Journal of Algebraic Structures and Their Applications
ISSN: 2382-9761
www.as.yazd.ac.ir

Algebraic Structures and Their Applications Vol. 3 No. 1 (2016), pp 17-24.

A NOTE ON THE ORDER GRAPH OF A GROUP

HAMID REZA DORBIDI

Communicated by Cheryl E. Praeger

Abstract. The order graph of a group G, denoted by $\Gamma^*(G)$, is a graph whose vertices are non trivial proper subgroups of G and two distinct vertices H and K are adjacent if and only if $|H|\big||K|$ or $|K|\big||H|$. In this paper, we study the connectivity and diameter of this graph. Also we give a relation between the order graph and prime graph of a group.

1. Introduction

Let G be a finite group. The order graph of G is the (undirected) graph $\Gamma^*(G)$, whose vertices are non-trivial proper subgroups of G and two distinct vertices H and K are adjacent if and only if either $|H|\big||K|$ or $|K|\big||H|$. So $\Gamma^*(G)$ is the empty graph if and only if $|G|$ is a prime number. So we only consider the groups whose orders are not prime. This graph has studied in [5]. In this paper, we study the connectivity and diameter of this graph. We show that the order graph of a non-simple group is a connected graph with diameter less than four with only one family exception. Also we give a connection between prime graph and order graph.

MSC(2010): Primary: 20A05. Secondary: 05C25.

Keywords: Connected graph, Frobenius group, Order graph, Prime graph.

Received: 30 April 2015, Accepted: 07 Feb2017.

∗Corresponding author

First we recall some facts and notations related to this paper. Throughout this paper G denotes a finite group. The normalizer of a subgroup H is denoted by $N_G(H)$. A subgroup N of G is called a characteristic subgroup if $\sigma(N) \subseteq N$ for all $\sigma \in Aut(G)$. A group G is characteristically simple if the only characteristic subgroups of G are $\{1\}$ and G. We denote gHg^{-1} by H^g. Let $\pi(n)$ be the set of prime divisors of n. We denote $\pi(|G|)$ by $\pi(G)$. The prime graph of a group G is a graph whose vertex set is $\pi(G)$ and two distinct primes p and q are adjacent if and only if G has an element of order pq. The cyclic group of order n is denoted by C_n. The symmetric group on n letters is denoted by S_n. D_n is the dihedral group of order $2n$. The alternating group is denoted by A_n. The finite field with q elements is denoted by $\mathbb{F}_q$.

Let V be a vector space and $A : V \longrightarrow V$ be a linear mapping. A subspace W of V is an $A-$invariant subspace if $A(W) \subseteq W$. The vector space V is called $A-$simple if the only $A-$invariant subspaces of V are 0 and V.

Let Γ be a simple graph with vertex set V. A path between two vertices x and y is a sequence $x = v_0, \cdots, v_n = y$ of vertices such that v_i is adjacent to v_{i-1}. A graph is connected if for any two vertices x, y there is a path between x and y. The length of the shortest path between x and y is denoted by $d(x, y)$. Also $diam(\Gamma) = sup\{d(x, y) : x$ and y are two vertices of $\Gamma\}$ is called the diameter of the graph Γ.

2. Preliminaries

Before proving the main theorems, we need the following definitions, lemma and theorems.

Definition 2.1. If $f : H \longrightarrow Aut(N)$ is a homomorphism then $G = N \times H$ with the operation $(n, h)(n', h') = (f(h^{-1})(n)n', hh')$ is called semi-direct product of H and N and is denoted $G = N \rtimes H$. If $G = HN$ and $H \cap N = \{1\}$ where $N \lhd G$ then $G \cong N \rtimes H$. If f is the trivial homomorphism then $G = H \times N$ is the direct product of groups.

