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z°-FILTERS AND RELATED IDEALS IN $C(X)$

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Abstract. In this article we introduce the concept of z°-filter on a topological space X. We study and investigate the behavior of z°-filters and compare them with corresponding ideals, namely, z°-ideals of $C(X)$, the ring of real-valued continuous functions on a completely regular Hausdorff space X. It is observed that X is a compact space if and only if every z°-filter is ci-fixed. Finally, by using z°-ultrafilters, we prove that any arbitrary product of i-compact spaces is i-compact.

1. Introduction and Preliminaries

We consider X to be a completely regular Hausdorff space and we denote by $C(X)$ the ring of all real-valued continuous functions on the space X. For each $f \in C(X)$, the set $Z(f) = \{x \in X : f(x) = 0\}$ is the zero-set of f and $Z(X) = \{Z(f) : f \in C(X)\}$. For $A \subseteq X$, by A° and $\overline{A}$ we mean the interior and the closure of A, respectively. An ideal I of $C(X)$ is called a z-ideal if f, $g \in C(X)$ with $Z(f) \subseteq Z(g)$ and $f \in I$ imply that $g \in I$. A nonempty subfamily $\mathcal{F}$ of $Z(X)$ is called a z-filter on X, if $\emptyset \notin \mathcal{F}$, the intersection of any two members of $\mathcal{F}$ is again a member of $\mathcal{F}$, and any member of $Z(X)$ containing

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a member of $\mathcal{F}$ also belongs to $\mathcal{F}$. In commutative rings, an ideal consisting entirely of zero divisors is called nonregular. Note that $f \in C(X)$ is a zero divisor if and only if $Z^\circ(f) = (Z(f))^\circ \neq \emptyset$.

A z-ultrafilter on X is meant a maximal z-filter, i.e., one not contained in any other z-filter. Also a z-filter $\mathcal{F}$ on X is called a prime z-filter if whenever $Z_1 \cup Z_2 \in \mathcal{F}$, then $Z_1 \in \mathcal{F}$ or $Z_2 \in \mathcal{F}$. If I is an ideal in $C(X)$ and $\mathcal{F}$ is a z-filter on X, then we note $Z[I] = \{Z(f) : f \in I\}$ and $Z^{-1}[\mathcal{F}] = \{f : Z(f) \in \mathcal{F}\}$. For $p \in X$, we have a z-ideal $O_p = \{f \in C(X) : p \in Z^\circ(f)\}$ contained in the maximal ideal $M_p = \{f \in C(X) : p \in Z(f)\}$. A point $p \in X$ is called a P-point if $O_p = M_P$. A space X is said to be P-space if its every point is a P-point, or equivalently, $Z(f) = Z^\circ(f)$, for every $f \in C(X)$.

A space X is called an almost P-space if every nonempty zero-set has a nonempty interior, or equivalently, for every $f \in C(X)$, $Z(f)$ is regular closed, i.e., $\overline{Z^\circ(f)} = Z(f)$, or equivalently, every nonregular z-ideal of $C(X)$ is a z°-ideal, see Theorem 4.2 in [4]. X is called basically disconnected if each open set has an open closure, or equivalently, if $\overline{Z^\circ(f)} = Z^\circ(f)$, for every $f \in C(X)$. For more information about P-spaces, almost P-spaces and basically disconnected spaces see [7] and [1], and for undefined terms and notations, the reader is referred to [6] and [7].

2. z°-filters and z°-ideals in $C(X)$

In this section, we introduce z°-filters and strongly prime z°-filters on a topological space X. We study and investigate the behavior of z°-filters and strongly prime z°-filters and compare them with z°-ideals and prime z°-ideals of $C(X)$. Recall that an ideal I in $C(X)$ is a z°-ideal if $f \in I$, $g \in C(X)$ and $Z^\circ(f) \subseteq Z^\circ(g)$ imply that $g \in I$, so every proper z°-ideal of $C(X)$ is a nonregular ideal. For more details and examples of z°-ideals in commutative rings and in $C(X)$, the reader is referred to [8], [2], [3], [4] and [5]. We begin with the following definition.

Definition 2.1. A nonempty subfamily $\mathcal{F}$ of $Z^\circ(X) = \{Z^\circ : Z \in Z(X)\}$ is called a z°-filter on X if it satisfies the following conditions:

a) $\emptyset \notin \mathcal{F}$.
b) If Z_1°, $Z_2^\circ \in \mathcal{F}$, then $Z_1^\circ \cap Z_2^\circ = (Z_1 \cap Z_2)^\circ \in \mathcal{F}$.
c) If $Z^\circ \in \mathcal{F}$ and $Z^\circ \subseteq F^\circ \in Z^\circ(X)$, then $F^\circ \in \mathcal{F}$.

