  $R$  for which  $\mathbb{A}\mathbb{G}_P(R)$  has ACC (respectively, DCC) on vertices.*

**Example 3.15.** Let  $R = \mathbb{Z}_2[\{X_i \mid i \in \mathbb{N}\}] / \langle \{X_i X_j \mid i, j \in \mathbb{N}\} \rangle$ . Then  $R$  is a non-Noetherian local ring with  $\mathcal{M} = \bigoplus_{i=1}^{\infty} R x_i$  (where  $x_i = X_i + \langle \{X_i X_j \mid i, j \in \mathbb{N}\} \rangle$ ). It is clear that  $\mathcal{M}^2 = (0)$  and  $\mathcal{M}$  is a  $\frac{R}{\mathcal{M}}$ -vector space. For every  $I \in \mathbb{P}(R)$ ,  $I$  is a  $\frac{R}{\mathcal{M}}$ -vector space of one dimension. Thus  $I$  is a minimal ideal of  $R$ , and so  $I$  is a simple  $R$ -module. Since every principal ideal of  $R$  is a simple  $R$ -module, we can conclude that  $R$  has ACC (respectively, DCC) on  $\mathbb{A}_P(R)$  but  $R$  is not a Noetherian ring. E

Let  $R$  be a ring. Then every non-zero proper principal ideal of  $R$  is a vertex of  $\mathbb{A}\mathbb{G}(R)$ , i.e.,  $\mathbb{A}_P(R) = \mathbb{P}(R)$  if and only if every element in  $R$  is a unit or zero-divisor (see [12, Proposition 1.13]). The following example gives a Noetherian ring such that  $\mathbb{A}\mathbb{G}_P(R) \cong K_{1,\infty}$  and  $\mathbb{A}_P(R) \subsetneq \mathbb{P}(R)$ .

**Example 3.16.** Let  $R = F \times \mathbb{Z}$  where  $F$  is a field. Assume that  $X = \{(0) \times n\mathbb{Z}\}$ , where  $n \in \mathbb{N}$ . Then  $F \times (0)$  is adjacent to each vertex in  $X$ , so  $\mathbb{A}\mathbb{G}_P(R)$  is an infinity star graph and  $F \times 2\mathbb{Z}$  is a principal ideal in  $\mathbb{P}(R)$  which is not belong to  $\mathbb{A}_P(R)$ . E

#### 4. THE PRINCIPAL IDEAL GRAPH AND THE SPECTRUM GRAPH

In this section, we have a comparison between the principal ideal graph and the spectrum graph. We begin with the following remark.

**Remark 4.1.** By [12, Example 1.9], there exists a local zero-dimensional ring  $R$  such that  $\mathbb{A}^*(R) \neq \emptyset$  and  $\mathbb{A}\mathbb{G}_s(R)$  is an empty graph, but  $\mathbb{A}\mathbb{G}_P(R)$  is non empty. Also, [18, Example 2.7] gives a non-connected spectrum graph of a local ring, but if  $R$  is a non-domain ring, then by Proposition 3.3,  $\mathbb{A}\mathbb{G}_P(R)$  is a connected graph.

**Lemma 4.2.** ([3, Theorem 10]) *Let  $R$  be a ring such that  $|\text{Min}(R)| = 1$ . If  $\mathbb{A}\mathbb{G}(R)$  is a triangle-free graph, then  $\mathbb{A}\mathbb{G}(R)$  is a star graph.*

**Proposition 4.3.** *Let  $R$  be a non-domain ring. Then*

- (1) *If  $\mathbb{A}\mathbb{G}_P(R)$  is a finite graph, then  $\mathbb{A}\mathbb{G}_s(R)$  is a finite graph.*
- (2) *If  $\mathbb{A}\mathbb{G}_P(R)$  is a complete graph, then  $\mathbb{A}\mathbb{G}_s(R) \cong K_1$  or  $\mathbb{A}\mathbb{G}_s(R) \cong K_2$ .*
- (3) *Assume that  $R$  is an Artinian ring such that  $\mathbb{A}\mathbb{G}(R)$  is a triangle-free graph, then  $\mathbb{A}\mathbb{G}_P(R)$  is a star graph if and only if  $\mathbb{A}\mathbb{G}_s(R)$  is a star graph.*

$\vec{\mathcal{E}}$

*Proof.* (1) It is clear by Proposition 3.1 .

