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EXACT SEQUENCES OF EXTENDED d -HOMOLOGY

M.Z. KAZEMI BANEH*, S.N. HOSSEINI

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ABSTRACT. In this article, we show the existence of certain exact sequences with respect to two homology theories, called d -homology and extended d -homology. We present sufficient conditions for the existence of long exact extended d -homology sequence. Also we give some illustrative examples.

1. INTRODUCTION AND PRELIMINARIES

In [2, 4], the authors have introduced the notion of a kernel transformation d and have defined two homology functors with respect to d , called d -homology and extended d -homology. In this paper, we have proved the existence of some exact sequences containing these homologies.

In this section, we recall some of the necessary results obtained in [2, 4] and in section 2, we give the exact sequences involving d -homology and extended d -homology. If there are some conditions as some objects are subobject or some morphisms are epic or some morphisms are monic, then we have

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*Corresponding author

exact sequences involving these two homology theories and at the end there is the long exact extended d-homology sequence.

To this end, for a pointed category \mathcal{C} , following the notation of [2], we recall:

- For $f : A \rightarrow B$, the maps $K_f \xrightarrow{k_f} A$, $B \xrightarrow{c_f} C_f$ and $P_f \xrightarrow[\pi_2]{\pi_1} A$ are respectively the kernel, the cokernel and the kernel pair of f .
- The image I_f of f is the coequalizer of the kernel pair of f . f can be factorized to $f = m_f e_f$ that e_f is the coequalizer of the kernel pair of f .
- For a pair of maps $A \xrightarrow[g]{f} B$, the maps $Equ(f, g) \xrightarrow{equ(f, g)} A$ and $B \xrightarrow{coe(f, g)} Coe(f, g)$ are respectively the equalizer and the coequalizer of (f, g) .

Lemma 1.1. *Given the below diagram in which the squares are commutative and the rows are coequalizers, i is the unique map making the right square commute. Furthermore, i is a regular epi.*

$$\begin{array}{ccccc}
 A & \xrightarrow{f} & B & \xrightarrow{q} & C \\
 \downarrow r & & \downarrow g & & \downarrow i \\
 A' & \xrightarrow{f'} & B & \xrightarrow{1_B} & C' \\
 & & \downarrow g' & & \downarrow q'
 \end{array}$$

For a category \mathcal{C} with a zero object, pullback and pushout, let $\bar{\mathcal{C}}$ be the arrow category and $\widehat{\mathcal{C}}$ be the pair-chain category of \mathcal{C} . Let $K : \bar{\mathcal{C}} \rightarrow \mathcal{C}$ be the kernel functor and $I : \bar{\mathcal{C}} \rightarrow \mathcal{C}$ be the image functor.

Proposition 1.2. *The functor $j : \widehat{\mathcal{C}} \rightarrow \bar{\mathcal{C}}$ takes the object $(f, g) \in \widehat{\mathcal{C}}$ to j_{fg} and the morphism (α, β, γ) to $(I(\alpha, \beta), K(\beta, \gamma))$ and we have the following commutative diagram.*

$$\begin{array}{ccccccc}
 & & f & & & & \\
 & & \curvearrowright & & & & \\
 A & \xrightarrow{e_f} & I_f & \xrightarrow{j_{fg}} & K_g & \xrightarrow{k_g} & B \\
 \downarrow \alpha & & \downarrow I(\alpha, \beta) & & \downarrow K(\beta, \gamma) & & \downarrow \beta \\
 A' & \xrightarrow{e_{f'}} & I_{f'} & \xrightarrow{j_{f'g'}} & K_{g'} & \xrightarrow{k_{g'}} & B' \\
 & & \curvearrowleft & & & & \\
 & & f' & & & &
 \end{array}$$

Remark 1.3. For $(f, g) \in \widehat{\mathcal{C}}$, $f = m_f e_f$ in which $e = coeq(\pi_1, \pi_2)$. Let $(f, g) \in \widehat{\mathcal{C}}$. By the above diagram, $k_g j_{fg} = m_f$ and k_f is monic. So j_{fg} is monic if and only if m_f is monic. In a pointed regular (homological, semiabelian or abelian) category j_{fg} is monic, see [1, 2].

Definition 1.4. With $S : \mathcal{C} \rightarrow \mathcal{C}$ the squaring functor, taking $a \xrightarrow{f} b$ to $a^2 \xrightarrow{f^2} b^2$, a kernel transformation is a natural transformation $d : S \circ K \rightarrow K : \widehat{\mathcal{C}} \rightarrow \mathcal{C}$ such that for all (f, g) in $\widehat{\mathcal{C}}$, the pullback, $j_{fg}^* : R_{fg} \rightarrow K_g^2$, of j_{fg} along d_g and the coequalizer of the pair $j_1 = pr_1 j_{fg}^*$ and $j_2 = pr_2 j_{fg}^*$ exist, where pr_1 and pr_2 are the projection maps.

Theorem 1.5. The d -homology functor $H^d : \widehat{\mathcal{C}} \rightarrow \mathcal{C}$ takes $(f, g) \in \widehat{\mathcal{C}}$ to $H_{fg}^d = Coe(j_1, j_2)$ and the morphism (α, β, γ) to $H^d(\alpha, \beta, \gamma)$. We have the following commutative diagram.

$$\begin{array}{ccc} K_g & \xrightarrow{q_1} & H_{fg}^d \\ K(\beta, \gamma) \downarrow & & \downarrow H^d(\alpha, \beta, \gamma) \\ K_{g'} & \xrightarrow{q'_1} & H_{f'g'}^d \end{array}$$

Proposition 1.6. Let R be a commutative ring with unity. Any kernel transformation in $R\text{mod}$ is of the form $d = rpr_1 + spr_2 = +(r \times s)$, for some $r, s \in R$. In particular any kernel transformation d in Abgrp is of the form $d = +(r \times s)$, for some $r, s \in \mathbb{Z}$.