If $A \subseteq C(X)$ and $\mathcal{F} \subseteq Z^\circ(X)$, then we note $Z^\circ[A] = \{Z^\circ(f) : f \in A\}$ and $Z^{\circ^{-1}}[\mathcal{F}] = \{f : Z^\circ(f) \in \mathcal{F}\}$.

Proposition 2.2. *An ideal I in $C(X)$ is a z°-ideal if and only if $Z^\circ(f) \in Z^\circ[I]$ implies that $f \in I$, for each $f \in C(X)$.*

Proof. For each $f \in C(X)$, let $Z^\circ(f) \in Z^\circ[I]$ implies that $f \in I$. Suppose that $g \in C(X)$ and $f \in I$ with $Z^\circ(f) \subseteq Z^\circ(g)$, then $Z^\circ(f) = Z^\circ(f) \cap Z^\circ(g) = (Z(f) \cap Z(g))^\circ = Z^\circ(f^2 + g^{\frac{2}{3}}) \in Z^\circ[I]$. By our hypothesis, we have $f^2 + g^{\frac{2}{3}} \in I$. Since $f \in I$, we infer that $g^{\frac{2}{3}} \in I$ and hence $g \in I$. This shows that I is a z°-ideal. The converse is evident. ■

Proposition 2.3. *a) If $\mathcal{F}$ is a z°-filter on X, then $Z^{\circ^{-1}}[\mathcal{F}]$ is a z°-ideal in $C(X)$.*

b) If I is a z°-ideal in $C(X)$, then $Z^\circ[I]$ is a z°-filter on X.

Proof. It is straightforward. ■

Clearly, O_p is a z°-ideal in $C(X)$, so $Z^\circ[O_p] = \{Z^\circ : p \in Z^\circ\}$ is a z°-filter on X. It is clear that an intersection of z°-ideals in $C(X)$ is a z°-ideal. As in [4] and [5], we denote by $I_\circ$ the smallest z°-ideal containing the ideal I. For any nonregular ideal I, $I_\circ$ exists and in fact it is the intersection of all z°-ideals containing I. For more details about nonregular ideals in $C(X)$, see [4] and [5].

Proposition 2.4. *a) If I is a nonregular ideal in $C(X)$, then $I_\circ = Z^{\circ^{-1}}Z^\circ[I]$.*

b) An ideal I in $C(X)$ is a z°-ideal if and only if $I = Z^{\circ^{-1}}Z^\circ[I]$.

Proof. Evident. ■

Remark 2.5. a) Let I be an ideal in $C(X)$. It is clear that $Z^{-1}Z[I] \subseteq Z^{\circ^{-1}}Z^\circ[I]$. The reverse inclusion does not hold, in general. For example we consider the ideal $I = \{f \in C(\mathbb{R}) : [0,1] \cup \{2\} \subseteq Z(f)\}$ in $C(\mathbb{R})$. Assume that $Z(g) = [0,1]$ and $Z(f) = [0,1] \cup \{2\}$. Clearly, $Z^\circ(g) = Z^\circ(f)$ and $f \in I$, hence $g \in Z^{\circ^{-1}}Z^\circ[I]$. But, since $2 \notin Z(g)$, we infer that $Z(g) \notin Z[I]$ and therefore $g \notin Z^{-1}Z[I]$.

b) X is an almost P-space if and only if $Z^{\circ^{-1}}Z^\circ[I] = Z^{-1}Z[I]$, for every nonregular ideal I in $C(X)$.

Definition 2.6. A z°-filter $\mathcal{F}$ on X is said to be strongly prime (resp., prime) if for any Z_1°, $Z_2^\circ \in Z^\circ(X)$, $(Z_1 \cup Z_2)^\circ \in \mathcal{F}$ (resp., $Z_1^\circ \cup Z_2^\circ \in \mathcal{F}$) implies that $Z_1^\circ \in \mathcal{F}$ or $Z_2^\circ \in \mathcal{F}$.

Clearly, if $\mathcal{F}$ is a strongly prime z°-filter on X, then $Z^{\circ^{-1}}[\mathcal{F}]$ is a prime z°-ideal in $C(X)$ and if P is a prime z°-ideal in $C(X)$, then $Z^\circ[P]$ is a strongly prime z°-filter on X. Every strongly prime z°-filter is prime, but the converse is not true in general. Moreover if $\mathcal{F}$ is a prime z°-filter, $Z^{\circ^{-1}}[\mathcal{F}]$ is not necessarily prime z°-ideal. In fact $Z^\circ[O_p]$ is a prime z°-filter, for each $p \in X$, however O_p is not prime and above argument implies that $Z^\circ[O_p]$ is not strongly prime.

