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EXACT SEQUENCES OF EXTENDED d-HOMOLOGY

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Abstract. In this article, we show the existence of certain exact sequences with respect to two homology theories, called d-homology and extended d-homology. We present sufficient conditions for the existence of long exact extended d- homology sequence. Also we give some illustrative examples.

1. Introduction and Preliminaries

In [2, 4], the authors have introduced the notion of a kernel transformation d and have defined two homology functors with respect to d, called d-homology and extended d-homology. In this paper, we have proved the existence of some exact sequences containing these homologies.

In this section, we recall some of the necessary results obtained in [2, 4] and in section 2, we give the exact sequences involving d-homology and extended d-homology. If there are some conditions as some objects are subobject or some morphisms are epic or some morphisms are monic, then we have

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exact sequences involving these two homology theories and at the end there is the long exact extended d-homology sequence.

To this end, for a pointed category $\mathcal{C}$, following the notation of [2], we recall:

- For $f : A \longrightarrow B$, the maps $K_f \xrightarrow{k_f} A$, $B \xrightarrow{c_f} C_f$ and $P_f \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} A$ are respectively the kernel, the cokernel and the kernel pair of f.
- The image I_f of f is the coequalizer of the kernel pair of f. f can be factorized to $f = m_f e_f$ that e_f is the coequalizer of the kernel pair of f.
- For a pair of maps $A \underset{g}{\overset{f}{\rightrightarrows}} B$, the maps $Equ(f,g) \xrightarrow{equ(f,g)} A$ and $B \xrightarrow{coe(f,g)} Coe(f,g)$ are respectively the equalizer and the coequalizer of (f,g).

Lemma 1.1. *Given the below diagram in which the squares are commutative and the rows are coequalizers, i is the unique map making the right square commute. Furthermore, i is a regular epi.*

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, "r"'] & B \arrow[r, "q"] \arrow[d, "1_B"] & C \arrow[d, dotted, "i"] \\
A' \arrow[r, shift left, "f'"] \arrow[r, shift right, "g'"'] & B \arrow[r, "q'"'] & C'
\end{tikzcd}$$

For a category $\mathcal{C}$ with a zero object, pullback and pushout, let $\bar{\mathcal{C}}$ be the arrow category and $\widehat{\mathcal{C}}$ be the pair-chain category of $\mathcal{C}$. Let $K : \bar{\mathcal{C}} \longrightarrow \mathcal{C}$ be the kernel functor and $I : \bar{\mathcal{C}} \longrightarrow \mathcal{C}$ be the image functor.

Proposition 1.2. *The functor $j : \widehat{\mathcal{C}} \longrightarrow \bar{\mathcal{C}}$ takes the object $(f,g) \in \widehat{\mathcal{C}}$ to j_{fg} and the morphism (α, β, γ) to $(I(\alpha,\beta), K(\beta,\gamma))$ and we have the following commutative diagram.*

$$\begin{tikzcd}
A \arrow[r, two heads, "e_f"] \arrow[d, "\alpha"'] \arrow[rrr, bend left, "f"] \arrow[rrr, phantom, bend left=20, "///"] \arrow[dr, phantom, "///"] & I_f \arrow[r, "j_{fg}"] \arrow[d, "{I(\alpha,\beta)}"] \arrow[dr, phantom, "///"] & K_g \arrow[r, tail, "k_g"] \arrow[d, "{K(\beta,\gamma)}"] \arrow[dr, phantom, "///"] & B \arrow[d, "\beta"] \\
A' \arrow[r, two heads, "e_{f'}"'] \arrow[rrr, bend right, "f'"'] \arrow[rrr, phantom, bend right=20, "///"] & I_{f'} \arrow[r, "j_{f'g'}"'] & K_{g'} \arrow[r, tail, "k_{g'}"'] & B'
\end{tikzcd}$$

Remark 1.3. For $(f,g) \in \widehat{\mathcal{C}}$, $f = m_f e_f$ in which $e = coeq(\pi_1, \pi_2)$. Let $(f,g) \in \widehat{\mathcal{C}}$. By the above diagram, $k_g j_{fg} = m_f$ and k_f is monic. So j_{fg} is monic if and only if m_f is monic. In a pointed regular (homological, semiabelian or abelian) category j_{fg} is monic, see [1, 2].

Definition 1.4. With $S : \mathcal{C} \longrightarrow \mathcal{C}$ the squaring functor, taking $a \xrightarrow{f} b$ to $a^2 \xrightarrow{f^2} b^2$, a kernel transformation is a natural transformation $d : S \circ K \longrightarrow K : \bar{\mathcal{C}} \longrightarrow \mathcal{C}$ such that for all (f, g) in $\widehat{\mathcal{C}}$, the pullback, $j^*_{fg} : R_{fg} \longrightarrow K^2_g$, of j_{fg} along d_g and the coequalizer of the pair $j_1 = pr_1 j^*_{fg}$ and $j_2 = pr_2 j^*_{fg}$ exist, where pr_1 and pr_2 are the projection maps.