Example 1.7. Let $\mathcal{C} = R\text{mod}$ and $d = rpr_1 + spr_2 = +(r \times s)$ with $r, s \in R$. Let (f, g) be a pair-chain. Then $R_{fg} = \{(a, b) \in K_g^2 | ra + sb \in I_f\}$, j^* is the inclusion and $H_{fg} = \frac{K_g}{(j_1 - j_2)(R_{fg})} = \{[a] : a \in K_g\}$, where $[a] = \{b | r(a - b) \in (r + s)K_g + I_f\} = \{b | s(a - b) \in (r + s)K_g + I_f\}$ is the equivalence class under the equivalence relation $a \sim b$ if and only if $\exists m, n \in K_g$ such that $a - b = m - n$ and $rm + sn \in I_f$.

Example 1.8. As a special case of above Example, for $d = +(r \times 1)$ or $d = +(1 \times r)$ with $r \in R$, we have $H_{fg}^d = \frac{K_g}{(1+r)K_g + I_f}$.

We call the homology which is defined in the [1, 6] the standard homology.

Definition 1.9. The standard homology or s -homology functor H^s , takes $(f, g) \in \widehat{\mathcal{C}}$ to $Coker(j_{fg})$, and for a pair chain map $(\alpha, \beta, \gamma) : (f, g) \rightarrow (f', g')$, we have the following commutative diagram.

$$\begin{array}{ccc} K_g & \xrightarrow{q} & H_{fg}^s \\ k(\beta, \gamma) \downarrow & & \downarrow H^s(\alpha, \beta, \gamma) \\ K_{g'} & \xrightarrow{q'} & H_{f'g'}^s \end{array}$$

where $q = coker(j_{fg})$ and $q' = coker(j_{f'g'})$.

Theorem 1.10. See [3]. For a kernel transformation d in \mathcal{C} there is a natural transformation $p : H^s \rightarrow H^d$. Furthermore p is pointwise regular epic.

In an abelian category \mathcal{C} , If the kernel transformation $d = pr_1 - pr_2 = -$, then we have: $H^- = H^s$. For a pointed category \mathcal{C} with pulback and pushout, following the notation of [4], we recall:

Notation 1. Let $m : A \longrightarrow C$ and $j : B \longrightarrow C$ be two maps in \mathcal{C} . Define $A+_C B$ also denoted by $A + B$ by the pushout:

$$\begin{array}{ccc} P_{jm} & \xrightarrow{\alpha} & A \\ \gamma \downarrow & \text{po} & \downarrow h \\ B & \xrightarrow{i} & A+_C B \end{array}$$

where $B \xleftarrow{\gamma} P_{jm} \xrightarrow{\alpha} A$ is the pullback of (j, m) .

With $S : \mathcal{C} \longrightarrow \mathcal{C}$ the squaring functor and a kernel transformation d , for all $(f, g) \in \widehat{\mathcal{C}}$ and diagonal map Δ we have the maps $m_{d_g \Delta_g} : I_{d_g \Delta_g} \longrightarrow K_g$ (such that $m_{d_g \Delta_g} e_{d_g \Delta_g} = d_g \Delta_g$) and $j_{fg} : I_f \longrightarrow K_g$. The sum $I_{d_g \Delta_g} + I_f$ is therefore obtained by the following diagrams:

$$\begin{array}{ccc} P_{jm} \xrightarrow{\alpha} I_{d_g \Delta_g} & \text{and} & P_{jm} \xrightarrow{\alpha} I_{d_g \Delta_g} \\ \gamma \downarrow \quad \text{pb} \quad \downarrow m_{d_g \Delta_g} & & \gamma \downarrow \quad \text{po} \quad \downarrow h \\ I_f \xrightarrow{j_{fg}} K_g & & I_f \xrightarrow{i} I_{d_g \Delta_g} + I_f \end{array}$$

Since $m_{d_g \Delta_g} \alpha = j_{fg} \gamma$, there is a unique map $\beta : I_{d_g \Delta_g} + I_f \longrightarrow K_g$ making the following triangles commutative.

$$\begin{array}{ccc} P_{jm} & \xrightarrow{\alpha} & I_{d_g \Delta_g} \\ \gamma \downarrow & \text{po} & \downarrow h \\ I_f & \xrightarrow{i} & I_{d_g \Delta_g} + I_f \\ & \searrow j_{fg} & \downarrow \beta \\ & & K_g \end{array}$$

$m_{d_g \Delta_g}$

Theorem 1.11. The extended d -homology functor $\bar{H}^d : \widehat{\mathcal{C}} \longrightarrow \mathcal{C}$ takes $(f, g) \in \widehat{\mathcal{C}}$ to $\bar{H}_{fg}^d = C_\beta$ and the morphism $(\sigma, \delta, \zeta) : (f, g) \longrightarrow (f', g')$ in $\widehat{\mathcal{C}}$ to the unique morphism $\bar{H}^d(\sigma, \delta, \zeta)$ that we have the following commutative diagram.

$$\begin{array}{ccc} K_g & \xrightarrow{c_\beta} & \bar{H}_{fg}^d \\ K(\delta, \zeta) \downarrow & & \downarrow \bar{H}^d(\sigma, \delta, \zeta) \\ K_{g'} & \xrightarrow{c_{\beta'}} & \bar{H}_{f'g'}^d \end{array}$$

Theorem 1.12. Let d be a kernel transformation in \mathcal{C} . There is a natural transformation $p : H^s \longrightarrow \bar{H}^d$. Furthermore p is pointwise regular epic.

