A NOTE ON THE ORDER GRAPH OF A GROUP

HAMID REZA DORBIDI

Communicated by Cheryl E. Praeger

Abstract. The order graph of a group $G$, denoted by $\Gamma^*(G)$, is a graph whose vertices are non trivial proper subgroups of $G$ and two distinct vertices $H$ and $K$ are adjacent if and only if $|H|||K|$ or $|K|||H|$. In this paper, we study the connectivity and diameter of this graph. Also we give a relation between the order graph and prime graph of a group.

1. Introduction

Let $G$ be a finite group. The order graph of $G$ is the (undirected) graph $\Gamma^*(G)$, whose vertices are non-trivial proper subgroups of $G$ and two distinct vertices $H$ and $K$ are adjacent if and only if either $|H|||K|$ or $|K|||H|$. So $\Gamma^*(G)$ is the empty graph if and only if $|G|$ is a prime number. So we only consider the groups whose orders are not prime. This graph has studied in [5]. In this paper, we study the connectivity and diameter of this graph. We show that the order graph of a non-simple group is a connected graph with diameter less than four with only one family exception. Also we give a connection between prime graph and order graph.


Keywords: Connected graph, Frobenius group, Order graph, Prime graph.

Received: 30 April 2015, Accepted: 07 Feb 2017.

© 2016 Yazd University.
First we recall some facts and notations related to this paper. Throughout this paper $G$ denotes a finite group. The normalizer of a subgroup $H$ is denoted by $N_G(H)$. A subgroup $N$ of $G$ is called a characteristic subgroup if $\sigma(N) \subseteq N$ for all $\sigma \in Aut(G)$. A group $G$ is characteristically simple if the only characteristic subgroups of $G$ are $\{1\}$ and $G$. We denote $gHg^{-1}$ by $H^g$. Let $\pi(n)$ be the set of prime divisors of $n$. We denote $\pi(|G|)$ by $\pi(G)$. The prime graph of a group $G$ is a graph whose vertex set is $\pi(G)$ and two distinct primes $p$ and $q$ are adjacent if and only if $G$ has an element of order $pq$. The cyclic group of order $n$ is denoted by $C_n$. The symmetric group on $n$ letters is denoted by $S_n$. $D_n$ is the dihedral group of order $2n$. The alternating group is denoted by $A_n$. The finite field with $q$ elements is denoted by $\mathbb{F}_q$.

Let $V$ be a vector space and $A : V \rightarrow V$ be a linear mapping. A subspace $W$ of $V$ is an $A$–invariant subspace if $A(W) \subseteq W$. The vector space $V$ is called $A$–simple if the only $A$–invariant subspaces of $V$ are 0 and $V$.

Let $\Gamma$ be a simple graph with vertex set $V$. A path between two vertices $x$ and $y$ is a sequence $x = v_0, \ldots, v_n = y$ of vertices such that $v_i$ is adjacent to $v_{i-1}$. A graph is connected if for any two vertices $x, y$ there is a path between $x$ and $y$. The length of the shortest path between $x$ and $y$ is denoted by $d(x, y)$. Also $diam(\Gamma) = sup\{d(x, y) : x$ and $y$ are two vertices of $\Gamma\}$ is called the diameter of the graph $\Gamma$.

### 2. Preliminaries

Before proving the main theorems, we need the following definitions, lemma and theorems.

**Definition 2.1.** If $f : H \rightarrow Aut(N)$ is a homomorphism then $G = N \times H$ with the operation $(n, h)(n', h') = (f(h^{-1})(n)n', hh')$ is called semi-direct product of $H$ and $N$ and is denoted $G = N \rtimes H$. If $G = HN$ and $H \cap N = \{1\}$ where $N \triangleleft G$ then $G \cong N \rtimes H$. If $f$ is the trivial homomorphism then $G = H \times N$ is the direct product of groups.