Theorem 1.5. *The d-homology functor $H^d : \widehat{\mathcal{C}} \longrightarrow \mathcal{C}$ takes $(f, g) \in \widehat{\mathcal{C}}$ to $H^d_{fg} = Coe(j_1, j_2)$ and the morphism (α, β, γ) to $H^d(\alpha, \beta, \gamma)$. We have the following commutative diagram.*

$$\begin{tikzcd}
K_g \arrow[r, "q_1"] \arrow[d, "K(\beta,\gamma)"'] & H^d_{fg} \arrow[d, "H^d(\alpha,\beta,\gamma)"] \\
K_{g'} \arrow[r, "q'_1"'] & H^d_{f'g'}
\end{tikzcd}$$

Proposition 1.6. *Let R be a commutative ring with unity. Any kernel transformation in Rmod is of the form $d = rpr_1 + spr_2 = +(r \times s)$, for some $r, s \in R$. In particular any kernel transformation d in Abgrp is of the form $d = +(r \times s)$, for some $r, s \in \mathbb{Z}$.*

Example 1.7. Let $\mathcal{C} = Rmod$ and $d = rpr_1 + spr_2 = +(r \times s)$ with $r, s \in R$. Let (f, g) be a pair-chain. Then $R_{fg} = \{(a, b) \in K^2_g | ra + sb \in I_f\}$, j^* is the inclusion and $H_{fg} = \frac{K_g}{(j_1 - j_2)(R_{fg})} = \{[a] : a \in K_g\}$, where $[a] = \{b | r(a - b) \in (r + s)K_g + I_f\} = \{b | s(a - b) \in (r + s)K_g + I_f\}$ is the equivalence class under the equivalence relation $a \sim b$ if and only if $\exists m, n \in K_g$ such that $a - b = m - n$ and $rm + sn \in I_f$.

Example 1.8. As a special case of above Example, for $d = +(r \times 1)$ or $d = +(1 \times r)$ with $r \in R$, we have $H^d_{fg} = \frac{K_g}{(1+r)K_g + I_f}$.

We call the homology which is defined in the [1, 6] the standard homology.

Definition 1.9. The standard homology or s-homology functor H^s, takes $(f, g) \in \widehat{\mathcal{C}}$ to $Coker(j_{fg})$, and for a pair chain map $(\alpha, \beta, \gamma) : (f, g) \longrightarrow (f', g')$, we have the following commutative diagram.

$$\begin{tikzcd}
K_g \arrow[r, "q"] \arrow[d, "k(\beta,\gamma)"'] & H^s_{fg} \arrow[d, "H^s(\alpha,\beta,\gamma)"] \\
K_{g'} \arrow[r, "q'"'] & H^s_{f'g'}
\end{tikzcd}$$

where $q = coker(j_{fg})$ and $q' = coker(j_{f'g'})$.

Theorem 1.10. *See* [3]. *For a kernel transformation d in $\mathcal{C}$ there is a natural transformation $p : H^s \longrightarrow H^d$. Furthermore p is pointwise regular epic.*

In an abelian category $\mathcal{C}$, If the kernel transformation $d = pr_1 - pr_2 = -$, then we have: $H^{-} = H^{s}$. For a pointed category $\mathcal{C}$ with pulback and pushout, following the notation of [4], we recall:

Notation 1. *Let $m : A \longrightarrow C$ and $j : B \longrightarrow C$ be two maps in $\mathcal{C}$. Define $A +_C B$ also denoted by $A + B$ by the pushout:*

$$\begin{tikzcd} P_{jm} \arrow[r, "\alpha"] \arrow[d, "\gamma"'] \arrow[dr, phantom, "po"] & A \arrow[d, "h"] \\ B \arrow[r, "i"'] & A +_C B \end{tikzcd}$$

where $B \xleftarrow{\gamma} P_{jm} \xrightarrow{\alpha} A$ *is the pullback of (j, m).*

With $S : \mathcal{C} \longrightarrow \mathcal{C}$ the squaring functor and a kernel transformation d, for all $(f, g) \in \widehat{\mathcal{C}}$ and diagonal map Δ we have the maps $m_{d_g\Delta_g} : I_{d_g\Delta_g} \longrightarrow K_g$ (such that $m_{d_g\Delta_g} e_{d_g\Delta_g} = d_g\Delta_g$) and $j_{fg} : I_f \longrightarrow K_g$. The sum $I_{d_g\Delta_g} + I_f$ is therefore obtained by the following diagrams:

$$\begin{tikzcd} P_{jm} \arrow[r, "\alpha"] \arrow[d, "\gamma"'] \arrow[dr, phantom, "\text{pb}"] & I_{d_g\Delta_g} \arrow[d, "m_{d_g\Delta_g}"] \\ I_f \arrow[r, "j_{fg}"'] & K_g \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} P_{jm} \arrow[r, "\alpha"] \arrow[d, "\gamma"'] \arrow[dr, phantom, "\text{po}"] & I_{d_g\Delta_g} \arrow[d, "h"] \\ I_f \arrow[r, "i"'] & I_{d_g\Delta_g} + I_f \end{tikzcd}$$

Since $m_{d_g\Delta_g}\alpha = j_{fg}\gamma$, there is a unique map $\beta : I_{d_g\Delta_g} + I_f \longrightarrow K_g$ making the following triangles commutative.

$$\begin{tikzcd} P_{jm} \arrow[r, "\alpha"] \arrow[d, "\gamma"'] \arrow[dr, phantom, "\text{po}"] & I_{d_g\Delta_g} \arrow[d, "h"] \arrow[ddr, bend left, "m_{d_g\Delta_g}"] & \\ I_f \arrow[r, "i"'] \arrow[drr, bend right, "j_{fg}"'] & I_{d_g\Delta_g} + I_f \arrow[dr, "\beta"] & \\ & & K_g \end{tikzcd}$$

Theorem 1.11. *The extended d-homology functor $\bar{H}^d : \widehat{\mathcal{C}} \longrightarrow \mathcal{C}$ takes $(f, g) \in \widehat{\mathcal{C}}$ to $\bar{H}^d_{fg} = C_\beta$ and the morphism $(\sigma, \delta, \zeta) : (f, g) \longrightarrow (f', g')$ in $\widehat{\mathcal{C}}$ to the unique morphism $\bar{H}^d(\sigma, \delta, \zeta)$ that we have the following commutative diagram.*

$$\begin{tikzcd} K_g \arrow[r, "c_\beta"] \arrow[d, "K(\delta,\zeta)"'] & \bar{H}^d_{fg} \arrow[d, "\bar{H}^d(\sigma,\delta,\zeta)"] \\ K_{g'} \arrow[r, "c_{\beta'}"'] & \bar{H}^d_{f'g'} \end{tikzcd}$$

