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z° -FILTERS AND RELATED IDEALS IN $C(X)$

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ABSTRACT. In this article we introduce the concept of z° -filter on a topological space X . We study and investigate the behavior of z° -filters and compare them with corresponding ideals, namely, z° -ideals of $C(X)$, the ring of real-valued continuous functions on a completely regular Hausdorff space X . It is observed that X is a compact space if and only if every z° -filter is ci-fixed. Finally, by using z° -ultrafilters, we prove that any arbitrary product of i-compact spaces is i-compact.

1. INTRODUCTION AND PRELIMINARIES

We consider X to be a completely regular Hausdorff space and we denote by $C(X)$ the ring of all real-valued continuous functions on the space X . For each $f \in C(X)$, the set $Z(f) = \{x \in X : f(x) = 0\}$ is the zero-set of f and $Z(X) = \{Z(f) : f \in C(X)\}$. For $A \subseteq X$, by A° and \bar{A} we mean the interior and the closure of A , respectively. An ideal I of $C(X)$ is called a z -ideal if $f, g \in C(X)$ with $Z(f) \subseteq Z(g)$ and $f \in I$ imply that $g \in I$. A nonempty subfamily \mathcal{F} of $Z(X)$ is called a z -filter on X , if $\emptyset \notin \mathcal{F}$, the intersection of any two members of \mathcal{F} is again a member of \mathcal{F} , and any member of $Z(X)$ containing

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a member of \mathcal{F} also belongs to \mathcal{F} . In commutative rings, an ideal consisting entirely of zero divisors is called nonregular. Note that $f \in C(X)$ is a zero divisor if and only if $Z^\circ(f) = (Z(f))^\circ \neq \emptyset$.

A z -ultrafilter on X is meant a maximal z -filter, i.e., one not contained in any other z -filter. Also a z -filter \mathcal{F} on X is called a prime z -filter if whenever $Z_1 \cup Z_2 \in \mathcal{F}$, then $Z_1 \in \mathcal{F}$ or $Z_2 \in \mathcal{F}$. If I is an ideal in $C(X)$ and \mathcal{F} is a z -filter on X , then we note $Z[I] = \{Z(f) : f \in I\}$ and $Z^{-1}[\mathcal{F}] = \{f : Z(f) \in \mathcal{F}\}$. For $p \in X$, we have a z -ideal $O_p = \{f \in C(X) : p \in Z^\circ(f)\}$ contained in the maximal ideal $M_p = \{f \in C(X) : p \in Z(f)\}$. A point $p \in X$ is called a P -point if $O_p = M_p$. A space X is said to be P -space if its every point is a P -point, or equivalently, $Z(f) = Z^\circ(f)$, for every $f \in C(X)$.

A space X is called an almost P -space if every nonempty zero-set has a nonempty interior, or equivalently, for every $f \in C(X)$, $Z(f)$ is regular closed, i.e., $\overline{Z^\circ(f)} = Z(f)$, or equivalently, every nonregular z -ideal of $C(X)$ is a z° -ideal, see Theorem 4.2 in [4]. X is called basically disconnected if each open set has an open closure, or equivalently, if $\overline{Z^\circ(f)} = Z^\circ(f)$, for every $f \in C(X)$. For more information about P -spaces, almost P -spaces and basically disconnected spaces see [7] and [1], and for undefined terms and notations, the reader is referred to [6] and [7].

2. z° -FILTERS AND z° -IDEALS IN $C(X)$

In this section, we introduce z° -filters and strongly prime z° -filters on a topological space X . We study and investigate the behavior of z° -filters and strongly prime z° -filters and compare them with z° -ideals and prime z° -ideals of $C(X)$. Recall that an ideal I in $C(X)$ is a z° -ideal if $f \in I$, $g \in C(X)$ and $Z^\circ(f) \subseteq Z^\circ(g)$ imply that $g \in I$, so every proper z° -ideal of $C(X)$ is a nonregular ideal. For more details and examples of z° -ideals in commutative rings and in $C(X)$, the reader is referred to [8], [2], [3], [4] and [5]. We begin with the following definition.

Definition 2.1. A nonempty subfamily \mathcal{F} of $Z^\circ(X) = \{Z^\circ : Z \in Z(X)\}$ is called a z° -filter on X if it satisfies the following conditions:

- a) $\emptyset \notin \mathcal{F}$.
- b) If $Z_1^\circ, Z_2^\circ \in \mathcal{F}$, then $Z_1^\circ \cap Z_2^\circ = (Z_1 \cap Z_2)^\circ \in \mathcal{F}$.
- c) If $Z^\circ \in \mathcal{F}$ and $Z^\circ \subseteq F^\circ \in Z^\circ(X)$, then $F^\circ \in \mathcal{F}$.

If $A \subseteq C(X)$ and $\mathcal{F} \subseteq Z^\circ(X)$, then we note $Z^\circ[A] = \{Z^\circ(f) : f \in A\}$ and $Z^{\circ^{-1}}[\mathcal{F}] = \{f : Z^\circ(f) \in \mathcal{F}\}$.