**Lemma 2.1.**

1. If $N_1$ is a characteristic subgroup of $N$ and $N \triangleleft G$ then $N_1 \triangleleft G$.

2. If $G$ is an abelian characteristically simple group then $G$ is an elementary abelian group i.e $G$ is a vector space over a finite field $\mathbb{F}_q$ where $q$ is a prime number.

**Proof.**

1. Let $\sigma_g$ be the conjugation by $g \in G$. Since $N \triangleleft G$, so $\sigma_g$ is an automorphism of $N$ for all $g \in G$. Hence $N_1^g = \sigma_g(N_1) \subseteq N_1$ i.e $N_1 \triangleleft G$.

2. Let $q$ be a prime divisor of $|G|$. So $G_1 = \{g \in G : g^q = 1\}$ is a characteristic subgroup. Thus $G_1 = G$. Hence $G$ is an elementary abelian group i.e $G$ is a vector space over the finite field $\mathbb{F}_q$. 0.1cm□
Definition 2.2. If there exists a proper subgroup $H \neq \{1\}$ of $G$ such that $H^g \cap H = \{1\}$ for all $g \in G\setminus H$ then $G$ is called a Frobenius group with respect to $H$.

Remark 2.1. If $G$ is a Frobenius group with respect to $H$ and $H < M < G$ then $M$ is also a Frobenius group with respect to $H$. It is clear that $Z(G) = 1$. So a Frobenius group $G$ is not a nilpotent group.

The following theorem states some properties of the Frobenius groups.

Theorem 2.2. [4, chapter 16] If $G$ is a Frobenius group with respect to $H$ then $F = G\setminus (\cup_{g \in G} H^g \setminus \{1\})$ is a normal subgroup of $G$ which is called Frobenius kernel of $G$. Since $G = FH$ and $F \cap H = \{1\}$, $H$ is called a complement of $F$. Also $F$ is a nilpotent subgroup and $|H||F| - 1$.

Definition 2.3. A Frobenius group $G$ is called a minimal Frobenius group if $G$ has no proper Frobenius subgroup.

3. Main results

In this section we state our main results.

Theorem 3.1. A group $G$ is a minimal Frobenius group if and only if $G$ has a non normal maximal subgroup of prime order.

Proof. Let $G$ be a minimal Frobenius group with complement $H$ and kernel $F$. If $H < M$ then $M$ is a Frobenius group. So $M = G$ by minimality of $G$. So $H$ is a maximal subgroup. If $H_1 < H$ is a non-trivial subgroup then $H_1 F \cap H = H_1$. This implies that $H_1 F$ is a Frobenius group with respect to $H_1$ which is a contradiction. So $H$ has no proper non-trivial subgroup. Hence $|H|$ is a prime number. Conversely, assume $G$ has a non normal maximal subgroup $H$ of prime order. Since $|H|$ is a prime number, so $H \cap H^g = \{1\}$ for all $g \in G\setminus N_G(H)$. But $N_G(H) = H$ by maximality and non normality of $H$. So $G$ is a Frobenius group with respect to $H$. Let $G_1 < G$ be a subgroup of $G$. If $G_1 \cap H^g = \{1\}$ for all $g \in G$ then $G_1 \subseteq F$ is a nilpotent subgroup. So $G_1$ is not a Frobenius group. Assume $G_1 \cap H^g \neq \{1\}$ for some $g \in G$. Since $|H^g|$ is a prime number, we conclude that $H^g \subseteq G_1$. So $H^g = G_1$ by maximality of $H^g$. Hence $G_1$ is not a Frobenius group. Therefore $G$ is a minimal Frobenius group.