Theorem 1.12. *Let d be a kernel transformation in $\mathcal{C}$. There is a natural transformation $p : H^s \longrightarrow \bar{H}^d$. Furthermore p is pointwise regular epic.*

Proof. By 1.1 for $(f,g) \in \widehat{\mathcal{C}}$, p_{fg} is the unique map in which the following diagram commutes.

$$\begin{tikzcd} I_f \arrow[r, "j"] \arrow[d, "i"'] & K_g \arrow[r, "c_j"] \arrow[d, "1_{K_g}"] & H^s_{fg} \arrow[d, dotted, "p_{fg}"] \\ I_f + I_{d_g\Delta_g} \arrow[r, "\beta"'] & K_g \arrow[r, "c_\beta"'] & \bar{H}^d_{fg} \end{tikzcd}$$

Naturality of p is straightforward. ■

Lemma 1.13. *If for $(f,g) \in \widehat{\mathcal{C}}$, $i : I_f \longrightarrow I_f + I_{d_g\Delta_g}$ is epic, then $p_{fg} : H^s_{fg} \cong \bar{H}^d_{fg}$ is an isomorphism.*

Proposition 1.14. *Let $\mathcal{C}$ be an abelian category. For $(f,g) \in \widehat{\mathcal{C}}$, $p_{fg} : H^s_{fg} \cong \bar{H}^d_{fg}$ is an isomorphism if and only if $I_{d_g\Delta_g}$ is a subobject of I_f.*

Lemma 1.15. *Let $\mathcal{C}$ be an abelian catergory and $d = +(r \times s)$. If for all A, $r : A \to A$ or $s : A \to A$ is monic, then $H^d \cong_n \bar{H}^d$.*

2. EXACT SEQUENCES

Let $\mathcal{C}$ be an abelian catergory and d be a kernel transformation. For a $\mathcal{C}$-chain:

$$\dots \longrightarrow C_{n+1} \xrightarrow{\delta_{n+1}} C_n \xrightarrow{\delta_n} C_{n-1} \longrightarrow \dots$$

we establish the following notation.

$k_n = k_{\delta_n} : K_n \longrightarrow C_n$, $I_n = I_{\delta_{n+1}}$, $\Delta_n : K_n \longrightarrow K_n^2$ is the diagonal map.

We have the following commutative diagram.

$$\begin{tikzcd} K_n \arrow[r, "\Delta_n"] \arrow[d, "e_{d_n\Delta_n}"'] & K_n^2 \arrow[r, "d_n"] & K_n \\ I_{d_n\Delta_n} \arrow[urr, "m_{d_n\Delta_n}"'] & & \end{tikzcd}$$

We also define:

$$\begin{tikzcd} I_{d_n\Delta_n} \bigcap I_n \arrow[r, "\alpha_n"] \arrow[d, "\gamma_n"'] \arrow[dr, phantom, "\text{po}"] & I_{d_n\Delta_n} \arrow[d, "h_n"] \arrow[ddr, bend left, "m_{d_n\Delta_n}"] & \\ I_n \arrow[r, "i_n"'] \arrow[drr, bend right, "j_n"'] & I_{d_n\Delta_n} + I_n \arrow[dr, "\beta_n"] & \\ & & K_n \end{tikzcd}$$

where the outer square is a pullback.

Let $q_n = coker(k_n\beta_n) : C_n \longrightarrow E_n$. Since $\delta_n k_n = 0$, $\delta_n k_n \beta_n = 0$, and so there is a unique map

$p_n : E_n \longrightarrow C_{n-1}$, such that $p_n q_n = \delta_n$. Since $\delta_{n-1}\delta_n = 0$, $\delta_{n-1} p_n q_n = 0$ and since q_n

is epic, we get $\delta_{n-1}p_n = 0$. Hence there is a unique map $\psi_n : E_n \longrightarrow K_{n-1}$ such that $k_{n-1}\psi_n = p_n$.

Now $\delta_n = m_{\delta_n}e_{\delta_n}$ and $\delta_n k_n = 0$, so $m_{\delta_n}e_{\delta_n}k_n = 0$. Since m_{δ_n} is monic, $e_{\delta_n}k_n = 0$ and so $e_{\delta_n}k_n\beta_n = 0$. Hence there is a unique map $\varepsilon_n : E_n \longrightarrow I_{n-1}$ such that $\varepsilon_n q_n = e_{\delta_n}$. Since e_{δ_n} is epic, so is ε_n. We have $k_{n-1}\psi_n q_n = p_n q_n = \delta_n = m_{\delta_n}e_{\delta_n} = k_{n-1}j_{n-1}\varepsilon_n q_n$ and since k_{n-1} is monic and q_n is epic, $\psi_n = j_{n-1}\varepsilon_n$.

The following diagram summarizes the above arguments.

$$\begin{tikzcd}
& & I_{n-1} \arrow[r, "j_{n-1}"] & K_{n-1} \arrow[dr, "k_{n-1}"] & & \\
\cdots \arrow[r] & C_{n+1} \arrow[r, "\delta_{n+1}"] & C_n \arrow[u, "e_{\delta_n}"] \arrow[rr, "\delta_n"] \arrow[dr, "q_n"] & & C_{n-1} \arrow[r, "\delta_{n-1}"] & \cdots \\
& I_{d_n\Delta_n} + I_n \arrow[r, "\beta_n"'] \arrow[ur, "k_n\beta_n"] & K_n \arrow[u, "k_n"] \arrow[r, "q_nk_n"'] & E_n \arrow[uul, "\varepsilon_n"'] \arrow[uu, "\psi_n"'] \arrow[ur, "p_n"'] & &
\end{tikzcd}$$

Proposition 2.1. *See* [1, 6]. *Let $\mathcal{C}$ be an abelian category. Suppose the following diagram in $\mathcal{C}$ commutes and has exact rows.*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\varphi"] & B \arrow[r, "g"] \arrow[d, "\psi"] & C \arrow[r] \arrow[d, "\eta"] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' \arrow[r] & 0
\end{tikzcd}$$

Then

(a) If φ is an iso, then ψ is monic if and only if η is monic.