(2) Assume that  $\mathbb{A}\mathbb{G}_P(R)$  is a complete graph. Then by Proposition 3.4,  $\mathbb{A}\mathbb{G}(R)$  is a complete graph. Since  $\mathbb{A}\mathbb{G}_s(R)$  is an induced subgraph of  $\mathbb{A}\mathbb{G}(R)$ ,  $\mathbb{A}\mathbb{G}_s(R)$  is a complete graph, too. Therefore, by [18, Proposition 2.5],  $\mathbb{A}\mathbb{G}_s(R) \cong K_1$  or  $\mathbb{A}\mathbb{G}_s(R) \cong K_2$ .

(3) ( $\Rightarrow$ ) Suppose that  $\mathbb{A}\mathbb{G}_P(R)$  is a star graph. Thus,  $\mathbb{A}\mathbb{G}(R)$  is a star graph (see Proposition 3.13). Since  $R$  is an Artinian ring, by [12, Lemma 2.5], either  $R = F_1 \times F_2$ , where  $F_1, F_2$  are fields, or  $R$  is a local ring. Therefore, either  $\mathbb{A}\mathbb{G}_s(R) \cong K_1$  or  $\mathbb{A}\mathbb{G}_s(R) \cong K_2$ .

( $\Leftarrow$ ) Assume that  $\mathbb{A}\mathbb{G}_s(R)$  is a star graph. Since  $R$  is an Artinian ring, by [18, Theorem 3.10], either  $R \cong F_1 \times F_2$  or  $R$  is a local ring. If  $R \cong F_1 \times F_2$ ,  $\mathbb{A}\mathbb{G}_P(R) \cong K_2$ . Let  $R$  be a local ring. Since  $R$  is an Artinian ring,  $|\text{Min}(R)| = 1$ . So by Lemma 4.2,  $\mathbb{A}\mathbb{G}(R)$  is a star graph and hence  $\mathbb{A}\mathbb{G}_P(R)$  is star, too.  $\square$

*The following example shows that the converse of Proposition 4.3 (1) and (2) does not hold.*

**Example 4.4.** Let  $R = F \times \mathbb{Z}$ , where  $F$  is a field. Then  $\mathbb{A}_s(R) = \{(0) \times \mathbb{Z}, F \times (0)\}$  and hence  $\mathbb{A}\mathbb{G}_s(R) \cong K_2$ . But  $\mathbb{A}\mathbb{G}_P(R)$  is an infinite star graph.  $\square$

**Corollary 4.5.** *Let  $R$  be an Artinian ring such that  $\mathbb{A}\mathbb{G}_P(R)$  is a complete graph. Then, either  $R$  is a local ring or  $R \cong F_1 \times F_2$ , where  $F_1, F_2$  are fields.*

*Proof.* It is clear with Proposition 4.3 and [18, Theorem 3.10].  $\square$

In the next proposition, we have a compression between diameter and girth of spectrum graph and principal ideal graph.

**Proposition 4.6.** *Let  $R$  be a non-domain ring. Then*

- (1)  $\text{gr}(\mathbb{A}\mathbb{G}_P(R)) \leq \text{gr}(\mathbb{A}\mathbb{G}_s(R))$ .
- (2) *If  $\mathbb{A}\mathbb{G}_s(R)$  is a connected graph, then  $\text{diam}(\mathbb{A}\mathbb{G}_s(R)) \leq \text{diam}(\mathbb{A}\mathbb{G}_P(R))$ .*
- (3) *Assume that  $R$  is a Noetherian ring and  $\text{diam}(\mathbb{A}\mathbb{G}_s(R)) = 2$ , then  $\text{diam}(\mathbb{A}\mathbb{G}_P(R)) = 2$ .*
- (4) *Assume that  $R$  is a Noetherian ring such that  $R$  is not Artinian and  $\text{diam}(\mathbb{A}\mathbb{G}_s(R)) = 1$ , then  $\text{diam}(\mathbb{A}\mathbb{G}_P(R)) = 2$ .*

$\vec{\mathcal{E}}$

*Proof.* (1) It is clear since for every ring  $R$ ,  $\text{gr}(\mathbb{A}\mathbb{G}_s(R)) = \infty$  (see [18, Corollary 2.4]).