Theorem 3.2. Let $p \neq q$ be two prime numbers and $n$ be an integer. $x^{(p-q^n-1)} - 1$ has an irreducible factor of degree $n$ over $\mathbb{F}_q$ if and only if there is an $n$-dimensional vector space $V$ over $\mathbb{F}_q$ and a linear mapping $A$ such that $A^p = 1$ and $V$ is $A$-simple.
Proof. Assume $x^{(p,q^n-1)} - 1$ has an irreducible factor $f(x)$ of degree $n$ over $\mathbb{F}_q$. Let $V = \mathbb{F}_{q^n} = \mathbb{F}_q[\alpha]$ be the splitting field of $f(x)(\alpha$ is a root of $f(x))$. Let $A : V \to V$ be the multiplication by $\alpha$. Then the minimal polynomial of $A$ is $f(x)$. Since $f(x)|x^p - 1$, so $A^p = 1$. Let $0 \neq v \in V$ and $f_v(x)$ be a polynomial of minimum degree such that $f_v(A)(v) = 0$. Let $V_1 = \langle v, Av, \ldots, A^n(v) \rangle$ be the minimal $A$–invariant subspace of $V$ which contains $v$. It is clear that $f_v(x)|f(x)$ and $\dim(V_1) = \deg(f_v(x))$. Since $f(x)$ is irreducible, so $f_v(x) = f(x)$. Thus $V = V_1$. Hence $V$ has no non trivial $A$–invariant subspaces. Conversely, Assume $V$ is an $n$–dimensional vector space over $\mathbb{F}_q$ and there is a linear mapping $A$ such that $A^p = 1$ and $V$ is $A$–simple. Let $f_A(x)$ be the minimal polynomial of $A$ over $\mathbb{F}_q$ of degree $m$. Since $A^p = 1$, so $f_A(x)|x^p - 1$. If $f_A(x) = h(x)g(x)$ then $\ker(g(A))$ is an $A$–invariant subspace. So $\ker(g(A)) = V$ or $\ker(g(A)) = 0$. If $\ker(g(A)) = V$ then $g(A) = 0$. So $g(x) = f_A(x)$ by definition of $f_A(x)$. If $\ker(g(A)) = 0$ then $h(A) = 0$ since $\im(h(A) \subseteq \ker(g(A))) = 0$. So $h(x) = f_A(x)$ by definition of $f_A(x)$. Hence $f_A(x)$ is an irreducible polynomial. If $v \neq 0$ then $V_1 = \langle v, Av, \ldots, A^{m-1}(v) \rangle$ is an $A$–invariant subspace. So $V_1 = V$ and $m = n$. It is a standard fact in finite field theory that every irreducible polynomial of degree $n$ over $\mathbb{F}_q$ is a factor of $x^{q^n} - x$. Since $A$ is invertible, $f_A(x)|x^{q^n-1} - 1$. Hence $f_A(x)|(x^{q^n-1} - 1, x^p - 1) = x(q^n-1,p) - 1$. □

Definition 3.1. Let $p \neq q$ be two prime numbers and $n$ be an integer. Assume that $x^{(q^n-1,p)} - 1$ has an irreducible factor $f(x)$ of degree $n$ over $\mathbb{F}_q$. Let $V = \mathbb{F}_{q^n} = \mathbb{F}_q[\alpha]$ be the splitting field of $f(x)$. Let $A : V \to V$ be the multiplication by $\alpha$. Since $A^p = 1$, so there is a homomorphism from $C_p$ to $\text{Aut}(V)$. Let $A(p,q,n) = V \rtimes C_p$ be the semi direct product of action of $C_p$ on $V$. The set of all $A(p,q,n)$ is denoted by $A$. Every element of $A$ is called an $A$–group.

Remark 3.1. If $n = 1$ and $p \nmid q - 1$ then $A(p,q,1) = C_p \times C_q \cong C_{pq}$. If $n = 1$ and $p|q - 1$ then $A(p,q,1) = C_{pq}$ or $A(p,q,1) = C_q \rtimes C_p$. Also if $n > 1$ then $p|q^n - 1$. Note that $C_p$ is normal in $A(p,q,n)$ if and only if $n = 1$ and $A(p,q,1) = C_{pq}$.

The following theorem gives the structure of minimal Frobenius groups.

Theorem 3.3. A group $G$ is a minimal Frobenius group if and only if $G$ is an $A$–group and $G \not\cong C_{pq}$.