Lemma 2.1. (1) *If N_1 is a characteristic subgroup of N and $N \lhd G$ then $N_1 \lhd G$.*

(2) *If G is an abelian characteristically simple group then G is an elementary abelian group i.e G is a vector space over a finite field $\mathbb{F}_q$ where q is a prime number.*

Proof. (1) Let σ_g be the conjugation by $g \in G$. Since $N \lhd G$, so σ_g is an automorphism of N for all $g \in G$. Hence $N_1^g = \sigma_g(N_1) \subseteq N_1$ i.e $N_1 \lhd G$.

(2) Let q be a prime divisor of $|G|$. So $G_1 = \{g \in G : g^q = 1\}$ is a characteristic subgroup. Thus $G_1 = G$. Hence G is an elementary abelian group i.e G is a vector space over the finite field $\mathbb{F}_q$.

0.1cm□

Definition 2.2. If there exists a proper subgroup $H \neq \{1\}$ of G such that $H^g \cap H = \{1\}$ for all $g \in G \backslash H$ then G is called a Frobenius group with respect to H.

Remark 2.1. If G is a Frobenius group with respect to H and $H < M < G$ then M is also a Frobenius group with respect to H. It is clear that $Z(G) = 1$. So a Frobenius group G is not a nilpotent group.

The following theorem states some properties of the Frobenius groups.

Theorem 2.2. [4, chapter 16] *If G is a Frobenius group with respect to H then $F = G \backslash (\cup_{g \in G} H^g \backslash \{1\})$ is a normal subgroup of G which is called Frobenius kernel of G. Since $G = FH$ and $F \cap H = \{1\}$, H is called a complement of F. Also F is a nilpotent subgroup and $|H| \big| |F| - 1$.*

Definition 2.3. A Frobenius group G is called a minimal Frobenius group if G has no proper Frobenius subgroup.

3. Main results

In this section we state our main results.

Theorem 3.1. *A group G is a minimal Frobenius group if and only if G has a non normal maximal subgroup of prime order.*

Proof. Let G be a minimal Frobenius group with complement H and kernel F. If $H < M$ then M is a Frobenius group. So $M = G$ by minimality of G. So H is a maximal subgroup. If $H_1 < H$ is a non-trivial subgroup then $H_1 F \cap H = H_1$. This implies that $H_1 F$ is a Frobenius group with respect to H_1 which is a contradiction. So H has no proper non-trivial subgroup. Hence $|H|$ is a prime number. Conversely, assume G has a non normal maximal subgroup H of prime order. Since $|H|$ is a prime number, so $H \cap H^g = \{1\}$ for all $g \in G \backslash N_G(H)$. But $N_G(H) = H$ by maximality and non normality of H. So G is a Frobenius group with respect to H. Let $G_1 < G$ be a subgroup of G. If $G_1 \cap H^g = \{1\}$ for all $g \in G$ then $G_1 \subseteq F$ is a nilpotent subgroup. So G_1 is not a Frobenius group. Assume $G_1 \cap H^g \neq \{1\}$ for some $g \in G$. Since $|H^g|$ is a prime number, we conclude that $H^g \subseteq G_1$. So $H^g = G_1$ by maximality of H^g. Hence G_1 is not a Frobenius group. Therefore G is a minimal Frobenius group. □

Theorem 3.2. *Let $p \neq q$ be two prime numbers and n be an integer. $x^{(p,q^n-1)} - 1$ has an irreducible factor of degree n over $\mathbb{F}_q$if and only if there is an $n-$dimensional vector space V over $\mathbb{F}_q$ and a linear mapping A such that $A^p = 1$ and V is $A-$simple.*