(b) If η is an iso, then ψ is epic if and only if φ is epic.

Theorem 2.2. *Let d be a kernel transformation. For each n, we have:*

(a) $K_{\psi_n} = \bar{H}^d_n$.

(b) $C_{\psi_n} = H^s_{n-1}$.

Proof. (a) Since $q_nk_n\beta_n = 0$, there exists a unique morphism ξ_n, such that $\xi_n c_{\beta_n} = q_nk_n$.

So we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & I_{d_n\Delta_n} + I_n \arrow[r, "\beta_n"] \arrow[d, "1"'] & K_n \arrow[r, "c_{\beta_n}"] \arrow[d, "k_n"] & \bar{H}^d_n \arrow[r] \arrow[d, "\xi_n"] & 0 \\
0 \arrow[r] & I_{d_n\Delta_n} + I_n \arrow[r, "k_n\beta_n"'] & C_n \arrow[r, "q_n"'] & E_n \arrow[r] & 0
\end{tikzcd}$$

and therefore by above Proposition, ξ_n is monic.

Now $k_{n-1}\psi_n\xi_n c_{\beta_n} = p_nq_nk_n = \delta_nk_n = 0$, k_{n-1} is monic and c_{β_n} is epic, so $\psi_n\xi_n = 0$. Since k_{ψ_n} is a kernel, there is a unique map $\lambda_n : \bar{H}^d_n \longrightarrow K_{\psi_n}$ such that $k_{\psi_n}\lambda_n = \xi_n$. Since ξ_n is monic, so is λ_n.

On the other hand, $k_{n-1}\psi_n q_n k_n = 0$, $\psi_n q_n k_n = 0$, and so there is a unique map ξ'_n, such that $k_{\psi_n}\xi'_n = q_n k_n$. We have $k_n = ker(\delta_n) = ker(p_n q_n) = ker(k_{n-1}j_{n-1}\varepsilon_n q_n) = ker(\varepsilon_n q_n)$ and $ker(\psi_n) = ker(j_{n-1}\varepsilon_n) = ker(\varepsilon_n)$. So we have the following commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & K_n \arrow[r, "k_n"] \arrow[d, "\xi'_n"'] & C_n \arrow[r, "\varepsilon_n q_n"] \arrow[d, "q_n"] & D \arrow[r] \arrow[d, "1"] & 0 \\ 0 \arrow[r] & K_{\psi_n} \arrow[r, "k_{\psi_n}"'] & E_n \arrow[r, "\varepsilon_n"'] & D \arrow[r] & 0 \end{tikzcd}$$

and therefore by above Proposition, ξ'_n is epic.

Since $k_{\psi_n}\lambda_n c_{\beta_n} = \xi_n c_{\beta_n} = q_n k_n = k_{\psi_n}\xi'_n$ and k_{ψ_n} is monic, $\lambda_n c_{\beta_n} = \xi'_n$. Since ξ'_n is epic, so is λ_n. So λ_n is a bimorphism and therefore an isomorphism. Hence $\bar{H}^d_n \cong K_{\psi_n}$.

(b) We have $c_{\psi_n} = coker(\psi_n) = coker(j_{n-1}\varepsilon_n) = coker(j_{n-1}) = H^s_{n-1}$. ■

Example 2.3. In the catergory of $Rmod$, $\psi_n : \frac{C_n}{I_{d_n\Delta_n}+I_n} \longrightarrow K_{n-1}$ is the map defined by $\psi_n(x + I_{d_n\Delta_n} + I_n) = \delta_n(x)$. We have $\bar{H}^d_n = K_{\psi_n} = \frac{K_n}{I_{d_n\Delta_n}+I_n}$. Also $H^s_{n-1} = C_{\psi_n} = \frac{K_{n-1}}{I_{n-1}}$, as expected.

Lemma 2.4. *Let $\mathcal{C}$ be an abelian catergory, d be a kernel transformation and the sequence*

$$0 \longrightarrow C'_* \xrightarrow{\nu_*} C_* \xrightarrow{\pi_*} C''_* \longrightarrow 0$$

of $\mathcal{C}$-chains be short exact. Then:

(a) With $f_n = K(\nu_n, \nu_{n-1})$, there is a unique map $l_n : I'_n + I_{d'_n\Delta'_n} \longrightarrow I_n + I_{d_n\Delta_n}$ such that the following diagram commutes.

$$\begin{tikzcd} I'_n + I_{d'_n\Delta'_n} \arrow[r, "\beta'_n"] \arrow[d, "l_n"'] & K'_n \arrow[d, "f_n"] \\ I_n + I_{d_n\Delta_n} \arrow[r, "\beta_n"'] & K_n \end{tikzcd}$$

(b) With $f'_n = K(\pi_n, \pi_{n-1})$, there is a unique map $l'_n : I_n + I_{d_n\Delta_n} \longrightarrow I''_n + I_{d''_n\Delta''_n}$ such that the following diagram commutes.

$$\begin{tikzcd} I_n + I_{d_n\Delta_n} \arrow[r, "\beta_n"] \arrow[d, "l'_n"'] & K_n \arrow[d, "f'_n"] \\ I''_n + I_{d''_n\Delta''_n} \arrow[r, "\beta''_n"'] & K''_n \end{tikzcd}$$