Proof. Let $G$ be a minimal Frobenius group with kernel $F$ and complement $H$. So $H$ is a maximal subgroup of prime order $p$. If $F_1$ is a characteristic subgroup of $F$ then $F_1 < G$ by Lemma 2.1 and $H < HF_1 < G$ which is a contradiction. Hence $F$ is a characteristically simple group. Since $F$ is a nilpotent group and $Z(F)$ is a characteristic subgroup, so $Z(F) = F$. Thus it is an elementary abelian group by Lemma 2.1 i.e it is a vector space over a finite field $\mathbb{F}_q$ where $q$ is a prime number. Also conjugation by the generator of $H$ is a linear map of order $p$.
Since $G$ is a minimal Frobenius group, so $F$ has no invariant subspaces. So $G$ is an $A$–group. Conversely, assume $G = V \rtimes C_p$ is an $A$–group. So $H = C_p$ is a non normal subgroup of prime order by remark 3.1. If $H < T < G$ then $T \cap V < V$ is a non-trivial invariant subspace of $V$ which is a contradiction. So $H$ is a maximal subgroup of prime order. Hence $G$ is a minimal Frobenius group by Theorem 3.1.

**Theorem 3.4.** Assume $G$ is not an $A$–group. Let $N$ be a nontrivial normal subgroup of $G$ and $H$ be a subgroup of $G$. Then $d(H, N) \leq 3$ where $d$ is distance in the order graph. Moreover, if $d(H, N) = 3$ then $G = HN$ and $(|H|, |N|) = 1$.

**Proof.** Assume $d(H, N) \geq 3$. If $p | (|H|, |N|)$ and $P$ is a subgroup of order $p$ then $H - P - N$ is a path of length two between $H, N$ which is a contradiction. Hence $(|H|, |N|) = 1$. If $HN < G$ then $H - HN - N$ is a path of length two between $H$ and $N$ which is a contradiction. Thus $G = HN$. Now assume $d(H, N) > 3$. If $|H|$ is not a prime number then $H$ has a subgroup $H_1$. Then $H - H_1 - H_1N - N$ is a path between $H, N$. So $d(H, N) \leq 3$. Hence $|H|$ is a prime number. If $H < M < G$ then $M = H(M \cap N)$. So $H - M - M \cap N - N$ is a path between $H, N$. Thus $d(H, N) \leq 3$. Hence $H$ is a maximal subgroup of $G$. If $H < G$ then $|N|$ is also a prime number similarly. So $G \cong C_{pq}$. Hence $G$ is an $A$–group which is a contradiction. If $H \not\triangleleft G$ then $G$ is a minimal Frobenius group by Theorem 3.1. So $G$ is an $A$–group by Theorem 3.3 which is a contradiction.

**Theorem 3.5.** Let $G$ be a non simple group. Then $\Gamma^*(G)$ is connected if and only if $G$ is not an $A$–group. moreover if $\Gamma^*(G)$ is connected then its diameter is at most 4.

**Proof.** Let $N$ be a nontrivial normal subgroup of $G$ and $H, K$ be two subgroups of $G$. First assume $G$ is not an $A$–group. Suppose $d(H, N) = 3$. Hence $G = HN$ and $(|H|, |N|) = 1$ by Theorem 3.4. If $p | (|H|, |K|) > 1$ and $P$ is a subgroup of order $p$ then $H - P - K$ is a path of length two between $H, K$. So $d(H, K) \leq 2$. Now assume $(|H|, |K|) = 1$. Thus $|K||G| = |H||N|$. So $|K|||N|$. Hence $K$ and $N$ are adjacent and $d(H, K) \leq 4$. By symmetry, if $d(K, N) = 3$ then $d(H, K) \leq 4$. If $d(H, N), d(K, N) \leq 2$ then $d(H, K) \leq 4$. Conversely suppose that $G = A(p, q, n)$ is an $A$–group then the subgroups of $G$ are of order $p$ and $q^i$. So $\Gamma^*(G)$ is a disconnected graph.