Proof. Assume $x^{(p,q^n-1)}-1$ has an irreducible factor $f(x)$ of degree n over $\mathbb{F}_q$. Let $V=\mathbb{F}_{q^n}=\mathbb{F}_q[\alpha]$ be the splitting field of $f(x)(\alpha$ is a root of $f(x))$. Let $A:V\to V$ be the multiplication by α. Then the minimal polynomial of A is $f(x)$. Since $f(x)|x^p-1$, so $A^p=1$. Let $0\neq v\in V$ and $f_v(x)$ be a polynomial of minimum degree such that $f_v(A)(v)=0$. Let $V_1=\langle v,Av,\cdots A^n(v)\rangle$ be the minimal $A-$invariant subspace of V which contains v. It is clear that $f_v(x)|f(x)$ and $dim(V_1)=deg(f_v(x))$. Since $f(x)$ is irreducible, so $f_v(x)=f(x)$. Thus $V=V_1$. Hence V has no non trivial $A-$invariant subspaces. Conversely, Assume V is an $n-$dimensional vector space over $\mathbb{F}_q$ and there is a linear mapping A such that $A^p=1$ and V is $A-$simple. Let $f_A(x)$ be the minimal polynomial of A over $\mathbb{F}_q$ of degree m. Since $A^p=1$, so $f_A(x)|x^p-1$. If $f_A(x)=h(x)g(x)$ then $Ker(g(A))$ is an $A-$invariant subspace. So $Ker(g(A))=V$ or $Ker(g(A))=0$. If $Ker(g(A))=V$ then $g(A)=0$. So $g(x)=f_A(x)$ by definition of $f_A(x)$. If $Ker(g(A))=0$ then $h(A)=0$ since $Im(h(A)\subseteq Ker(g(A))=0$. So $h(x)=f_A(x)$ by definition of $f_A(x)$. Hence $f_A(x)$ is an irreducible polynomial. If $v\neq 0$ then $V_1=\langle v,Av,\cdots,A^{m-1}(v)\rangle$ is an $A-$invariant subspace. So $V_1=V$ and $m=n$. It is a standard fact in finite field theory that every irreducible polynomial of degree n over $\mathbb{F}_q$ is a factor of $x^{q^n}-x$. Since A is invertible, $f_A(x)|x^{q^n-1}-1$. Hence $f_A(x)|(x^{q^n-1}-1,x^p-1)=x^{(q^n-1,p)}-1$. □

Definition 3.1. Let $p\neq q$ be two prime numbers and n be an integer. Assume that $x^{(q^n-1,p)}-1$ has an irreducible factor $f(x)$ of degree n over $\mathbb{F}_q$. Let $V=F_{q^n}=\mathbb{F}_q[\alpha]$ be the splitting field of $f(x)$. Let $A:V\to V$ be the multiplication by α. Since $A^p=1$, so there is a homomorphism from C_p to $Aut(V)$. Let $\mathcal{A}(p,q,n)=V\rtimes C_p$ be the semi direct product of action of C_p on V. The set of all $A(p,q,n)$ is denoted by $\mathcal{A}$. Every element of $\mathcal{A}$ is called an $\mathcal{A}-$group.

Remark 3.1. If $n=1$ and $p\nmid q-1$ then $A(p,q,1)=C_p\times C_q\cong C_{pq}$. If $n=1$ and $p|q-1$ then $A(p,q,1)=C_{pq}$ or $A(p,q,1)=C_q\rtimes C_p$. Also if $n>1$ then $p|q^n-1$. Note that C_p is normal in $A(p,q,n)$ if and only if $n=1$ and $A(p,q,1)=C_{pq}$.

The following theorem gives the structure of minimal Frobenius groups.

Theorem 3.3. *A group G is a minimal Frobenius group if and only if G is an $\mathcal{A}-$group and $G\ncong C_{pq}$.*

Proof. Let G be a minimal Frobenius group with kernel F and complement H. So H is a maximal subgroup of prime order p. If F_1 is a characteristic subgroup of F then $F_1\lhd G$ by Lemma 2.1 and $H<HF_1<G$ which is a contradiction. Hence F is a characteristically simple group. Since F is a nilpotent group and $Z(F)$ is a characteristic subgroup, so $Z(F)=F$. Thus it is an elementary abelian group by Lemma 2.1 i.e it is a vector space over a finite field $\mathbb{F}_q$ where q is a prime number. Also conjugation by the generator of H is a linear map of order p.