Proof. (a) By definition of Δ_n we have $f^2_n\Delta'_n = \Delta_n f_n$. Naturality of d implies $f_n d'_n = d_n f^2_n$. So $f_n d'_n\Delta'_n = d_n f^2_n\Delta_n = d_n\Delta_n f_n$. So there is a unique map $\rho_n = I(f_n, f_n)$ such that $\rho_n e_{d'_n\Delta'_n} = e_{d_n\Delta_n} f_n$ as the following diagram shows.

$$\begin{tikzcd}
K'_n \arrow[rr, "\Delta'_n"] \arrow[dr, "e_{d'_n\Delta'_n}"] \arrow[dd, "f_n"'] & & K'^2_n \arrow[r, "d'_n"] \arrow[dd, "f^2_n"] & K'_n \arrow[dd, "f_n"] \\
& I_{d'_n\Delta'_n} \arrow[urr, "m_{d'_n\Delta'_n}"] \arrow[dd, dotted, "\rho_n"] & & \\
K_n \arrow[rr, "\Delta_n"] \arrow[dr, "e_{d_n\Delta_n}"'] & & K^2_n \arrow[r, "d_n"] & K_n \\
& I_{d_n\Delta_n} \arrow[urr, "m_{d_n\Delta_n}"'] & &
\end{tikzcd}$$

We have $f_n m'_{d_n\Delta_n} e_{d'_n\Delta'_n} = f_n d'_n \Delta'_n = d_n \Delta_n f_n = m_{d_n\Delta_n} e_{d_n\Delta_n} f_n = m_{d_n\Delta_n} \rho_n e_{d'_n\Delta'_n}$. Since $e_{d'_n\Delta'_n}$ is epic, we get:

$$f_n m_{d'_n\Delta'_n} = m_{d_n\Delta_n} \rho_n \qquad (1)$$

Since $k_n f_n = \nu_n k'_n$, f_n is monic and since $m_{d'_n\Delta'_n}$ is also monic, ρ_n is monic.

There is a unique map $x_n = I(\nu_{n+1}, \nu_n)$, such that:

$$x_n e'_{n+1} = e_{n+1} \nu_{n+1} \qquad (2)$$

as the following diagram shows.

$$\begin{tikzcd}
C'_{n+1} \arrow[rr, "\delta'_{n+1}"] \arrow[dr, "e'_{n+1}"] \arrow[dd, "\nu_{n+1}"'] & & C'_n \arrow[rr, "\delta'_n"] \arrow[dd, "\nu_n"] & & C'_{n-1} \arrow[dd, "\nu_{n-1}"] \\
& I'_n \arrow[ur, "m_{\delta'_{n+1}}"] \arrow[rr, "j'_n"'] \arrow[dd, dotted, "x_n"] & & K'_n \arrow[ul, "k'_n"'] \arrow[dd, "f_n"] & \\
C_{n+1} \arrow[rr, "\delta_{n+1}"] \arrow[dr, "e_{n+1}"'] & & C_n \arrow[rr, "\delta_n"] & & C_{n-1} \\
& I_n \arrow[ur, "m_{\delta_{n+1}}"] \arrow[rr, "j_n"'] & & K_n \arrow[ul, "k_n"'] &
\end{tikzcd}$$

Next we show:

$$f_n j'_n = j_n x_n \qquad (3)$$

We know $k_n f_n = \nu_n k'_n$, $k'_n j'_n = m_{\delta'_{n+1}}$ and $k_n j_n = m_{\delta_{n+1}}$. So $k_n f_n j'_n e'_{n+1} = \nu_n k'_n j'_n e'_{n+1}$ $= \nu_n m_{\delta'_{n+1}} e'_{n+1} = \nu_n \delta'_{n+1} = \delta_{n+1} \nu_{n+1} = m_{\delta_{n+1}} e_{n+1} \nu_{n+1} = k_n j_n e_{n+1} \nu_{n+1} = k_n j_n x_n e'_{n+1}$. k_n is monic and e'_{n+1} is epic, so $f_n j'_n = j_n x_n$. Since f_n and j'_n are monic, x_n is monic.

By (1), (2) and (3) we have $m_{d_n\Delta_n} \rho_n \alpha'_n = f_n m_{d'_n\Delta'_n} \alpha'_n = f_n j'_n \gamma'_n = j_n x_n \gamma'_n$. Since the front and back squares in the following diagram are pullbacks,

$$\begin{tikzcd}
I_{d'_n\Delta'_n}\bigcap I'_n \arrow[rr, "\alpha'_n"] \arrow[dr, dotted, "z_n"] \arrow[dd, "\gamma'_n"'] & & I_{d'_n\Delta'_n} \arrow[dr, "\rho_n"] \arrow[dd, "m_{d'_n\Delta'_n}"] & \\
& I_{d_n\Delta_n}\bigcap I_n \arrow[rr, "\alpha_n"'] \arrow[dd, "\gamma_n"] & & I_{d_n\Delta_n} \arrow[dd, "m_{d_n\Delta_n}"] \\
I'_n \arrow[rr, "j'_n"'] \arrow[dr, "x_n"'] & & K'_n \arrow[dr, "f_n"] & \\
& I_n \arrow[rr, "j_n"'] & & K_n
\end{tikzcd}$$

there is a unique map z_n, such that:

$$\gamma_n z_n = x_n \gamma'_n \tag{4}$$

$$\alpha_n z_n = \rho_n \alpha'_n \tag{5}$$

α'_n, being pullback of a monic, is monic, and since ρ_n is monic, z_n is monic.