**Example 3.6.** Let $G$ be group of order $147 = 3 \times 7^2$. Since $x^3 - 1 = (x - 1)(x - 2)(x - 4)$ over $\mathbb{F}_7$, so $G$ is not an $A$–group by Definition 3.1. Hence $\Gamma^*(G)$ is a connected graph.
Theorem 3.7. [4] P. Hall’s theorem,p 7|Let $G$ be solvable and $|G| = mn$ with $(m, n) = 1$. Then $G$ contains subgroups of order $m$, and all these are conjugate in $G$. If $U \leq G$ and $|U|$ divides $m$, then there exists $H \leq G$ such that $|H| = m$ and $U \leq H$.

Theorem 3.8. Let $G$ be a finite solvable group which is not an $A$–group. Then $\text{diam}(\Gamma^*(G)) \leq 4$. If $|\pi(G)| \geq 3$ then $\text{diam}(\Gamma^*(G)) \leq 3$.

Proof. Since $G$ is solvable, so it is not a simple group. So $\text{diam}(\Gamma^*(G)) \leq 4$ by Theorem 3.5. Assume $|\pi(G)| \geq 3$ and $H, K$ be two subgroups of $G$. If $p|(|H||K|) > 1$ and $P$ is a subgroup of order $p$ then $H - P - K$ is a path of length two between $H, K$. So $d(H, K) \leq 2$. Assume now that $p|(|H||K|) = 1$. Without loss of generality, assume $|\pi(H)| \geq |\pi(K)|$. Let $P$ be a subgroup of order $p||H|$. It is easily seen that $p|K||m$ where $(m, \frac{|G|}{m}) = 1$. Hence $G$ has a Hall-subgroup $M$ of order $m$. Thus $H - P - M - K$ is a path between $H, K$. So $d(H, K) \leq 3$. \[\square\]

In [5], it is proved that the diameter of the order graph of an abelian group is at most four. We improve this bound by the following theorem.

Theorem 3.9. Let $G \not\cong C_{pq}$ be a finite nilpotent group. Then $\text{diam}(\Gamma^*(G)) \leq 3$.

Proof. First note that if $G$ is a nilpotent group and $d||G|$ then $G$ has a (normal) subgroup of order $d$. If $|\pi(G)| \geq 3$ then $\text{diam}(\Gamma^*(G)) \leq 3$ by Theorem 3.8. So assume that $|\pi(G)| = 2$. Since $|G| \neq pq$, so $|G| = p^aq^b$ where $p, q$ are prime numbers and $a > 1$. Let $H$ be the unique subgroup of order $p^a$. Also $G$ has a subgroup $K$ of order $p^{a-1}q^b$. Let $K_1$ be a subgroup of $K$ of order $p^{a-1}$. Then every subgroup of $G$ is adjacent to $H$ or $K$. Also $d(H, K) = 2$ by the path $H - K_1 - K$. Let $N$ and $M$ be two subgroups of $G$. If $(|M|, |N|) > 1$ then $d(M, N) \leq 2$. If $(|M|, |N|) = 1$ then $|M| = p^{a'}$ and $|N| = q^{b'}$. If $a' < a$ then $M - K - N$ is a path of length two. If $a' = a$ then $M - K_1 - K - N$ is a path of length three. So $\text{diam}(\Gamma^*(G)) \leq 3$. If $G$ is a $p$–group then $\Gamma^*(G)$ is a complete graph and $\text{diam}(\Gamma^*(G)) = 1$. \[\square\]

Remark 3.2. Let $G = \mathbb{Z}_{12}$. So $G$ has 4 nontrivial proper subgroups. So $\Gamma^*(G)$ is the path $(2), (3), (4), (6)$. Hence $\Gamma^*(G)$ has diameter three.

The following theorem gives a relation between the order graph of a group and its prime graph.