Since G is a minimal Frobenius group, so F has no invariant subspaces. So G is an $\mathcal{A}-$group. Conversely, assume $G = V \rtimes C_p$ is an $\mathcal{A}-$group. So $H = C_p$ is a non normal subgroup of prime order by remark 3.1. If $H < T < G$ then $T \cap V < V$ is a non-trivial invariant subspace of V which is a contradiction. So H is a maximal subgroup of prime order. Hence G is a minimal Frobenius group by Theorem 3.1. $\square$

Theorem 3.4. *Assume G is not an $\mathcal{A}-$group. Let N be a nontrivial normal subgroup of G and H be a subgroup of G. Then $d(H, N) \leq 3$ where d is distance in the order graph. Moreover, if $d(H, N) = 3$ then $G = HN$ and $(|H|, |N|) = 1$.*

Proof. Assume $d(H, N) \geq 3$. If $p\big|(|H|, |N|)$ and P is a subgroup of order p then $H - P - N$ is a path of length two between H, N which is a contradiction. Hence $(|H|, |N|) = 1$. If $HN < G$ then $H - HN - N$ is a path of length two between H and N which is a contradiction. Thus $G = HN$. Now assume $d(H, N) > 3$. If $|H|$ is not a prime number then H has a subgroup H_1. Then $H - H_1 - H_1N - N$ is a path between H, N. So $d(H, N) \leq 3$. Hence $|H|$ is a prime number. If $H < M < G$ then $M = H(M \cap N)$. So $H - M - M \cap N - N$ is a path between H, N. Thus $d(H, N) \leq 3$. Hence H is a maximal subgroup of G. If $H \lhd G$ then $|N|$ is also a prime number similarly. So $G \cong C_{pq}$. Hence G is an $\mathcal{A}-$group which is a contradiction. If $H \ntrianglelefteq G$ then G is a minimal Frobenius group by Theorem 3.1. So G is an $\mathcal{A}-$group by Theorem 3.3 which is a contradiction. $\square$

Theorem 3.5. *Let G be a non simple group. Then $\Gamma^*(G)$ is connected if and only if G is not an $\mathcal{A}-$group. moreover if $\Gamma^*(G)$ is connected then its diameter is at most 4.*

Proof. Let N be a nontrivial normal subgroup of G and H, K be two subgroups of G. First assume G is not an $\mathcal{A}-$group. Suppose $d(H, N) = 3$. Hence $G = HN$ and $(|H|, |N|) = 1$ by Theorem 3.4. If $p\big|(|H|, |K|) > 1$ and P is a subgroup of order p then $H - P - K$ is a path of length two between H, K. So $d(H, K) \leq 2$. Now assume $(|H|, |K|) = 1$. Thus $|K|\big||G| = |H||N|$. So $|K|\big||N|$. Hence K and N are adjacent and $d(H, K) \leq 4$. By symmetry, If $d(K, N) = 3$ then $d(H, K) \leq 4$. If $d(H, N), d(K, N) \leq 2$ then $d(H, K) \leq 4$. Conversely suppose that $G = \mathcal{A}(p, q, n)$ is an $\mathcal{A}-$group then the subgroups of G are of order p and q^i. So $\Gamma^*(G)$ is a disconnected graph. $\square$

Example 3.6. *Let G be group of order $147 = 3 \times 7^2$. Since $x^3 - 1 = (x - 1)(x - 2)(x - 4)$ over $\mathbb{F}_7$, so G is not an $\mathcal{A}-$group by Definition 3.1. Hence $\Gamma^*(G)$ is a connected graph.*