(2) Assume that  $\mathbb{A}\mathbb{G}_s(R)$  is a connected graph. So by [18, Proposition 2.5],  $\text{diam}(\mathbb{A}\mathbb{G}_s(R)) \in \{0, 1, 2\}$ . If  $\text{diam}(\mathbb{A}\mathbb{G}_s(R)) = 0$  or  $1$ , then there is nothing to prove. Let  $\text{diam}(\mathbb{A}\mathbb{G}_s(R)) = 2$ , so there are  $P_1, P_2 \in \mathbb{A}_s(R)$  such that  $P_1 P_2 \neq (0)$ . Since  $P_1, P_2 \in \mathbb{A}\mathbb{G}(R)$ ,  $\text{diam}(\mathbb{A}\mathbb{G}(R)) \geq 2$  and hence  $\text{diam}(\mathbb{A}\mathbb{G}_P(R)) \geq 2 = \text{diam}(\mathbb{A}\mathbb{G}_s(R))$ .

(3) Suppose that  $\text{diam}(\mathbb{A}\mathbb{G}_s(R)) = 2$ , by [18, Proposition 2.3],  $\mathbb{A}\mathbb{G}_s(R) \cong K_{1,\infty}$ . Thus, there is a vertex in  $\mathbb{A}\mathbb{G}_s(R)$  which is adjacent to every other vertex in  $\mathbb{A}\mathbb{G}_s(R)$ . Since  $|\mathbb{A}_s(R)| > 2$ , by [18, Proposition 3.2], there is a vertex in  $\mathbb{A}\mathbb{G}(R)$  which is adjacent to every other vertex in  $\mathbb{A}\mathbb{G}(R)$ , so  $\text{diam}(\mathbb{A}\mathbb{G}(R)) \leq 2$ . Therefore,  $\text{diam}(\mathbb{A}\mathbb{G}_P(R)) \leq 2$ . By (2),  $2 = \text{diam}(\mathbb{A}\mathbb{G}_s(R)) \leq \text{diam}(\mathbb{A}\mathbb{G}_P(R))$ , so  $\text{diam}(\mathbb{A}\mathbb{G}_P(R)) = 2$ .

(4) Since  $\text{diam}(\mathbb{A}\mathbb{G}_s(R)) = 1$ ,  $\mathbb{A}\mathbb{G}_s(R) \cong K_2$  (see [18, Proposition 2.3]). Therefore, by [18, Proposition 3.6],  $\mathbb{A}\mathbb{G}(R)$  is a complete bipartite graph, so  $\text{diam}(\mathbb{A}\mathbb{G}_P(R)) \leq 2$ , by Proposition 3.8 (1). Note that  $R$  is not Artinian. So by [12, Theorem 2.10],  $\mathbb{A}\mathbb{G}_s(R)$  is a proper subgraph of  $\mathbb{A}\mathbb{G}(R)$  and hence

$|\mathbb{A}^*(R)| \geq 3$ . If  $\text{diam}(\mathbb{A}\mathbb{G}_P(R)) = 0$ , then by Proposition 3.6 (1),  $|\mathbb{A}^*(R)| = 1$ , a contradiction. Now assume that  $\text{diam}(\mathbb{A}\mathbb{G}_P(R)) = 1$ , so  $\mathbb{A}\mathbb{G}_P(R) \cong K_n$ , where  $n \geq 2$ . If  $n = 2$ , then Proposition 3.6 (2) implies that  $|\mathbb{A}^*(R)| = 2$ , a contradiction. For case  $n > 2$ , we have  $\mathbb{A}\mathbb{G}(R) \cong K_n$ , where  $n > 2$  (see Proposition 3.4). Thus  $\mathbb{A}\mathbb{G}(R)$  is not a complete bipartite graph, a contradiction. Therefore  $\text{diam}(\mathbb{A}\mathbb{G}_P(R)) \neq 0, 1$  and hence  $\text{diam}(\mathbb{A}\mathbb{G}_P(R)) = 2$ .  $\square$