The following results are the counterparts of Theorem 2.9 and Exercise 2E in [7].

Proposition 2.7. *For any z°-ideal I in $C(X)$, the following statements are equivalent:*

a) I is prime.

b) I contains a prime ideal.

c) For all g, $h \in C(X)$, if $gh = 0$, then $g \in I$ or $h \in I$.

d) For every $f \in C(X)$, there is a $Z^\circ \in Z^\circ[I]$ on which f does not change sign.

Proof. Since I is also a z-ideal the implications $(a \Rightarrow b \Rightarrow c \Rightarrow d)$ are evident by Theorem 2.9 in [7].

$(d \Rightarrow a)$ Given $gh \in I$, consider the function $|g| - |h|$. By our hypothesis, there must be a $Z^\circ(f) \in Z^\circ[I]$ on which $|g| - |h|$ does not change sign. By Proposition 2.2 we have $f \in I$. Now without loss of generality, let us assume that $|g| - |h|$ is nonnegative on $Z^\circ(f)$. Hence $Z^\circ(f) \cap Z(g) \subseteq Z^\circ(f) \cap Z(h)$. We claim that $Z^\circ(f) \cap Z^\circ(h) = Z^\circ(f) \cap Z^\circ(gh)$. It is clear that $Z^\circ(f) \cap Z^\circ(h) \subseteq Z^\circ(f) \cap Z^\circ(gh)$. Now let $x \in Z^\circ(f) \cap Z^\circ(gh)$, so there exists an open set G in X such that $x \in G \subseteq Z(g) \cup Z(h)$. Hence $x \in G \cap Z^\circ(f) \subseteq Z^\circ(f) \cap (Z(g) \cup Z(h)) = (Z^\circ(f) \cap Z(g)) \cup (Z^\circ(f) \cap Z(h)) = Z^\circ(f) \cap Z(h)$. This implies that $x \in (Z^\circ(f) \cap Z(h))^\circ = Z^\circ(f) \cap Z^\circ(h)$. Thus $Z^\circ(f^2 + h^2) = Z^\circ(f^2 + g^2h^2)$ and our claim is settled. Since I is a z°-ideal and $f^2 + g^2h^2 \in I$ we have $f^2 + h^2 \in I$, and so $h \in I$. ■

Proposition 2.8. *For a z°-filter $\mathcal{F}$ on X, the following statements are equivalent:*

a) $\mathcal{F}$ is strongly prime.

b) Of any Z_1°, $Z_2^\circ \in Z^\circ(X)$ with $(Z_1 \cup Z_2)^\circ = X$, at least one is in $\mathcal{F}$.

c) Given Z_1, $Z_2 \in Z(X)$, there exists $Z^\circ \in \mathcal{F}$ such that one of $Z^\circ \cap Z_1$, $Z^\circ \cap Z_2$ contains the other.

Proof. The implication $(a \Rightarrow b)$ is clear.

$(b \Rightarrow c)$ Clearly, (b) is equivalent to part (c) of Proposition 2.7 which implies part (d) of the same proposition. Let Z_1, $Z_2 \in Z(X)$, where $Z_1 = Z(g)$ and $Z_2 = Z(h)$ for $g, h \in C(X)$. Now consider the function $|g| - |h|$. Hence by part (d) of Proposition 2.7 there exists $Z^\circ \in \mathcal{F}$ on which $|g| - |h|$ does not change sign, without loss of generality, let us assume that it is nonnegative on Z°. Then on Z°, whenever g vanishes, h must also vanish. In other words $Z^\circ \cap Z_1 \subseteq Z^\circ \cap Z_2$.

$(c \Rightarrow a)$ Suppose that for some Z_1, $Z_2 \in Z(X)$ we have $(Z_1 \cup Z_2)^\circ \in \mathcal{F}$. Then we can find a $Z^\circ \in \mathcal{F}$ such that, $Z^\circ \cap Z_1 \subseteq Z^\circ \cap Z_2$, say. This implies that $Z^\circ \cap (Z_1 \cup Z_2) = Z^\circ \cap Z_2$ and hence $Z^\circ \cap (Z_1 \cup Z_2)^\circ = Z^\circ \cap Z_2^\circ$. But $\mathcal{F}$ is a z°-filter, hence $Z^\circ \cap (Z_1 \cup Z_2)^\circ \in \mathcal{F}$ implies that $Z^\circ \cap Z_2^\circ \in \mathcal{F}$, and therefore $Z_2^\circ \in \mathcal{F}$, i.e., $\mathcal{F}$ is a strongly prime z°-filter. ■

If in the previous proposition we replace the condition "strongly prime" by "prime", then we have the following result.