Proof. By 1.1 for $(f, g) \in \widehat{\mathcal{C}}$, p_{fg} is the unique map in which the following diagram commutes.

$$\begin{array}{ccccc}
 I_f & \xrightarrow{j} & K_g & \xrightarrow{c_j} & H_{fg}^s \\
 \downarrow i & & \downarrow 1_{K_g} & & \downarrow p_{fg} \\
 I_f + I_{d_g \Delta_g} & \xrightarrow{\beta} & K_g & \xrightarrow{c_\beta} & \bar{H}_{fg}^d
 \end{array}$$

Naturality of p is straightforward. ■

Lemma 1.13. *If for $(f, g) \in \widehat{\mathcal{C}}$, $i : I_f \rightarrow I_f + I_{d_g \Delta_g}$ is epic, then $p_{fg} : H_{fg}^s \cong \bar{H}_{fg}^d$ is an isomorphism.*

Proposition 1.14. *Let \mathcal{C} be an abelian category. For $(f, g) \in \widehat{\mathcal{C}}$, $p_{fg} : H_{fg}^s \cong \bar{H}_{fg}^d$ is an isomorphism if and only if $I_{d_g \Delta_g}$ is a subobject of I_f .*

Lemma 1.15. *Let \mathcal{C} be an abelian category and $d = +(r \times s)$. If for all A , $r : A \rightarrow A$ or $s : A \rightarrow A$ is monic, then $H^d \cong_n \bar{H}^d$.*

2. EXACT SEQUENCES

Let \mathcal{C} be an abelian category and d be a kernel transformation. For a \mathcal{C} -chain:

$$\dots \longrightarrow C_{n+1} \xrightarrow{\delta_{n+1}} C_n \xrightarrow{\delta_n} C_{n-1} \longrightarrow \dots$$

we establish the following notation.

$k_n = k_{\delta_n} : K_n \rightarrow C_n$, $I_n = I_{\delta_{n+1}}$, $\Delta_n : K_n \rightarrow K_n^2$ is the diagonal map.

We have the following commutative diagram.

$$\begin{array}{ccccc}
 K_n & \xrightarrow{\Delta_n} & K_n^2 & \xrightarrow{d_n} & K_n \\
 \downarrow e_{d_n \Delta_n} & & \nearrow m_{d_n \Delta_n} & & \\
 I_{d_n \Delta_n} & & & &
 \end{array}$$

We also define:

$$\begin{array}{ccccc}
 I_{d_n \Delta_n} \cap I_n & \xrightarrow{\alpha_n} & I_{d_n \Delta_n} & & \\
 \downarrow \gamma_n & \text{po} & \downarrow h_n & \searrow m_{d_n \Delta_n} & \\
 I_n & \xrightarrow{i_n} & I_{d_n \Delta_n} + I_n & \xrightarrow{\beta_n} & K_n \\
 & \searrow j_n & & &
 \end{array}$$

where the outer square is a pullback.

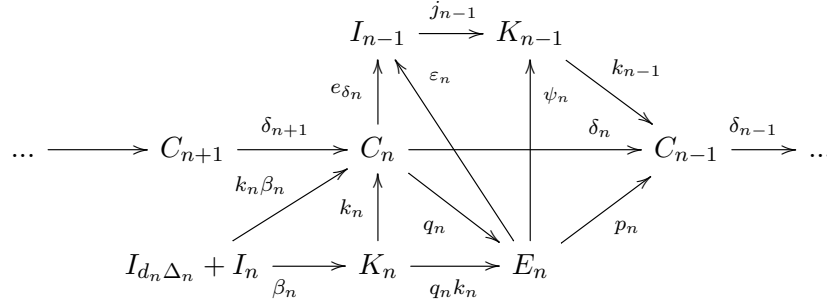
Let $q_n = \text{coker}(k_n \beta_n) : C_n \rightarrow E_n$. Since $\delta_n k_n = 0$, $\delta_n k_n \beta_n = 0$, and so there is a unique map

$p_n : E_n \rightarrow C_{n-1}$, such that $p_n q_n = \delta_n$. Since $\delta_{n-1} \delta_n = 0$, $\delta_{n-1} p_n q_n = 0$ and since q_n

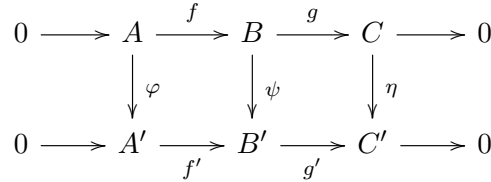
is epic, we get $\delta_{n-1}p_n = 0$. Hence there is a unique map $\psi_n : E_n \longrightarrow K_{n-1}$ such that $k_{n-1}\psi_n = p_n$.

Now $\delta_n = m_{\delta_n}e_{\delta_n}$ and $\delta_n k_n = 0$, so $m_{\delta_n}e_{\delta_n}k_n = 0$. Since m_{δ_n} is monic, $e_{\delta_n}k_n = 0$ and so $e_{\delta_n}k_n\beta_n = 0$. Hence there is a unique map $\varepsilon_n : E_n \longrightarrow I_{n-1}$ such that $\varepsilon_n q_n = e_{\delta_n}$. Since e_{δ_n} is epic, so is ε_n . We have $k_{n-1}\psi_n q_n = p_n q_n = \delta_n = m_{\delta_n}e_{\delta_n} = k_{n-1}j_{n-1}\varepsilon_n q_n$ and since k_{n-1} is monic and q_n is epic, $\psi_n = j_{n-1}\varepsilon_n$.

The following diagram summarizes the above arguments.



Proposition 2.1. See [1, 6]. Let \mathcal{C} be an abelian category. Suppose the following diagram in \mathcal{C} commutes and has exact rows.



Then

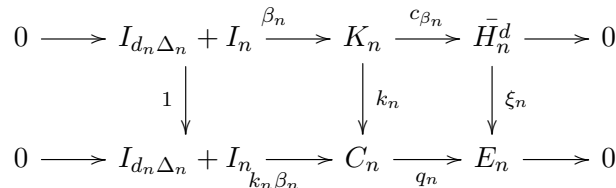
- (a) If φ is an iso, then ψ is monic if and only if η is monic.
- (b) If η is an iso, then ψ is epic if and only if φ is epic.