by (4) and (5) we have $h_n\rho_n\alpha'_n = h_n\alpha_n z_n = i_n\gamma_n z_n = i_n x_n\gamma'_n$. Since the following front and back squares are pushouts,

$$\begin{tikzcd}
I_{d'_n\Delta'_n}\bigcap I'_n \arrow[rr, "\alpha'_n"] \arrow[dr, "z_n"] \arrow[dd, "\gamma'_n"'] & & I_{d'_n\Delta'_n} \arrow[dr, "\rho_n"] \arrow[dd, "h'_n"] & \\
& I_{d_n\Delta_n}\bigcap I_n \arrow[rr, "\alpha_n"'] \arrow[dd, "\gamma_n"] & & I_{d_n\Delta_n} \arrow[dd, "h_n"] \\
I'_n \arrow[rr, "i'_n"'] \arrow[dr, "x_n"'] & & I_{d'_n\Delta'_n} + I'_n \arrow[dr, dotted, "l_n"] & \\
& I_n \arrow[rr, "i_n"'] & & I_{d_n\Delta_n} + I_n
\end{tikzcd}$$

there is a unique map l_n, such that $l_n i'_n = i_n x_n$ and $l_n h'_n = h_n\rho_n$. Now $j_n x_n = f_n j'_n$, so $\beta_n l_n i'_n = \beta_n i_n x_n = j_n x_n = f_n j'_n = f_n\beta'_n i'_n$. Therefore:

$$\beta_n l_n i'_n = f_n\beta'_n i'_n \tag{6}$$

Also $m_{d_n\Delta_n}\rho_n = f_n m_{d'_n\Delta'_n}$, so $\beta_n l_n h'_n = \beta_n h_n\rho_n = m_{d_n\Delta_n}\rho_n = f_n m_{d'_n\Delta'_n} = f_n\beta'_n h'_n$. Therefore:

$$\beta_n l_n h'_n = f_n\beta'_n h'_n \tag{7}$$

By (6) and (7) and pushoutness of $I_{d'_n\Delta'_n} + I'_n$, we have $\beta_n l_n = f_n\beta'_n$ as desired. The uniqueness is obvious by the fact that β_n is monic.

(b) Similar to part (a), there exists a unique morphism $\rho'_n = I(f'_n, f'_n)$ such that:

$$\rho'_n e_{d_n\Delta_n} = e_{d''_n\Delta''_n} f'_n$$

$$f'_n m_{d_n\Delta_n} = m_{d''_n\Delta''_n}\rho'_n$$

and there exists a unique morphism $x'_n = I(\pi_{n+1}, \pi_n)$ such that:

$$x'_n e_{n+1} = e''_{n+1}\pi_{n+1}$$

Since e''_{n+1} and π_{n+1} are epic, x'_n is epic.

Also there exists a unique morphism $z'_n : I_{d_n\Delta_n} \bigcap I_n \longrightarrow I_{d''_n\Delta''_n} \bigcap I''_n$ such that the following diagram commutes.

$$\begin{tikzcd}
I_{d_n\Delta_n} \bigcap I_n \arrow[rr, "\alpha_n"] \arrow[dr, dotted, "z'_n"] \arrow[dd, "\gamma_n"'] & & I_{d_n\Delta_n} \arrow[dr, "\rho'_n"] \arrow[dd, "m_{d_n\Delta_n}" near start] & \\
& I_{d''_n\Delta''_n} \bigcap I''_n \arrow[rr, "\alpha''_n"' near start] \arrow[dd, "\gamma''_n" near start] & & I_{d''_n\Delta''_n} \arrow[dd, "m_{d''_n\Delta''_n}"] \\
I_n \arrow[rr, "j_n"' near start] \arrow[dr, "x'_n"'] & & K_n \arrow[dr, "f'_n"] & \\
& I''_n \arrow[rr, "j''_n"'] & & K''_n
\end{tikzcd}$$

One can easily verify that:

$$\gamma''_n z'_n = x'_n \gamma_n$$
$$\alpha''_n z'_n = \rho'_n \alpha_n$$

Finally we have a unique morphism $l'_n : I_{d_n\Delta_n} + I_n \longrightarrow I_{d''_n\Delta''_n} + I''_n$ such that the following diagram commutes.

$$\begin{tikzcd}
I_{d_n\Delta_n} \bigcap I_n \arrow[rr, "\alpha_n"] \arrow[dr, "z'_n"] \arrow[dd, "\gamma_n"'] & & I_{d_n\Delta_n} \arrow[dr, "\rho'_n"] \arrow[dd, "h_n" near start] & \\
& I_{d''_n\Delta''_n} \bigcap I''_n \arrow[rr, "\alpha''_n"' near start] \arrow[dd, "\gamma''_n" near start] & & I_{d''_n\Delta''_n} \arrow[dd, "h''_n"] \\
I_n \arrow[rr, "i_n"' near start] \arrow[dr, "x'_n"'] & & I_{d_n\Delta_n} + I_n \arrow[dr, dotted, "l'_n"] & \\
& I''_n \arrow[rr, "i''_n"'] & & I_{d''_n\Delta''_n} + I''_n
\end{tikzcd}$$

The following equalities are an easy consequence.

$$\beta''_n l'_n i_n = f'_n \beta_n i_n$$
$$\beta''_n l'_n h_n = f'_n \beta_n h_n$$

By the above equalities and pushoutness of $I_{d_n\Delta_n} + I_n$, we have $\beta''_n l'_n = f'_n \beta_n$ as desired. ■

Lemma 2.5. *Let $\mathcal{C}$ be an abelian category and the sequence*

$$0 \longrightarrow C'_* \xrightarrow{\nu_*} C_* \xrightarrow{\pi_*} C''_* \longrightarrow 0$$

of $\mathcal{C}$-chains be short exact. Then we have the following exact sequence.

$$0 \longrightarrow K'_n \xrightarrow{f_n} K_n \xrightarrow{f'_n} K''_n$$

Proof. See [6]. ■

Lemma 2.6. *Let $\mathcal{C}$ be an abelian category, d be a kernel transformation and the sequence*

$$0 \longrightarrow C'_* \xrightarrow{\nu_*} C_* \xrightarrow{\pi_*} C''_* \longrightarrow 0$$

of $\mathcal{C}$-chains be short exact. Then we have the following sequence with g'_n epic and $g'_n g_n = 0$.