Corollary 2.9. *For a z°-filter $\mathcal{F}$ on X, the following statements are equivalent:*

a) $\mathcal{F}$ is prime.

b) Of any Z_1°, $Z_2^\circ \in Z^\circ(X)$ with $Z_1^\circ \cup Z_2^\circ = X$, at least one is in $\mathcal{F}$.

c) Given Z_1, $Z_2 \in Z(X)$, there exists $Z^\circ \in \mathcal{F}$ such that one of $Z^\circ \cap Z_1^\circ$, $Z^\circ \cap Z_2^\circ$ contains the other.

The remainder of this section devotes to the outline of the theory of convergence of z°-filters.

Definition 2.10. a) A point $p \in X$ is called a cluster point of a z°-filter $\mathcal{F}$ if p belongs to the closure of every member of $\mathcal{F}$.

b) The z°-filter $\mathcal{F}$ is said to be convergent to the limit $p \in X$ and we write $\mathcal{F} \to p$, if every neighborhood of p contains a member of $\mathcal{F}$.

By the above definition, p is a cluster point of a z°-filter $\mathcal{F}$ if and only if every neighborhood of p intersects all members of $\mathcal{F}$. The set of all cluster points of a z°-filter $\mathcal{F}$ is denoted by $\mathrm{cl}(\mathcal{F})$. Hence $p \in \mathrm{cl}(\mathcal{F})$ if and only if $p \in \bigcap_{Z^\circ \in \mathcal{F}} \overline{Z^\circ}$. Also, if $\mathcal{F} \to p$, then $p \in \mathrm{cl}(\mathcal{F})$, in fact in this case we have $\mathrm{cl}(\mathcal{F}) = \{p\}$, the converse is true for prime z°-filter. Furthermore, $\mathcal{F} \to p$ if and only if $Z^\circ[O_p] \subseteq \mathcal{F}$. It is obvious that $Z^\circ[O_p] \to p$. To see this if G is a neighborhood of p, then by completely regularity of X, there exists $Z^\circ \in Z^\circ(X)$ such that $p \in Z^\circ \subseteq G$.

Example 2.11. Let $\emptyset \neq A \subseteq X$. It is clear that $\mathcal{F} = \{Z^\circ : A \subseteq Z^\circ\}$ is a z°-filter on X. Furthermore, $\mathrm{cl}(\mathcal{F}) = \overline{A}$. To see this suppose that $Z^\circ \in \mathcal{F}$, then $A \subseteq Z^\circ$ and hence $\overline{A} \subseteq \overline{Z^\circ}$, consequently $\overline{A} \subseteq \mathrm{cl}(\mathcal{F})$. Now let there exists $p \in \mathrm{cl}(\mathcal{F})$ which $p \notin \overline{A}$. Therefore there exists $f \in C(X)$ such that $\overline{A} \subseteq Z^\circ(f)$ and $f(p) \neq 0$. Hence $Z^\circ(f) \in \mathcal{F}$ and $p \in \mathrm{cl}(\mathcal{F})$ implies that $p \in \overline{Z^\circ(f)} \subseteq Z(f)$, which is a contradiction.

The next proposition is the counterpart of Theorem 3.17 in [7].

Proposition 2.12. *Let $p \in X$ and $\mathcal{F}$ be a prime z°-filter on X. Then the following statements are equivalent:*

a) $p \in \mathrm{cl}(\mathcal{F})$.

b) $\mathcal{F} \to p$.

c) $\mathrm{cl}(\mathcal{F}) = \{p\}$.

Proof. $(a \Rightarrow b)$ It suffices to show that $Z^\circ[O_p] \subseteq \mathcal{F}$. Let $p \in Z^\circ$, since X is completely regular, there exist Z_1, $Z_2 \in Z(X)$ such that $p \in Z_1^\circ$ and $X \setminus Z^\circ \subseteq Z_2^\circ$ with $Z_1^\circ \cap Z_2^\circ = \emptyset$. Clearly $X = Z^\circ \cup Z_2^\circ \in \mathcal{F}$. Therefore either Z° or Z_2° belongs to the prime z°-filter $\mathcal{F}$. But $Z_2^\circ \notin \mathcal{F}$, because $p \notin \overline{Z_2^\circ}$, so $Z^\circ \in \mathcal{F}^\circ$.