Theorem 3.7. [4, P. Hall's theorem,p 7]*Let G be solvable and $|G| = mn$ with $(m, n) = 1$. Then G contains subgroups of order m, and all these are conjugate in G. If $U \leq G$ and $|U|$ divides m, then there exists $H \leq G$ such that $|H| = m$ and $U \leq H$.*

Theorem 3.8. *Let G be a finite solvable group which is not an $\mathcal{A}-$group. Then $diam(\Gamma^*(G)) \leq 4$. If $|\pi(G)| \geq 3$ then $diam(\Gamma^*(G)) \leq 3$.*

Proof. Since G is solvable, so it is not a simple group. So $diam(\Gamma^*(G)) \leq 4$ by Theorem 3.5. Assume $|\pi(G)| \geq 3$ and H, K be two subgroups of G. If $p\big|(|H|, |K|) > 1$ and P is a subgroup of order p then $H - P - K$ is a path of length two between H, K. So $d(H, K) \leq 2$. Assume now that $(|H|, |K|) = 1$. Without loss of generality, assume $|\pi(H)| \geq |\pi(K)|$. Let P be a subgroup of order $p\big||H|$. It is easily seen that $p|K|\big|m$ where $(m, \frac{|G|}{m}) = 1$. Hence G has a Hall-subgroup M of order m. Thus $H - P - M - K$ is a path between H, K. So $d(H, K) \leq 3$. $\square$

In [5], it is proved that the diameter of the order graph of an abelian group is at most four. We improve this bound by the following theorem.

Theorem 3.9. *Let $G \ncong C_{pq}$ be a finite nilpotent group. Then $diam(\Gamma^*(G)) \leq 3$.*

Proof. First note that if G is a nilpotent group and $d\big||G|$ then G has a (normal) subgroup of order d. If $|\pi(G)| \geq 3$ then $diam(\Gamma^*(G)) \leq 3$ by Theorem 3.8. So assume that $|\pi(G)| = 2$. Since $|G| \neq pq$, so $|G| = p^a q^b$ where p, q are prime numbers and $a > 1$. Let H be the unique subgroup of order p^a. Also G has a subgroup K of order $p^{a-1}q^b$. Let K_1 be a subgroup of K of order p^{a-1}. Then every subgroup of G is adjacent to H or K. Also $d(H, K) = 2$ by the path $H - K_1 - K$. Let N and M be two subgroups of G. If $(|M|, |N|) > 1$ then $d(M, N) \leq 2$. If $(|M|, |N|) = 1$ then $|M| = p^{a'}$ and $|N| = q^{b'}$. If $a' < a$ then $M - K - N$ is a path of length two. If $a' = a$ then $M - K_1 - K - N$ is a path of length three. So $diam(\Gamma^*(G)) \leq 3$. If G is a $p-$group then $\Gamma^*(G)$ is a complete graph and $diam(\Gamma^*(G)) = 1$. $\square$

Remark 3.2. Let $G = \mathbb{Z}_{12}$. So G has 4 nontrivial proper subgroups. So $\Gamma^*(G)$ is the path $\langle 2\rangle, \langle 3\rangle, \langle 4\rangle, \langle 6\rangle$. Hence $\Gamma^*(G)$ has diameter three.

The following theorem gives a relation between the order graph of a group and its prime graph.

Theorem 3.10. *If the prime graph of G is a connected graph then $\Gamma^*(G)$ is a connected graph.*

Proof. Let S be the set of subgroups of prime orders. So every subgroup of G is adjacent to an element of S in $\Gamma^*(G)$. It suffices to prove that there is a path between any two elements of S. Let $p \neq q \in \pi(G)$ be two adjacent vertices of the prime graph of G. So there is an element g_{pq} in G of order pq. Hence $P - \langle g_{pq} \rangle - Q$ for every $P, Q \in S$ such that $|P| = p, |Q| = q$. Since the prime graph is a connected graph, so there is a path between any two elements of S. Hence $\Gamma^*(G)$ is a connected graph. $\square$

Remark 3.3. Let $G = A_5$ and $H \cong A_4$ and $K \cong D_5$ be the subgroups of order 12 and 10 of G. The prime graph of G has three isolated vertices $2, 3, 5$. So it is not a connected graph. The orders of subgroups of A_5 are $2, 3, 4, 5, 6, 10, 12$. So every subgroup is adjacent to H or K. Also H and K are adjacent to any subgroup of order two. Hence $\Gamma^*(A_5)$ is a connected graph. So the converse of Theorem 3.10 is not true.