Theorem 2.2. Let d be a kernel transformation. For each n , we have:

- (a) $K_{\psi_n} = \bar{H}_n^d$.
- (b) $C_{\psi_n} = H_{n-1}^s$.

Proof. (a) Since $q_n k_n \beta_n = 0$, there exists a unique morphism ξ_n , such that $\xi_n c_{\beta_n} = q_n k_n$.

So we have the following commutative diagram with exact rows:



and therefore by above Proposition, ξ_n is monic.

Now $k_{n-1}\psi_n \xi_n c_{\beta_n} = p_n q_n k_n = \delta_n k_n = 0$, k_{n-1} is monic and c_{β_n} is epic, so $\psi_n \xi_n = 0$. Since k_{ψ_n} is a kernel, there is a unique map $\lambda_n : \bar{H}_n^d \longrightarrow K_{\psi_n}$ such that $k_{\psi_n} \lambda_n = \xi_n$. Since ξ_n is monic, so is λ_n .

On the other hand, $k_{n-1}\psi_n q_n k_n = 0$, $\psi_n q_n k_n = 0$, and so there is a unique map ξ'_n , such that $k_{\psi_n} \xi'_n = q_n k_n$. We have $k_n = \ker(\delta_n) = \ker(p_n q_n) = \ker(k_{n-1} j_{n-1} \varepsilon_n q_n) = \ker(\varepsilon_n q_n)$ and $\ker(\psi_n) = \ker(j_{n-1} \varepsilon_n) = \ker(\varepsilon_n)$. So we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_n & \xrightarrow{k_n} & C_n & \xrightarrow{\varepsilon_n q_n} & D & \longrightarrow & 0 \\ & & \xi'_n \downarrow & & \downarrow q_n & & \downarrow 1 & & \\ 0 & \longrightarrow & K_{\psi_n} & \xrightarrow{k_{\psi_n}} & E_n & \xrightarrow{\varepsilon_n} & D & \longrightarrow & 0 \end{array}$$

and therefore by above Proposition, ξ'_n is epic.

Since $k_{\psi_n} \lambda_n c_{\beta_n} = \xi_n c_{\beta_n} = q_n k_n = k_{\psi_n} \xi'_n$ and k_{ψ_n} is monic, $\lambda_n c_{\beta_n} = \xi'_n$. Since ξ'_n is epic, so is λ_n . So λ_n is a bimorphism and therefore an isomorphism. Hence $\bar{H}_n^d \cong K_{\psi_n}$.

(b) We have $c_{\psi_n} = \text{coker}(\psi_n) = \text{coker}(j_{n-1} \varepsilon_n) = \text{coker}(j_{n-1}) = H_{n-1}^s$. ■

Example 2.3. In the category of $R\text{mod}$, $\psi_n : \frac{C_n}{I_{d_n \Delta_n} + I_n} \longrightarrow K_{n-1}$ is the map defined by $\psi_n(x + I_{d_n \Delta_n} + I_n) = \delta_n(x)$. We have $\bar{H}_n^d = K_{\psi_n} = \frac{K_n}{I_{d_n \Delta_n} + I_n}$. Also $H_{n-1}^s = C_{\psi_n} = \frac{K_{n-1}}{I_{n-1}}$, as expected.

Lemma 2.4. Let \mathcal{C} be an abelian category, d be a kernel transformation and the sequence

$$0 \longrightarrow C'_* \xrightarrow{\nu_*} C_* \xrightarrow{\pi_*} C''_* \longrightarrow 0$$

of \mathcal{C} -chains be short exact. Then:

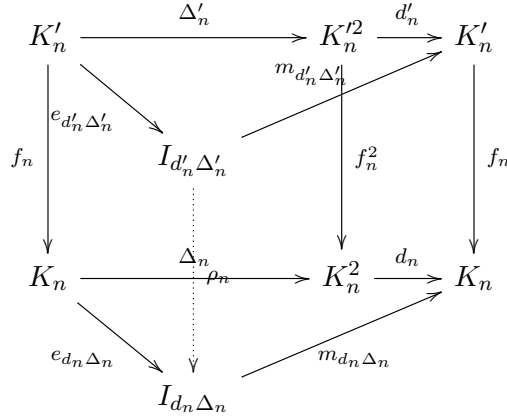
(a) With $f_n = K(\nu_n, \nu_{n-1})$, there is a unique map $l_n : I'_n + I_{d'_n \Delta'_n} \longrightarrow I_n + I_{d_n \Delta_n}$ such that the following diagram commutes.

$$\begin{array}{ccc} I'_n + I_{d'_n \Delta'_n} & \xrightarrow{\beta'_n} & K'_n \\ l_n \downarrow & & \downarrow f_n \\ I_n + I_{d_n \Delta_n} & \xrightarrow{\beta_n} & K_n \end{array}$$

(b) With $f'_n = K(\pi_n, \pi_{n-1})$, there is a unique map $l'_n : I_n + I_{d_n \Delta_n} \longrightarrow I''_n + I_{d''_n \Delta''_n}$ such that the following diagram commutes.

$$\begin{array}{ccc} I_n + I_{d_n \Delta_n} & \xrightarrow{\beta_n} & K_n \\ l'_n \downarrow & & \downarrow f'_n \\ I''_n + I_{d''_n \Delta''_n} & \xrightarrow{\beta''_n} & K''_n \end{array}$$

Proof. (a) By definition of Δ_n we have $f_n^2 \Delta'_n = \Delta_n f_n$. Naturality of d implies $f_n d'_n = d_n f_n^2$. So $f_n d'_n \Delta'_n = d_n f_n^2 \Delta_n = d_n \Delta_n f_n$. So there is a unique map $\rho_n = I(f_n, f_n)$ such that $\rho_n e_{d'_n \Delta'_n} = e_{d_n \Delta_n} f_n$ as the following diagram shows.