$(b \Rightarrow c)$ Let $Z^\circ \in \mathcal{F}$ and $p \notin \overline{Z^\circ}$. Then there exist Z_1, $Z_2 \in Z(X)$ such that $p \in Z_1^\circ$ and $\overline{Z^\circ} \subseteq Z_2^\circ$ with $Z_1^\circ \cap Z_2^\circ = \emptyset$. By our hypothesis, there exists $Z_3^\circ \in \mathcal{F}$ such that $Z_3^\circ \subseteq Z_1^\circ$. Since $Z^\circ \in \mathcal{F}$ we have $Z_2^\circ \in \mathcal{F}$. But, $Z_3^\circ \cap Z_2^\circ = \emptyset$, a contradiction. Now suppose that $q \in \mathrm{cl}(\mathcal{F})$ and $q \neq p$. Then there exist F_1, $F_2 \in Z(X)$ such that $p \in F_1^\circ$ and $q \in F_2^\circ$ with $F_1^\circ \cap F_2^\circ = \emptyset$. Obviously, $q \notin \overline{F_1^\circ}$ and since $\mathcal{F} \to p$, there exists $F_3^\circ \in \mathcal{F}$ such that $F_3^\circ \subseteq F_1^\circ$. Therefore $q \notin \overline{F_3^\circ}$. This implies that $q \notin \mathrm{cl}(\mathcal{F})$ which is a contradiction.

$(c \Rightarrow a)$ Evident. ■

3. z°-ultrafilters and ci-free z°-filters

Recall that a z-filter $\mathcal{F}$ on X is free (resp., fixed) if $\bigcap_{Z \in \mathcal{F}} Z = \emptyset$ (resp., $\bigcap_{Z \in \mathcal{F}} Z \neq \emptyset$). Also an ideal I in $C(X)$ is free (resp., fixed) if $\bigcap_{f \in I} Z(f) = \emptyset$ (resp., $\bigcap_{f \in I} Z(f) \neq \emptyset$). We say that a z°-filter $\mathcal{F}$

is i-free (resp., i-fixed) if $\bigcap_{Z^\circ\in\mathcal{F}} Z^\circ = \emptyset$ (resp., $\bigcap_{Z^\circ\in\mathcal{F}} Z^\circ \neq \emptyset$). Also a z°-filter $\mathcal{F}$ is called a ci-free (resp., ci-fixed) if $\bigcap_{Z^\circ\in\mathcal{F}} \overline{Z^\circ} = \emptyset$ (resp., $\bigcap_{Z^\circ\in\mathcal{F}} \overline{Z^\circ} \neq \emptyset$). Similarly an ideal I in $C(X)$ is said to be i-free (resp., i-fixed) if $\bigcap_{f\in I} Z^\circ(f) = \emptyset$ (resp., $\bigcap_{f\in I} Z^\circ(f) \neq \emptyset$). Also I is said to be ci-free (resp., ci-fixed) if $\bigcap_{f\in I} \overline{Z^\circ(f)} = \emptyset$ (resp., $\bigcap_{f\in I} \overline{Z^\circ(f)} \neq \emptyset$). We show in this section that X is a compact space if and only if every z°-ultrafilter on X is ci-fixed. It is also shown that an arbitrary product of i-compact (i.e., every z°-filter on X is i-fixed) spaces is an i-compact space. We begin by the following definition.

Definition 3.1. By a z°-ultrafilter on X we mean a maximal z°-filter, i.e., one not contained in any other z°-filter.

Remark 3.2. a) If M is a nonregular maximal ideal in $C(X)$, then $Z^\circ[M]$ is a z°-ultrafilter on X. Furthermore, if $Z^\circ(f)$ meets every member of $Z^\circ[M]$, then $f \in M$.

b) If $\mathcal{A}$ is a z°-ultrafilter on X, then ${Z^\circ}^{-1}[\mathcal{A}]$ is a maximal z°-ideal (maximal in the realm of z°-ideals) in $C(X)$. Furthermore, if Z° meets every member of $\mathcal{A}$, then $Z^\circ \in \mathcal{A}$.

c) Every z°-ultrafilter is a prime z°-filter.

Remark 3.3. a) Every ci-free z°-filter is i-free and every i-fixed z°-filter is ci-fixed.

b) Every free ideal is ci-free and every ci-free ideal is i-free.

c) Every i-fixed ideal is ci-fixed and every ci-fixed ideal is fixed.

d) If X is a P-space, then the set of free ideals, the set of i-free ideals, and the set of ci-free ideals in $C(X)$ coincide.

e) If X is an almost P-space, then an ideal I is free if and only if it is ci-free.

f) If X is a basically disconnected space, then an ideal I is i-free if and only if it is ci-free.

An i-free z°-filter may not be ci-free. In other words a ci-fixed z°-filter may not be i-fixed, see the next example.