$$E'_n \xrightarrow{g_n} E_n \xrightarrow{g'_n} E''_n \longrightarrow 0$$

Proof. Set $f_n = K(\nu_n, \nu_{n-1})$. Then $\nu_n k'_n = k_n f_n$ and since k'_n and ν_1 are monic, f_n is monic. By Lemma 2.4, there is a unique map l_n such that $\beta_n l_n = f_n \beta'_n$. Since $q_n \nu_n k'_n \beta'_n = q_n k_n f_n \beta'_n = q_n k_n \beta_n l_n = 0 l_n = 0$, there is a unique map g_n, such that $g_n q'_n = q_n \nu_n$. Since q''_n and π_n are epic and $g'_n q_n = q''_n \pi_n$, g'_n is epic. We have $g'_n g_n q'_n = g'_n q_n \nu_n = q''_n \pi_n \nu_n = q''_n 0 = 0$. Since q'_n is epic, $g'_n g_n = 0$. ■

Lemma 2.7. *If l'_n epic, then the following sequence is exact.*

$$E'_n \xrightarrow{g_n} E_n \xrightarrow{g'_n} E''_n \longrightarrow 0$$

Proof. It suffices to show that g'_n is a cokernel of g_n. We make use of the following diagram.

$$\begin{tikzcd}
 & & 0 \arrow[d] & & & 0 \arrow[d] & \\
 & {} \arrow[r, "\delta'_{n+1}"] & C'_n \arrow[rrr, "\delta'_n"] \arrow[dr, "q'_n"] \arrow[ddd, "\nu_n"] & & & C'_{n-1} \arrow[r, "\delta'_{n-1}"] \arrow[ddd, "\nu_{n-1}"] & {} \\
I'_{d_n\Delta_n} + I'_n \arrow[r, "\beta'_n"'] \arrow[urr] \arrow[d, "l_n"] & K'_n \arrow[ur, "k'_n"] \arrow[d, "f_n"] & & E'_n \arrow[r, "\psi'_n"'] \arrow[urr, "p'_n"] \arrow[d, "g_n"] & K'_{n-1} \arrow[ur, "k'_{n-1}"] \arrow[d, "f_{n-1}"] & & \\
I_{d_n\Delta_n} + I_n \arrow[r, "\beta_n"] \arrow[drr] \arrow[dd, "l'_n"] & K_n \arrow[dr, "k_n"] \arrow[dd, "f'_n"] & & E_n \arrow[r, "\psi_n"] \arrow[drr, "p_n"] \arrow[dd, "g'_n"] & K_{n-1} \arrow[dr, "k_{n-1}"] \arrow[dd, "f'_{n-1}"] & & \\
 & {} \arrow[r, "\delta_{n+1}"'] & C_n \arrow[rrr, "\delta_n"'] \arrow[ur, "q_n"] \arrow[dd, "\pi_n"] & & & C_{n-1} \arrow[r, "\delta_{n-1}"] \arrow[dd, "\pi_{n-1}"] & {} \\
I''_{d_n\Delta_n} + I''_n \arrow[r, "\beta''_n"] \arrow[drr] & K''_n \arrow[dr, "k''_n"] & & E''_n \arrow[r, "\psi''_n"] \arrow[drr, "p''_n"] & K''_{n-1} \arrow[dr, "k''_{n-1}"] & & \\
 & {} \arrow[r, "\delta''_{n+1}"'] & C''_n \arrow[rrr, "\delta''_n"'] \arrow[ur, "q''_n"] \arrow[d] & & & C''_{n-1} \arrow[r, "\delta''_{n-1}"'] \arrow[d] & {} \\
 & & 0 & & & 0 &
\end{tikzcd}$$

Suppose $r : E_n \longrightarrow A$ such that $rg_n = 0$. Then $rg_n q'_n = 0$ and $rq_n \nu_n = 0$. Since $\pi_n = coker(\nu_n)$, there is $s : C''_n \longrightarrow A$ such that $s\pi_n = rq_n$. Since $q_n = coker(k_n \beta_n)$,

$sk''_n\beta''_n l'_n = s\pi_n k_n\beta_n = rq_n k_n\beta_n = 0$. By hypothesis l'_n is epic, so $sk''_n\beta''_n = 0$. Since $q''_n = coker(k''_n\beta''_n)$, there is a unique $t : E''_n \longrightarrow A$ such that $tq''_n = s$. It easily follows that t is the unique map such that $tg'_n = r$. Hence $g'_n = coker(g_n)$. ■

With f'_n, ρ'_n and l'_n as in the proof of Lemma 2.4, we have:

Lemma 2.8. *In the abelian catergory $\mathcal{C}$*

(a) If f'_n is epic, then so is ρ'_n.

(b) If ρ'_n is epic, then so is l'_n.

(c) If $I_{d''_n\Delta''_n}$ is a subobject of I''_n, then l'_n is epic.

Proof. (a) Since $\rho'_n e_{d_n\Delta_n} = e_{d''_n\Delta''_n} f'_n$ and $e_{d''_n\Delta''_n}$ is epic, the result follows.

(b) If $al'_n = bl'_n$, then $al'_n h''_n = bl'_n h''_n$ and so $ah'_n\rho'_n = bh'_n\rho'_n$. ρ'_n epic implies $ah'_n = bh'_n$. Since x'_n is epic, $ai''_n = bi''_n$. pushoutness of $I_{d''_n\Delta''_n} + I''_n$ implies $a = b$ and l'_n is epic.