**Proposition 4.7.** *Let  $R$  be an Artinian local ring such that  $\mathbb{A}\mathbb{G}_s(R)$  is a subgraph of  $\mathbb{A}\mathbb{G}_P(R)$ . Then  $\mathbb{A}\mathbb{G}_P(R)$  is a finite graph.  $\vec{\mathcal{E}}$*

*Proof.* Assume that  $(R, \mathcal{M})$  is an Artinian local ring. It is clear that  $\mathbb{A}\mathbb{G}_s(R) \cong K_1$ , where  $\mathbb{A}_s(R) = \{\mathcal{M}\}$ . Since  $\mathbb{A}\mathbb{G}_s(R)$  is a subgraph of  $\mathbb{A}\mathbb{G}_P(R)$ ,  $J(R) = \mathcal{M} = Rx$ , where  $0 \neq x \in R$ . Since  $R$  is an Artinian ring, there is  $k \in \mathbb{N}$ , such that  $x^k = 0$  and  $x^{k-1} \neq 0$ . Now, by [15, Theorem 9],  $(0) = Rx^k \subset Rx^{k-1} \subset \dots \subset Rx \subset R$  are the only ideals of  $R$ . Since  $R$  is an Artinian ring,  $\mathbb{A}_P(R) = \{J(R), Rx^2, \dots, Rx^{k-1}\}$ . Therefore,  $\mathbb{A}\mathbb{G}_P(R)$  is a finite graph.  $\square$

*The following proposition characterizes all Artinian rings  $R$  for which  $\mathbb{A}\mathbb{G}_s(R)$  is a subgraph of  $\mathbb{A}\mathbb{G}_P(R)$ .*

**Proposition 4.8.** *Let  $R$  be an Artinian ring. Then  $\mathbb{A}\mathbb{G}_s(R)$  is a subgraph of  $\mathbb{A}\mathbb{G}_P(R)$  if and only if  $R$  is a principal ideal ring.  $\vec{\mathcal{E}}$*

*Proof.* ( $\Leftarrow$ ) It is clear.

( $\Rightarrow$ ) Assume that  $\mathbb{A}\mathbb{G}_s(R)$  is a subgraph of  $\mathbb{A}\mathbb{G}_P(R)$ . Since  $R$  is an Artinian ring and  $\mathbb{A}\mathbb{G}_s(R)$  is a subgraph of  $\mathbb{A}\mathbb{G}_P(R)$ , by [12, Proposition 1.3],  $\text{Spec}(R) = \mathbb{A}_s(R) \subseteq \mathbb{A}_P(R)$ . Let  $\mathbf{X} = \{I : I \in \mathbb{I}(R) \setminus \mathbb{P}(R)\}$ . We claim that  $\mathbf{X} = \emptyset$ . Suppose on the contrary that  $\mathbf{X} \neq \emptyset$ . Since  $R$  is a Noetherian ring, we may assume that  $J$  is a Maximal element of  $\mathbf{X}$ . It is clear that  $J \notin \text{Spec}(R)$ , so there are  $x, y \in R$  such that  $xy \in J$  but  $x, y \notin J$ . Therefore,  $J \subsetneq J + Rx$ . Since  $J$  is a maximal element of  $\mathbf{X}$ ,  $J + Rx \in \mathbb{P}(R)$ . So there is  $z \in J + Rx$  such that  $J + Rx = Rz$ . Let  $\mathbf{Y} = \{r \in R : rz \in J\}$ . It is clear that  $J \subseteq \mathbf{Y}$  and  $z \in J + Rx$ . Since  $yz \in yJ + R(xy) \subseteq J$ ,  $y \in \mathbf{Y}$  and hence  $J \subsetneq \mathbf{Y}$ . Therefore  $\mathbf{Y} \in \mathbb{P}(R)$ . Let  $\mathbf{Y} = Rt$ , where  $t \in \mathbf{Y}$ . Now we claim that  $J = R(zt)$ . By definition,  $z\mathbf{Y} = zR(t) = R(zt) \subseteq J$ . On the other hand, if  $s \in J$ , then  $s \in J + Rx = Rz$ , so  $s = kz$  for some  $k$ , where clearly  $k \in \mathbf{Y}$ . Thus  $s \in z\mathbf{Y} = R(zt)$ . Therefore  $J = R(zt) \in \mathbb{P}(R)$ , a contradiction. We can conclude that  $\mathbf{X} = \emptyset$  as we claimed and we have  $R$  is a principal ideal ring.  $\square$