Example 3.4. Let $X = [0, 1]$. We consider the ideal $I = \{f \in C(X) : (0, r) \subseteq Z^\circ(f)$ for some $0 < r < 1\}$ in $C(X)$. Clearly, I is a z°-ideal and so $Z^\circ[I]$ is a z°-filter. Now take $f_r \in C(X)$ such that $Z(f_r) = [0, r]$, for each $0 < r < 1$. It is obvious that $f_r \in I$ and $\bigcap_{0<r<1} Z^\circ(f_r) = \emptyset$. This implies that $\bigcap_{Z^\circ\in Z^\circ[I]} Z^\circ = \emptyset$, i.e., $Z^\circ[I]$ is an i-free z°-filter. On the other hand, $\bigcap_{Z^\circ\in Z^\circ[I]} \overline{Z^\circ} \neq \emptyset$, for $0 \in \overline{Z^\circ}$, for each $Z^\circ \in Z^\circ[I]$. Hence $Z^\circ[I]$ is not a ci-free z°-filter.

Proposition 3.5. *Let I be a z°-ideal in $C(X)$. Then the following statements are equivalent:*

a) I is free.

b) I is ci-free.

c) $Z^\circ[I]$ is ci-free.

Proof. The implications ($a \Rightarrow b \Rightarrow c$) are evident.

($c \Rightarrow a$) Assume that $Z^\circ[I]$ is a ci-free z°-filter and on the contrary suppose $x \in \bigcap_{f\in I} Z(f)$. We claim that $x \in \bigcap_{f\in I} \overline{Z^\circ(f)}$. Otherwise there exists $f \in I$ such that $x \notin \overline{Z^\circ(f)}$. Now we can find a function $g \in C(X)$ such that $g(x) = 1$ and $\overline{Z^\circ(f)} \subseteq Z(g)$. Hence, $Z^\circ(f) \subseteq Z^\circ(g)$. Since I is a z°-ideal we have $g \in I$, which is a contradiction, for $x \notin Z(g)$. ■

Remark 3.6. Let $p \in X$ be a P-point. Then

a) p is a cluster point of a z°-filter $\mathcal{F}$ if and only if $\mathcal{F} \subseteq Z^\circ[O_p]$.

b) p is a cluster point of a prime z°-filter $\mathcal{F}$ if and only if $\mathcal{F} = Z^\circ[O_p]$.

c) $Z^\circ[O_p]$ is the unique z°-ultrafilter converging to p.

d) if $\mathcal{F}$ is a z°-filter converging to p, then $Z^\circ[O_p]$ is the unique z°-ultrafilter containing $\mathcal{F}$.

e) $Z^\circ[O_p]$ is a strongly prime z°-filter.

Now we are ready to prove the following result which is a counterpart of Lemma 4.10 in [7].

Theorem 3.7. *Let Z be a zero-set, then $\overline{Z^\circ}$ is a compact set if and only if Z° belongs to no ci-free z°-filter.*

Proof. Assume that $\overline{Z^\circ}$ is compact and there exists a ci-free z°-filter $\mathcal{F}$ such that $Z^\circ \in \mathcal{F}$. Hence $\bigcap_{F^\circ\in\mathcal{F}} \overline{F^\circ} = \emptyset$ and so $\overline{Z^\circ} \bigcap (\bigcap_{Z^\circ\neq F^\circ\in\mathcal{F}} \overline{F^\circ}) = \emptyset$, therefore $\overline{Z^\circ} \subseteq \bigcup_{Z^\circ\neq F^\circ\in\mathcal{F}} X \setminus \overline{F^\circ}$. Since $\overline{Z^\circ}$ is compact, there exist $F^\circ_{n_1}, \ldots, F^\circ_{n_m} \in \mathcal{F}$ such that $\overline{Z^\circ} \subseteq \bigcup_{i=1}^m X \setminus \overline{F^\circ_{n_i}}$ and hence $\overline{Z^\circ} \bigcap (\bigcap_{i=1}^m \overline{F^\circ_{n_i}}) = \emptyset$, which is a contradiction. Conversely, let $\mathcal{A}$ be any family of closed subsets of Z with the finite intersection property (note that if $Z^\circ = \emptyset$, clearly $\overline{Z^\circ}$ is compact). Then

$$\mathcal{F} = \{F^\circ : \bigcap_{A\in\mathcal{B}} A \subseteq F^\circ, \text{ for some finite subset } \mathcal{B} \text{ of } \mathcal{A}\}$$