We have $f_n m'_{d_n \Delta_n} e_{d'_n \Delta'_n} = f_n d'_n \Delta'_n = d_n \Delta_n f_n = m_{d_n \Delta_n} e_{d_n \Delta_n} f_n = m_{d_n \Delta_n} \rho_n e_{d'_n \Delta'_n}$. Since $e_{d'_n \Delta'_n}$ is epic, we get:

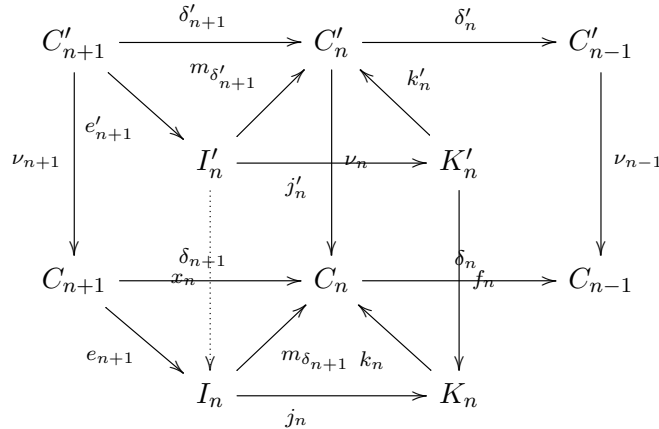
$$f_n m_{d'_n \Delta'_n} = m_{d_n \Delta_n} \rho_n \tag{1}$$

Since $k_n f_n = \nu_n k'_n$, f_n is monic and since $m_{d'_n \Delta'_n}$ is also monic, ρ_n is monic.

There is a unique map $x_n = I(\nu_{n+1}, \nu_n)$, such that:

$$x_n e'_{n+1} = e_{n+1} \nu_{n+1} \tag{2}$$

as the following diagram shows.

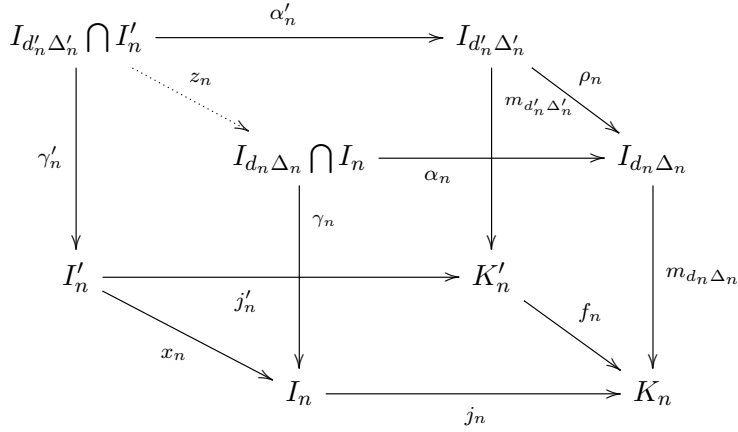


Next we show:

$$f_n j'_n = j_n x_n \tag{3}$$

We know $k_n f_n = \nu_n k'_n$, $k'_n j'_n = m_{\delta'_{n+1}}$ and $k_n j_n = m_{\delta_{n+1}}$. So $k_n f_n j'_n e'_{n+1} = \nu_n k'_n j'_n e'_{n+1} = \nu_n m_{\delta'_{n+1}} e'_{n+1} = \nu_n \delta'_{n+1} = \delta_{n+1} \nu_{n+1} = m_{\delta_{n+1}} e_{n+1} \nu_{n+1} = k_n j_n e_{n+1} \nu_{n+1} = k_n j_n x_n e'_{n+1}$. k_n is monic and e'_{n+1} is epic, so $f_n j'_n = j_n x_n$. Since f_n and j'_n are monic, x_n is monic.

By (1), (2) and (3) we have $m_{d_n \Delta_n} \rho_n \alpha'_n = f_n m_{d'_n \Delta'_n} \alpha'_n = f_n j'_n \gamma'_n = j_n x_n \gamma'_n$. Since the front and back squares in the following diagram are pullbacks,



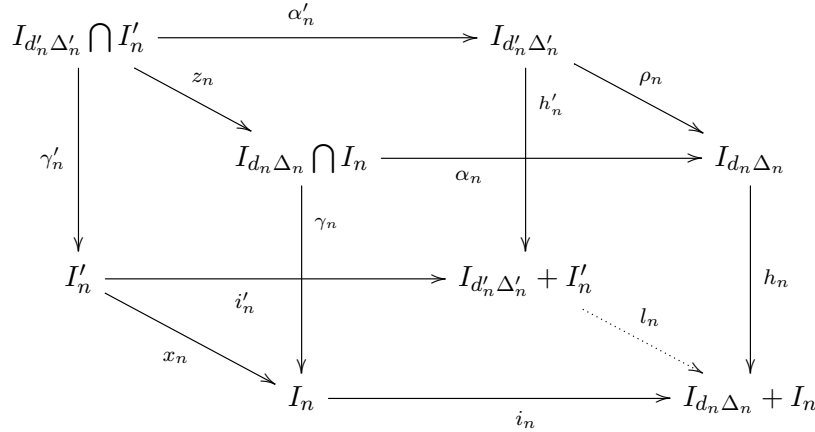
there is a unique map z_n , such that:

$$\gamma_n z_n = x_n \gamma'_n \tag{4}$$

$$\alpha_n z_n = \rho_n \alpha'_n \tag{5}$$

α'_n , being pullback of a monic, is monic, and since ρ_n is monic, z_n is monic.

by (4) and (5) we have $h_n \rho_n \alpha'_n = h_n \alpha_n z_n = i_n \gamma_n z_n = i_n x_n \gamma'_n$. Since the following front and back squares are pushouts,



there is a unique map l_n , such that $l_n i'_n = i_n x_n$ and $l_n h'_n = h_n \rho_n$. Now $j_n x_n = f_n j'_n$, so $\beta_n l_n i'_n = \beta_n i_n x_n = j_n x_n = f_n j'_n = f_n \beta'_n i'_n$. Therefore:

$$\beta_n l_n i'_n = f_n \beta'_n i'_n \tag{6}$$

Also $m_{d_n \Delta_n} \rho_n = f_n m_{d'_n \Delta'_n}$, so $\beta_n l_n h'_n = \beta_n h_n \rho_n = m_{d_n \Delta_n} \rho_n = f_n m_{d'_n \Delta'_n} = f_n \beta'_n h'_n$. Therefore:

$$\beta_n l_n h'_n = f_n \beta'_n h'_n \tag{7}$$

By (6) and (7) and pushoutness of $I'_{d'_n \Delta'_n} + I'_n$, we have $\beta_n l_n = f_n \beta'_n$ as desired. The uniqueness is obvious by the fact that β_n is monic.