Theorem 3.10. If the prime graph of $G$ is a connected graph then $\Gamma^*(G)$ is a connected graph.
Proof. Let S be the set of subgroups of prime orders. So every subgroup of G is adjacent to an element of S in \( \Gamma^*(G) \). It suffices to prove that there is a path between any two elements of S. Let \( p \neq q \in \pi(G) \) be two adjacent vertices of the prime graph of G. So there is an element \( g_{pq} \) in G of order \( pq \). Hence \( P - \langle g_{pq} \rangle - Q \) for every \( P, Q \in S \) such that \( |P| = p, |Q| = q \). Since the prime graph is a connected graph, so there is a path between any two elements of S. Hence \( \Gamma^*(G) \) is a connected graph. \( \square \)

Remark 3.3. Let \( G = A_5 \) and \( H \cong A_4 \) and \( K \cong D_5 \) be the subgroups of order 12 and 10 of G. The prime graph of G has three isolated vertices 2, 3, 5. So it is not a connected graph. The orders of subgroups of \( A_5 \) are 2, 3, 4, 5, 6, 10, 12. So every subgroup is adjacent to H or K. Also H and K are adjacent to any subgroup of order two. Hence \( \Gamma^*(A_5) \) is a connected graph. So the converse of Theorem 3.10 is not true.

**Theorem 3.11.** If \( n \geq 5 \) then \( \Gamma^*(A_n) \) is a connected graph. Moreover, If \( n \) is not a prime number then \( \text{diam}(\Gamma^*(A_n)) \leq 4 \) else \( \text{diam}(\Gamma^*(A_n)) \leq 5 \).

Proof. It is clear that \( A_{n-1} < A_n \). Also if \( p|n! \) then \( p \leq n \). Let H be a subgroup of \( A_n \) and P be a subgroup in H of prime order p. First assume \( p < n \). Hence \( p|\frac{(n-1)!}{2} \). So P is adjacent to \( A_{n-1} \). Thus \( H - P - A_{n-1} \) is a path between H and \( A_{n-1} \). So assume \( n = p \) and \( P < H \). Since \( p \nmid (p-1)! \), so \( \frac{|H|}{p} \) has a prime factor \( q \neq p \). Hence \( q|\text{order of } A_{n-1} \). Let Q be the subgroup of order q in H. So Q is adjacent to \( A_{n-1} \). Thus \( H - Q - A_{n-1} \) is a path between H and \( A_{n-1} \). If \( n = p \) and \( H = P \) then H is a subgroup of dihedral group \( D_p \) in \( A_p \). So H is not a maximal subgroup of \( A_n \). Let M be a subgroup which \( H < M < G \). So \( d(M, A_{n-1}) \leq 2 \) and \( d(H, A_{n-1}) \leq 3 \). Let K be another subgroup of \( A_n \). If \( n \) is not a prime number then \( d(H, A_{n-1}) \leq 2 \) and \( d(K, A_{n-1}) \leq 2 \). So \( d(H, K) \leq 4 \). If \( n = p \) is a prime number then without loss of generality \( |K| \neq p \). Hence \( d(H, A_{n-1}) \leq 3 \) and \( d(K, A_{n-1}) \leq 2 \). So \( d(H, K) \leq 5 \). \( \square \)

Remark 3.4. Since \( A_4 \) has no subgroup of order six, so \( \Gamma^*(A_4) \) is a disconnected graph. Also \( A_4 \cong (C_2 \oplus C_2) \rtimes C_3 = A(3, 2; 2) \) is an \( A \)-group.

4. **Conclusion and future research**

In this paper, we obtained some results about the order graph of a group. We have obtained a necessary and sufficient condition for a non simple group which ensures us that the order graph is connected. Also a relation between prime graph and order graph is given. It is proved that the order graph \( A_n \) is connected for \( n \geq 5 \). But for other simple groups the problem of connectedness of the order graph is unsolved.
Competing Interests

The author declares that no competing interests exist.

REFERENCES

(no 1) (2014), 1-10.

Hamid Reza Dorbidi

Department of Basic Sciences

University of Jiroft, P.O.Box 78671-61167, Jiroft, Kerman, Iran hr.dorbidi@ujiroft.ac.ir