is a z°-filter on X and it is clear that $Z^\circ \in \mathcal{F}$. By our hypothesis, $\mathcal{F}$ is ci-fixed, i.e., $\bigcap_{F^\circ\in\mathcal{F}} \overline{F^\circ} \neq \emptyset$. Now it is enough to show that $\bigcap_{F^\circ\in\mathcal{F}} \overline{F^\circ} \subseteq \bigcap \mathcal{A}$. Let $x \in \bigcap_{F^\circ\in\mathcal{F}} \overline{F^\circ} \setminus \bigcap \mathcal{A}$ and seek a contradiction. Clearly, there exists $A \in \mathcal{A}$ such that $x \notin A$. Since the zero-sets in a completely regular space form a base for the closed sets, we infer that $A = \bigcap_{\alpha\in\Lambda} F_\alpha$ for $F_\alpha \in Z(X)$. Therefore there are $\alpha \in \Lambda$ such that $x \notin F_\alpha$ and hence there exist F_1, $F_2 \in Z(X)$ such that $x \in F_1^\circ$ and $F_\alpha \subseteq F_2^\circ$ with $F_1^\circ \cap F_2^\circ = \emptyset$. Since $A \subseteq F_\alpha$, we have $A \subseteq F_2^\circ$ which implies that $F_2^\circ \in \mathcal{F}$. Since $F_1^\circ \cap F_2^\circ = \emptyset$ and $x \in F_1^\circ$, we infer that $x \notin \overline{F_2^\circ}$ which is a contradiction. ■

Corollary 3.8. *For a topological space X, the following statements are equivalent:*

a) X is a compact space.

b) Every z°-filter on X is ci-fixed.

c) Every z°-ultrafilter on X is ci-fixed.

d) Every z°-ideal in $C(X)$ is ci-fixed.

e) Every z°-ideal in $C(X)$ is fixed.

f) Every family of the form $\mathcal{A} = \{\overline{Z_\alpha^\circ} : Z_\alpha^\circ \in Z^\circ(X), \alpha \in S\}$ with the finite intersection property has nonempty intersection.

g) Every open cover of X of the form $\mathcal{A} = \{(X \setminus Z_\alpha^\circ)^\circ : Z_\alpha^\circ \in Z^\circ(X), \alpha \in S\}$ has a finite subcover.

In the previous corollary we can not replace ci-fixed by i-fixed, see our Example 3.4.

Now we are ready to prove a counterpart of similar result to Tychonoff's product theorem. We should also remind the reader that Tychonoff's product theorem is equivalent to the axiom of choice.

Definition 3.9. A space X is said to be i-compact if every z°-filter on X is i-fixed.

Every i-compact space is compact, the converse is not true in general , see our Example 3.4. But, whenever a compact space is basically disconnected, then it is easy to see that it is i-compact. In fact, if X is basically disconnected, then βX is i-compact. Recall that βX is the Stone-Čech compactification of X.

Proposition 3.10. *For a topological space X, the following statements are equivalent:*

a) X is an i-compact space.

b) Every z°-ultrafilter on X is i-fixed.

c) Every z°-ideal in $C(X)$ is i-fixed.

d) Every family of the form $\mathcal{A} = \{Z_\alpha^\circ \in Z^\circ(X) : \alpha \in S\}$ with the finite intersection property has nonempty intersection.

e) Every closed cover of X of the form $\mathcal{A} = \{X \setminus Z_\alpha^\circ : Z_\alpha^\circ \in Z^\circ(X), \alpha \in S\}$ has a finite subcover.

Proof. Evident. ■

Let τ be a continuous mapping from X to Y, where X, Y are topological spaces. It is clear that if $g \in C(Y)$, then $g \circ \tau \in C(X)$ and $\tau^{-1}(Z(g)) = Z(g \circ \tau)$. Furthermore, if τ is open, then $\tau^{-1}(Z^\circ(g)) = (\tau^{-1}(Z(g)))^\circ$. Consequently, $\tau^{-1}(Z^\circ(g)) = Z^\circ(g \circ \tau)$.

Proposition 3.11. *If $\mathcal{F}$ is a z°-filter on X, then $\tau^*\mathcal{F} = \{Z^\circ \in Z^\circ(Y) : \tau^{-1}(Z^\circ) \in \mathcal{F}\}$ is a z°-filter on Y. Furthermore, if $\mathcal{F}$ is a strongly prime (resp., prime) z°-filter, then $\tau^*\mathcal{F}$ is also strongly prime (resp., prime).*

Proof. It is evident that conditions (a) and (c) of Definition 2.1 hold. Hence, it suffices to show that condition (b) also holds. To see this let $Z^\circ(g_1)$, $Z^\circ(g_2) \in \tau^*\mathcal{F}$, where g_1, $g_2 \in C(Y)$. Therefore $\tau^{-1}(Z^\circ(g_1))$, $\tau^{-1}(Z^\circ(g_2)) \in \mathcal{F}$. Since $\mathcal{F}$ is a z°-filter, we have $\tau^{-1}(Z^\circ(g_1)) \cap \tau^{-1}(Z^\circ(g_2)) = \tau^{-1}(Z^\circ(g_1) \cap Z^\circ(g_2)) = \tau^{-1}((Z(g_1) \cap Z(g_2))^\circ) \in \mathcal{F}$. This shows that $(Z(g_1) \cap Z(g_2))^\circ = Z^\circ(g_1) \cap Z^\circ(g_2) \in \tau^*\mathcal{F}$. The proof of the second part of proposition is trivial. ■

In the following proposition we show that an open continuous image of every i-compact space is an i-compact space.