Theorem 3.11. *If $n \geq 5$ then $\Gamma^*(A_n)$ is a connected graph. Moreover, If n is not a prime number then $diam(\Gamma^*(A_n)) \leq 4$ else $diam(\Gamma^*(A_n)) \leq 5$.*

Proof. It is clear that $A_{n-1} < A_n$. Also if $p|n!$ then $p \leq n$. Let H be a subgroup of A_n and P be a subgroup in H of prime order p. First assume $p < n$. Hence $p|\frac{(n-1)!}{2}$. So P is adjacent to A_{n-1}. Thus $H - P - A_{n-1}$ is a path between H and A_{n-1}. So assume $n = p$ and $P < H$. Since $p \nmid (p-1)!$, so $\frac{|H|}{p}$ has a prime factor $q \neq p$. Hence $q \big| |A_{n-1}|$. Let Q be the subgroup of order q in H. So Q is adjacent to A_{n-1}. Thus $H - Q - A_{n-1}$ is a path between H and A_{n-1}. If $n = p$ and $H = P$ then H is a subgroup of dihedral group D_p in A_p. So H is not a maximal subgroup of A_n. Let M be a subgroup which $H < M < G$. So $d(M, A_{n-1}) \leq 2$ and $d(H, A_{n-1}) \leq 3$. Let K be another subgroup of A_n. If n is not a prime number then $d(H, A_{n-1}) \leq 2$ and $d(K, A_{n-1}) \leq 2$. So $d(H, K) \leq 4$. If $n = p$ is a prime number then without loss of generality $|K| \neq p$. Hence $d(H, A_{n-1}) \leq 3$ and $d(K, A_{n-1}) \leq 2$. So $d(H, K) \leq 5$. $\square$

Remark 3.4. Since A_4 has no subgroup of order six, so $\Gamma^*(A_4)$ is a disconnected graph. Also $A_4 \cong (C_2 \oplus C_2) \rtimes C_3 = \mathcal{A}(3, 2, 2)$ is an $\mathcal{A}-$group.

4. Conclusion and future research

In this paper, we obtained some results about the order graph of a group. we have obtained a necessary and sufficient condition for a non simple group which ensures us that the order graph is connected. Also a relation between prime graph and order graph is given. It is proved that the order graph A_n is connected for $n \geq 5$. But for other simple groups the problem of connectedness of the order graph is unsolved.

Competing Interests

The author declares that no competing interests exist.

References

[1] J. A. Bondi, J. S. Murty, Graph theory with applications, American Elsevier Publishing Co, INC, 1997.

[2] Y. Chen, On Thompson's conjecture, J. Algebra 15 (1996), 184-193.

[3] J. A. Gallian, Contemporary Abstract Algebra, D. C. Heath and company, 1994.

[4] B. Huppert, Character Theory of Finite Groups, De Gruyter Expositions in Mathematics, New York 1998.

[5] Sh. Payrovi, H. Pasebani, The Order Graphs of Groups, J Algebraic Structures and Their Applications, 1 (no 1) (2014), 1-10.

[6] J. S. Wiliams, Prime Graph Components of Finite Groups, J. Algebra 69 (1981), 487-513.

Hamid Reza Dorbidi

Department of Basic Sciences

University of Jiroft, P.O.Box 78671-61167, Jiroft, Kerman, Iran hr_dorbidi@ujiroft.ac.ir