(b) Similar to part (a), there exists a unique morphism $\rho'_n = I(f'_n, f'_n)$ such that:

$$\rho'_n e_{d_n \Delta_n} = e_{d'_n \Delta'_n} f'_n$$

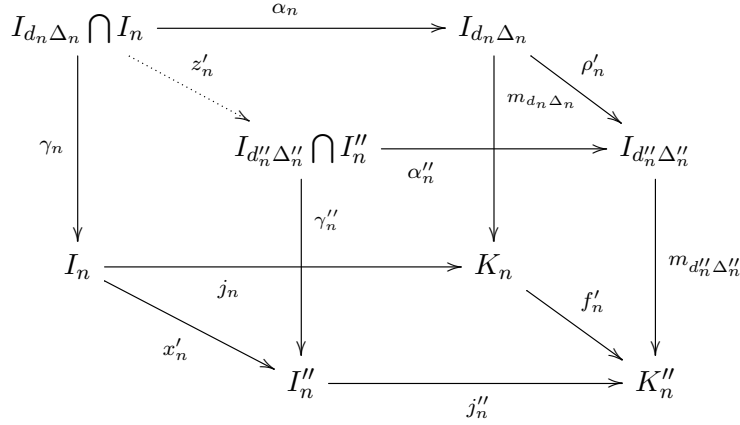
$$f'_n m_{d_n \Delta_n} = m_{d'_n \Delta'_n} \rho'_n$$

and there exists a unique morphism $x'_n = I(\pi_{n+1}, \pi_n)$ such that:

$$x'_n e_{n+1} = e''_{n+1} \pi_{n+1}$$

Since e''_{n+1} and π_{n+1} are epic, x'_n is epic.

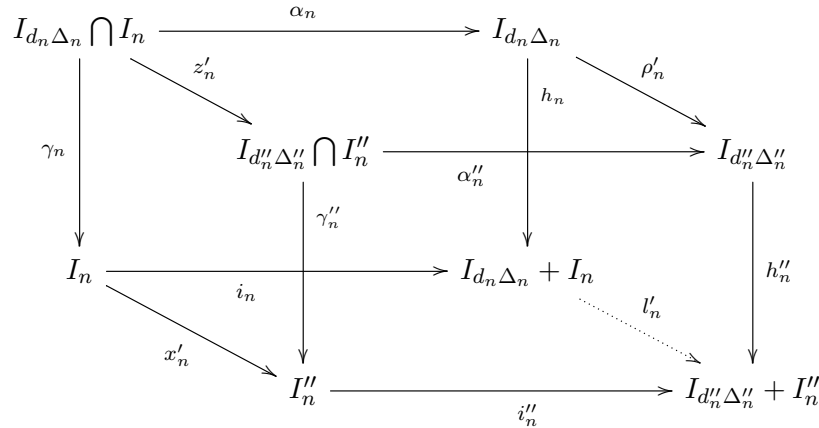
Also there exists a unique morphism $z'_n : I_{d_n \Delta_n} \cap I_n \longrightarrow I_{d''_n \Delta''_n} \cap I''_n$ such that the following diagram commutes.



One can easily verify that:

$$\begin{aligned}
 \gamma''_n z'_n &= x'_n \gamma_n \\
 \alpha''_n z'_n &= \rho'_n \alpha_n
 \end{aligned}$$

Finally we have a unique morphism $l'_n : I_{d_n \Delta_n} + I_n \longrightarrow I_{d''_n \Delta''_n} + I''_n$ such that the following diagram commutes.



The following equalities are an easy consequence.

$$\begin{aligned}
 \beta''_n l'_n i_n &= f'_n \beta_n i_n \\
 \beta''_n l'_n h_n &= f'_n \beta_n h_n
 \end{aligned}$$

By the above equalities and pushoutness of $I_{d_n \Delta_n} + I_n$, we have $\beta''_n l'_n = f'_n \beta_n$ as desired. ■

Lemma 2.5. *Let \mathcal{C} be an abelian category and the sequence*

$$0 \longrightarrow C'_* \xrightarrow{\nu_*} C_* \xrightarrow{\pi_*} C''_* \longrightarrow 0$$

of \mathcal{C} -chains be short exact. Then we have the following exact sequence.

$$0 \longrightarrow K'_n \xrightarrow{f_n} K_n \xrightarrow{f'_n} K''_n$$

Proof. See [6]. ■

Lemma 2.6. *Let \mathcal{C} be an abelian category, d be a kernel transformation and the sequence*

$$0 \longrightarrow C'_* \xrightarrow{\nu_*} C_* \xrightarrow{\pi_*} C''_* \longrightarrow 0$$

of \mathcal{C} -chains be short exact. Then we have the following sequence with g'_n epic and $g'_n g_n = 0$.