(c) If $I_{d''_n\Delta''_n}$ is a subobject of I''_n, then $i''_n = 1$. Since x'_n is epic, so is the morphism $l'_n : I_{d_n\Delta_n} + I_n \longrightarrow I_{d''_n\Delta''_n} + I''_n = I''_n$. ■

Theorem 2.9. *Let $\mathcal{C}$ be an abelian category, d be a kernel transformation and the sequence*

$$0 \longrightarrow C'_* \xrightarrow{\nu_*} C_* \xrightarrow{\pi_*} C''_* \longrightarrow 0$$

of $\mathcal{C}$-chains be short exact. Then we have the following commutative diagram.

$$\begin{tikzcd}
 & E'_n \arrow[r, "g_n"] \arrow[d, "\psi'_n"'] & E_n \arrow[r, "g'_n"] \arrow[d, "\psi_n"'] & E''_n \arrow[r] \arrow[d, "\psi''_n"] & 0 \\
0 \arrow[r] & K'_{n-1} \arrow[r, "f_{n-1}"'] & K_{n-1} \arrow[r, "f'_{n-1}"'] & K''_{n-1} &
\end{tikzcd}$$

Proof. All the maps have been introduced before, see Lemmas 2.4 and 2.6. We have $k_{n-1}f_{n-1}\psi'_n q'_n = \nu_{n-1}k'_{n-1}\psi'_n q'_n = \nu_{n-1}p'_n q'_n = \nu_{n-1}\delta'_{n-1} = \delta_n\nu_n = p_n q_n\nu_n = p_n g_n q'_n = k_{n-1}\psi_n g_n q'_n$.

Since k_{n-1} is monic and q'_n is epic, $f_{n-1}\psi'_n = \psi_n g_n$, showing the left square is commutative. We can similarly show $f'_{n-1}\psi_n = \psi''_n g'_n$. ■

Theorem 2.10. *Let $\mathcal{C}$ be an abelian category, d be a kernel transformation and the sequence*

$$0 \longrightarrow C'_* \xrightarrow{\nu_*} C_* \xrightarrow{\pi_*} C''_* \longrightarrow 0$$

of $\mathcal{C}$-chains be short exact. If l'_n be epic, then, for each n, we have the following exact sequence.

$$\bar{H'}^d_n \longrightarrow \bar{H}^d_n \longrightarrow \bar{H''}^d_n \longrightarrow H'^s_{n-1} \longrightarrow H^s_{n-1} \longrightarrow H''^s_{n-1}$$

Proof. By Lemmas 2.4 and 2.6, the diagram of Theorem 2.9 has exact rows. So there is a map $\chi_n : ker(\psi''_n) \longrightarrow coker(\psi'_n)$, see [6], such that the following sequence is exact:

$$K_{\psi'_n} \longrightarrow K_{\psi_n} \longrightarrow K_{\psi''_n} \xrightarrow{\chi_n} C_{\psi'_n} \longrightarrow C_{\psi_n} \longrightarrow C_{\psi''_n}$$

The result follows by Theorem 2.2. ■

Corollary 2.11. *Let $\mathcal{C}$ be an abelian catergory and $d = +(r \times s)$, $r, s \in \mathbb{Z}$. If for all $A \in \mathcal{C}$, $r : A \longrightarrow A$ or for all A, $s : A \longrightarrow A$ is monic, then for each short exact sequence of $\mathcal{C}$-chains:*

$$0 \longrightarrow C'_* \xrightarrow{\nu_*} C_* \xrightarrow{\pi_*} C''_* \longrightarrow 0$$

for which l'_n is epic, there is the following exact sequence.

$$H'^d_n \longrightarrow H^d_n \longrightarrow H''^d_n \longrightarrow H'^s_{n-1} \longrightarrow H^s_{n-1} \longrightarrow H''^s_{n-1}$$

Proof. The result follows by Lemma 1.15 and Theorem 2.10. ■

Corollary 2.12. *Let $I_{d''_n \Delta''_n}$ be a subobject of I''_n. Then the following diagram is exact.*

$$\bar{H}'^d_n \longrightarrow \bar{H}^d_n \longrightarrow \bar{H}''^d_n \longrightarrow H'^s_{n-1} \longrightarrow H^s_{n-1} \longrightarrow H''^s_{n-1}$$

Proof. Follows from Proposition 1.14 and Lemma 2.8. ■

Example 2.13. In the category of $Rmod$, $I_{d''_n \Delta''_n} = (r+s)K''_n$. So if $(r+s)K''_n$ is a submodule of I''_n, then the sequence in the above corollary is exact.

Proposition 2.14. *Let $\mathcal{C}$ be an abelian category, d be a kernel transformation and the sequence*

$$0 \longrightarrow C'_* \xrightarrow{\nu_*} C_* \xrightarrow{\pi_*} C''_* \longrightarrow 0$$

of $\mathcal{C}$-chains be short exact, for which l'_n is epic. Then there is the following chain that for each n is exact at $\bar{H}^d_n$ corner.

$$\ldots \longrightarrow \bar{H}'^d_n \longrightarrow \bar{H}^d_n \longrightarrow \bar{H}''^d_n \xrightarrow{\bar{\chi}_n} \bar{H}'^d_{n-1} \longrightarrow \bar{H}^d_{n-1} \longrightarrow \ldots$$

Proof. There is a natural transformation $p : H^s \longrightarrow \bar{H}^d$. Set $\bar{\chi}_n = p_{\delta_n \delta_{n-1}} \chi_n$, where χ_n is as in Theorem 2.10. We have $\bar{\chi}_n \bar{H}^d_n(\pi_*) = p_{\delta_n \delta_{n-1}} \chi_n \bar{H}^d_n(\pi_*) = 0$ and $\bar{H}^d_{n-1}(\nu_*)\bar{\chi}_n = \bar{H}^d_{n-1} p_{\delta_n \delta_{n-1}} \chi_n = p_{\delta_{n-1}\delta_{n-2}} H^s_{n-1} \chi_n = 0$, showing that the above sequence is a chain. The exactness at the $\bar{H}^d_n$ corner follows from Theorem 2.10. ■

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