*In view of the above proposition, one may naturally ask, is  $R$  a principal ideal ring, when  $\mathbb{A}\mathbb{G}_s(R)$  is a principal ideal graph? The answer is no! The following example shows that the Artinian hypothesis is necessary in Proposition 4.8.*

**Example 4.9.** Let  $F$  be a field and

$$R = F[[X, Y]]/\langle XY \rangle.$$

Then  $R$  is a local ring with maximal ideal  $\mathcal{M} = Rx + Ry$ , where  $x = X + \langle XY \rangle$  and  $y = Y + \langle XY \rangle$ . Note that  $\mathbb{A}_s(R) = \{Rx, Ry\} \subseteq \mathbb{A}_P(R)$ , but  $R$  is not a principal ideal ring. E

**Proposition 4.10.** *Let  $R$  be a non-domain reduced ring. Then every vertex of  $\mathbb{A}\mathbb{G}_P(R)$  has finite degree and  $\mathbb{A}\mathbb{G}_s(R)$  is a connected graph if and only if  $R \cong F_1 \times F_2$ , where  $F_1, F_2$  are fields.  $\vec{\mathcal{E}}$*

*Proof.* ( $\Leftarrow$ ) It is clear.

( $\Rightarrow$ ) Since every vertex of  $\mathbb{A}\mathbb{G}_P(R)$  has finite degree, by Corollary 3.2,  $R \cong F_1 \times F_2 \times \cdots \times F_n$ , where  $n \geq 2$  and  $F_i$ 's are field. If  $n > 2$ . Then  $\mathbb{A}\mathbb{G}_s(R)$  is not a connected graph, thus  $n = 2$  and  $R \cong F_1 \times F_2$ .  $\square$

*We conclude this paper with the following proposition.*

**Proposition 4.11.** *Let  $R$  be a non-local Artinian ring. Then  $\mathbb{A}\mathbb{G}_P(R)$  is a complete bipartite graph if and only if  $\mathbb{A}\mathbb{G}_s(R) \cong K_2$ .  $\vec{\mathcal{E}}$*

*Proof.* ( $\Leftarrow$ ) Assume that  $\mathbb{A}\mathbb{G}_s(R) \cong K_2$ . Since  $R$  is an Artinian ring, [18, Theorem 3.10] implies that  $R \cong F_1 \times F_2$ , so  $\mathbb{A}\mathbb{G}_P(R)$  is a complete bipartite graph.

( $\Rightarrow$ ) Suppose that  $\mathbb{A}\mathbb{G}_P(R)$  is a complete bipartite graph with two parts  $X$  and  $Y$ . Since  $R$  is an Artinian ring which is not local,  $R \cong R_1 \times R_2$ , where  $R_1, R_2$  are two non-zero rings. We claim that  $R_1$  is a field. Suppose on the contrary that  $R_1$  is not a field, so there exists non-zero proper ideal  $J$  of  $R_1$ . Let  $a \in J$  and  $I = R_1a$ . Since  $R_1$  is an Artinian ring, [12, Proposition 1.3] implies that  $J \in \mathbb{A}^*(R_1)$  and hence  $I \in \mathbb{A}_P(R_1)$ . Therefore  $I \times R_2 = R(a, 1)$  and  $I \times R_2, (0) \times R_2 \in \mathbb{A}_P(R)$ . Since  $(I \times R_2)((0) \times R_2) \neq (0) \times (0)$ , both  $I \times R_2$  and  $(0) \times R_2$  are in  $X$  or  $Y$ . But  $R_1 \times (0)$  is adjacent to  $(0) \times R_2$  and is not adjacent to  $I \times R_2$ , a contradiction. Thus,  $R_1$  is a field and similarly,  $R_2$  is a field, too. Consequently  $\mathbb{A}\mathbb{G}_s(R) \cong K_2$ .  $\square$

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