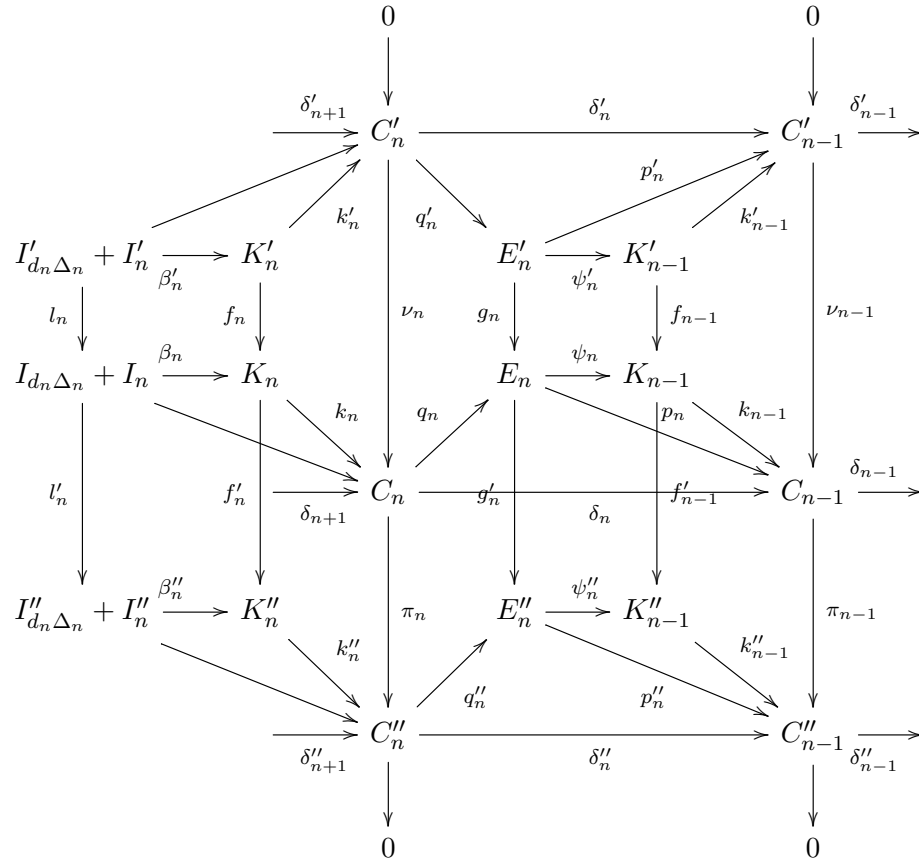
$$E'_n \xrightarrow{g_n} E_n \xrightarrow{g'_n} E''_n \longrightarrow 0$$

Proof. Set $f_n = K(\nu_n, \nu_{n-1})$. Then $\nu_n k'_n = k_n f_n$ and since k'_n and ν_1 are monic, f_n is monic. By Lemma 2.4, there is a unique map l_n such that $\beta_n l_n = f_n \beta'_n$. Since $q_n \nu_n k'_n \beta'_n = q_n k_n f_n \beta'_n = q_n k_n \beta_n l_n = 0 l_n = 0$, there is a unique map g_n , such that $g_n q'_n = q_n \nu_n$. Since q''_n and π_n are epic and $g'_n q_n = q''_n \pi_n$, g'_n is epic. We have $g'_n g_n q'_n = g'_n q_n \nu_n = q''_n \pi_n \nu_n = q''_n 0 = 0$. Since q'_n is epic, $g'_n g_n = 0$. ■

Lemma 2.7. *If l'_n epic, then the following sequence is exact.*

$$E'_n \xrightarrow{g_n} E_n \xrightarrow{g'_n} E''_n \longrightarrow 0$$

Proof. It suffices to show that g'_n is a cokernel of g_n . We make use of the following diagram.



Suppose $r : E_n \longrightarrow A$ such that $rg_n = 0$. Then $rg_n q'_n = 0$ and $rq_n \nu_n = 0$. Since $\pi_n = \text{coker}(\nu_n)$, there is $s : C''_n \longrightarrow A$ such that $s\pi_n = rq_n$. Since $q_n = \text{coker}(k_n \beta_n)$,

$sk''_n\beta''_nl'_n = s\pi_n k_n \beta_n = r q_n k_n \beta_n = 0$. By hypothesis l'_n is epic, so $sk''_n\beta''_n = 0$. Since $q''_n = \text{coker}(k''_n\beta''_n)$, there is a unique $t : E''_n \longrightarrow A$ such that $tq''_n = s$. It easily follows that t is the unique map such that $tg'_n = r$. Hence $g'_n = \text{coker}(g_n)$. ■

With f'_n, ρ'_n and l'_n as in the proof of Lemma 2.4, we have:

Lemma 2.8. *In the abelian category \mathcal{C}*

- (a) *If f'_n is epic, then so is ρ'_n .*
- (b) *If ρ'_n is epic, then so is l'_n .*
- (c) *If $I_{d''_n\Delta''_n}$ is a subobject of I''_n , then l'_n is epic.*

Proof. (a) Since $\rho'_n e_{d_n\Delta_n} = e_{d''_n\Delta''_n} f'_n$ and $e_{d''_n\Delta''_n}$ is epic, the result follows.

(b) If $al'_n = bl'_n$, then $al'_n h''_n = bl'_n h''_n$ and so $ah'_n \rho'_n = bh'_n \rho'_n$. ρ'_n epic implies $ah'_n = bh'_n$.

Since x'_n is epic, $ai''_n = bi''_n$. pushoutness of $I_{d''_n\Delta''_n} + I''_n$ implies $a = b$ and l'_n is epic.