Proposition 3.12. *If there exists a continuous and open function $\tau : X \to Y$ of an i-compact space X onto a space Y, then Y is an i-compact space.*

Proof. Suppose that the family $\mathcal{A} = \{Z^\circ(g_\alpha) \in Z^\circ(Y) : \alpha \in S\}$ has the finite intersection property. The family $\mathcal{B} = \{\tau^{-1}(Z^\circ(g_\alpha)) : Z^\circ(g_\alpha) \in \mathcal{A}\}$ is contained in $Z^\circ(X)$ with the finite intersection property. Since X is an i-compact, we have $\bigcap_{\alpha\in S} \tau^{-1}(Z^\circ(g_\alpha)) \neq \emptyset$. This implies that $\bigcap_{\alpha\in S} (Z^\circ(g_\alpha)) \neq \emptyset$. Thus by part (d) of Proposition 3.10, Y is i-compact. ■

Recall that the map $\pi_\beta : \prod_{\alpha\in S} X_\alpha \to X_\beta$, defined by $\pi_\beta(x) = x_\beta$ is called the projection map of $\prod_{\alpha\in S} X_\alpha$ onto X_β, where x_β is the βth coordinate of $x = (x_\alpha)_{\alpha\in S} \in \prod_{\alpha\in S} X_\alpha$. It is well-known that π_β is continuous and open map for any $\beta \in S$. The next result is similar to Lemma 4.13 in [7].

Lemma 3.13. *Let $\mathcal{A}$ be a z°-ultrafilter on product space $X = \prod_{\alpha\in S} X_\alpha$. If for any $\alpha \in S$, every z°-filter $\pi^*_\alpha\mathcal{A} := \mathcal{A}_\alpha$ is i-fixed on X_α, then $\mathcal{A}$ is i-fixed on X.*

Proof. For each $\alpha \in S$, take $x_\alpha \in \bigcap \mathcal{A}_\alpha$ and let $x = (x_\alpha)_{\alpha\in S} \in X$. We shall show that $x \in Z^\circ$, for each $Z^\circ \in \mathcal{A}$. If $Z^\circ = \pi_\alpha^{-1}(Z^\circ_\alpha)$, where $Z^\circ_\alpha \in Z^\circ(X_\alpha)$, then clearly $x \in Z^\circ$. If $Z^\circ = \pi^{-1}_{\alpha_1}(Z^\circ_{\alpha_1}) \cup \cdots \cup \pi^{-1}_{\alpha_n}(Z^\circ_{\alpha_n})$, where $n \in \mathbb{N}$ and $Z^\circ_{\alpha_k} \in Z^\circ(X_{\alpha_k})$ for $k = 1, 2, \cdots, n$, then there exists $1 \le k \le n$ such that $\pi^{-1}_{\alpha_k}(Z^\circ_{\alpha_k}) \in \mathcal{A}$, for $\mathcal{A}$ is prime. This implies that $x \in Z^\circ$. Finally, an arbitrary member of $\mathcal{A}$ is an intersection of sets of the latter form; consequently, x belongs to every $Z^\circ \in \mathcal{A}$. ■

Based on the results of this section, in the following theorem, we obtain a similar result to Tychonoff's product theorem.

Theorem 3.14. *The Cartesian product $\prod_{\alpha\in S} X_\alpha$, where $X_\alpha \neq \emptyset$ for $\alpha \in S$, is i-compact if and only if all spaces X_α are i-compact.*

Proof. Let $X = \prod_{\alpha\in S} X_\alpha$, where X_α is i-compact, for any $\alpha \in S$. Now consider any z°-ultrafilter $\mathcal{A}$ on X. Since each space X_α is i-compact, each z°-filter $\pi^*_\alpha\mathcal{A}$ is i-fixed. By the above lemma, $\mathcal{A}$ is i-fixed and therefore X is i-compact. Conversely, since the projection map $\pi_\alpha : X \to X_\alpha$ is a continuous and open function of X onto X_α, for any $\alpha \in S$, and X is an i-compact space, we infer that X_α is an i-compact space, by Proposition 3.12. ■

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