(c) If $I_{d''_n\Delta''_n}$ is a subobject of I''_n , then $i''_n = 1$. Since x'_n is epic, so is the morphism

$$l'_n : I_{d_n\Delta_n} + I_n \longrightarrow I_{d''_n\Delta''_n} + I''_n = I''_n .$$

Theorem 2.9. *Let \mathcal{C} be an abelian category, d be a kernel transformation and the sequence*

$$0 \longrightarrow C'_* \xrightarrow{\nu_*} C_* \xrightarrow{\pi_*} C''_* \longrightarrow 0$$

of \mathcal{C} -chains be short exact. Then we have the following commutative diagram.

$$\begin{array}{ccccccc} E'_n & \xrightarrow{g_n} & E_n & \xrightarrow{g'_n} & E''_n & \longrightarrow & 0 \\ \psi'_n \downarrow & & \psi_n \downarrow & & \downarrow \psi''_n & & \\ 0 \longrightarrow & K'_{n-1} & \xrightarrow{f_{n-1}} & K_{n-1} & \xrightarrow{f'_{n-1}} & K''_{n-1} & \end{array}$$

Proof. All the maps have been introduced before, see Lemmas 2.4 and 2.6. We have

$$k_{n-1} f_{n-1} \psi'_n q'_n = \nu_{n-1} k'_{n-1} \psi'_n q'_n = \nu_{n-1} p'_n q'_n = \nu_{n-1} \delta'_{n-1} = \delta_n \nu_n = p_n q_n \nu_n = p_n g_n q'_n = k_{n-1} \psi_n g_n q'_n.$$

Since k_{n-1} is monic and q'_n is epic, $f_{n-1} \psi'_n = \psi_n g_n$, showing the left square is commutative.

We can similarly show $f'_{n-1} \psi_n = \psi''_n g'_n$. ■

Theorem 2.10. *Let \mathcal{C} be an abelian category, d be a kernel transformation and the sequence*

$$0 \longrightarrow C'_* \xrightarrow{\nu_*} C_* \xrightarrow{\pi_*} C''_* \longrightarrow 0$$

of \mathcal{C} -chains be short exact. If l'_n be epic, then, for each n , we have the following exact sequence.

$$\bar{H}_n^d \longrightarrow \bar{H}_n^d \longrightarrow \bar{H}''_n^d \longrightarrow H'^s_{n-1} \longrightarrow H^s_{n-1} \longrightarrow H''^s_{n-1}$$

Proof. By Lemmas 2.4 and 2.6, the diagram of Theorem 2.9 has exact rows. So there is a map

$\chi_n : \ker(\psi''_n) \longrightarrow \text{coker}(\psi'_n)$, see [6], such that the following sequence is exact:

$$K_{\psi'_n} \longrightarrow K_{\psi_n} \longrightarrow K_{\psi''_n} \xrightarrow{\chi_n} C_{\psi'_n} \longrightarrow C_{\psi_n} \longrightarrow C_{\psi''_n}$$

The result follows by Theorem 2.2. ■

Corollary 2.11. *Let \mathcal{C} be an abelian category and $d = +(r \times s)$, $r, s \in \mathbb{Z}$. If for all $A \in \mathcal{C}$, $r : A \longrightarrow A$ or for all A , $s : A \longrightarrow A$ is monic, then for each short exact sequence of \mathcal{C} -chains:*

$$0 \longrightarrow C'_* \xrightarrow{\nu_*} C_* \xrightarrow{\pi_*} C''_* \longrightarrow 0$$

for which l'_n is epic, there is the following exact sequence.

$$H_n'^d \longrightarrow H_n^d \longrightarrow H_n''^d \longrightarrow H_{n-1}'^s \longrightarrow H_{n-1}^s \longrightarrow H_{n-1}''^s$$

Proof. The result follows by Lemma 1.15 and Theorem 2.10. ■

Corollary 2.12. *Let $I_{d''\Delta_n''}$ be a subobject of I_n'' . Then the following diagram is exact.*

$$\bar{H}_n'^d \longrightarrow \bar{H}_n^d \longrightarrow \bar{H}_n''^d \longrightarrow H_{n-1}'^s \longrightarrow H_{n-1}^s \longrightarrow H_{n-1}''^s$$

Proof. Follows from Proposition 1.14 and Lemma 2.8. ■

Example 2.13. In the category of $R\text{mod}$, $I_{d''\Delta_n''} = (r + s)K_n''$. So if $(r + s)K_n''$ is a submodule of I_n'' , then the sequence in the above corollary is exact.

Proposition 2.14. *Let \mathcal{C} be an abelian category, d be a kernel transformation and the sequence*

$$0 \longrightarrow C'_* \xrightarrow{\nu_*} C_* \xrightarrow{\pi_*} C''_* \longrightarrow 0$$

of \mathcal{C} -chains be short exact, for which l'_n is epic. Then there is the following chain that for each n is exact at \bar{H}_n^d corner.

$$\dots \longrightarrow \bar{H}_n'^d \longrightarrow \bar{H}_n^d \longrightarrow \bar{H}_n''^d \xrightarrow{\bar{\chi}_n} \bar{H}_{n-1}'^d \longrightarrow \bar{H}_{n-1}^d \longrightarrow \dots$$

Proof. There is a natural transformation $p : H^s \longrightarrow \bar{H}^d$. Set $\bar{\chi}_n = p_{\delta_n \delta_{n-1}} \chi_n$, where χ_n is as in Theorem 2.10. We have $\bar{\chi}_n \bar{H}_n^d(\pi_*) = p_{\delta_n \delta_{n-1}} \chi_n \bar{H}_n^d(\pi_*) = 0$ and $\bar{H}_{n-1}^d(\nu_*) \bar{\chi}_n = \bar{H}_{n-1}^d p_{\delta_n \delta_{n-1}} \chi_n = p_{\delta_{n-1} \delta_{n-2}} H_{n-1}^s \chi_n = 0$, showing that the above sequence is a chain. The exactness at the \bar{H}_n^d corner follows from Theorem 2.10. ■

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Mohammad Zaher Kazemi Baneh

Department of Mathematics, Faculty of Sciences,
University of Kurdistan, Sanandaj 66177-15175, I. R. Iran
zaherkazemi@uok.ac.ir

Seyed Naser Hosseini

Department of Pure Mathematics, Faculty of Mathematical Sciences,
Shahid Bahonar University of Kerman, Kerman, I. R. Iran
nhoseini@uk.ac.ir