Proposition 2.2. An ideal I in $C(X)$ is a z° -ideal if and only if $Z^\circ(f) \in Z^\circ[I]$ implies that $f \in I$, for each $f \in C(X)$.

Proof. For each $f \in C(X)$, let $Z^\circ(f) \in Z^\circ[I]$ implies that $f \in I$. Suppose that $g \in C(X)$ and $f \in I$ with $Z^\circ(f) \subseteq Z^\circ(g)$, then $Z^\circ(f) = Z^\circ(f) \cap Z^\circ(g) = (Z(f) \cap Z(g))^\circ = Z^\circ(f^2 + g^{\frac{2}{3}}) \in Z^\circ[I]$. By our

hypothesis, we have $f^2 + g^{\frac{2}{3}} \in I$. Since $f \in I$, we infer that $g^{\frac{2}{3}} \in I$ and hence $g \in I$. This shows that I is a z° -ideal. The converse is evident. ■

Proposition 2.3. *a) If \mathcal{F} is a z° -filter on X , then $Z^{\circ-1}[\mathcal{F}]$ is a z° -ideal in $C(X)$.*

b) If I is a z° -ideal in $C(X)$, then $Z^\circ[I]$ is a z° -filter on X .

Proof. It is straightforward. ■

Clearly, O_p is a z° -ideal in $C(X)$, so $Z^\circ[O_p] = \{Z^\circ : p \in Z^\circ\}$ is a z° -filter on X . It is clear that an intersection of z° -ideals in $C(X)$ is a z° -ideal. As in [4] and [5], we denote by I_\circ the smallest z° -ideal containing the ideal I . For any nonregular ideal I , I_\circ exists and in fact it is the intersection of all z° -ideals containing I . For more details about nonregular ideals in $C(X)$, see [4] and [5].

Proposition 2.4. *a) If I is a nonregular ideal in $C(X)$, then $I_\circ = Z^{\circ-1}Z^\circ[I]$.*

b) An ideal I in $C(X)$ is a z° -ideal if and only if $I = Z^{\circ-1}Z^\circ[I]$.

Proof. Evident. ■

Remark 2.5. *a) Let I be an ideal in $C(X)$. It is clear that $Z^{-1}Z[I] \subseteq Z^{\circ-1}Z^\circ[I]$. The reverse inclusion does not hold, in general. For example we consider the ideal $I = \{f \in C(\mathbb{R}) : [0, 1] \cup \{2\} \subseteq Z(f)\}$ in $C(\mathbb{R})$. Assume that $Z(g) = [0, 1]$ and $Z(f) = [0, 1] \cup \{2\}$. Clearly, $Z^\circ(g) = Z^\circ(f)$ and $f \in I$, hence $g \in Z^{\circ-1}Z^\circ[I]$. But, since $2 \notin Z(g)$, we infer that $Z(g) \notin Z[I]$ and therefore $g \notin Z^{-1}Z[I]$.*

b) X is an almost P -space if and only if $Z^{\circ-1}Z^\circ[I] = Z^{-1}Z[I]$, for every nonregular ideal I in $C(X)$.

Definition 2.6. A z° -filter \mathcal{F} on X is said to be strongly prime (resp., prime) if for any $Z_1^\circ, Z_2^\circ \in Z^\circ(X)$, $(Z_1 \cup Z_2)^\circ \in \mathcal{F}$ (resp., $Z_1^\circ \cup Z_2^\circ \in \mathcal{F}$) implies that $Z_1^\circ \in \mathcal{F}$ or $Z_2^\circ \in \mathcal{F}$.

Clearly, if \mathcal{F} is a strongly prime z° -filter on X , then $Z^{\circ-1}[\mathcal{F}]$ is a prime z° -ideal in $C(X)$ and if P is a prime z° -ideal in $C(X)$, then $Z^\circ[P]$ is a strongly prime z° -filter on X . Every strongly prime z° -filter is prime, but the converse is not true in general. Moreover if \mathcal{F} is a prime z° -filter, $Z^{\circ-1}[\mathcal{F}]$ is not necessarily prime z° -ideal. In fact $Z^\circ[O_p]$ is a prime z° -filter, for each $p \in X$, however O_p is not prime and above argument implies that $Z^\circ[O_p]$ is not strongly prime.

The following results are the counterparts of Theorem 2.9 and Exercise 2E in [7].

Proposition 2.7. *For any z° -ideal I in $C(X)$, the following statements are equivalent:*

a) I is prime.

b) I contains a prime ideal.

c) For all $g, h \in C(X)$, if $gh = 0$, then $g \in I$ or $h \in I$.

d) For every $f \in C(X)$, there is a $Z^\circ \in Z^\circ[I]$ on which f does not change sign.

Proof. Since I is also a z -ideal the implications $(a \Rightarrow b \Rightarrow c \Rightarrow d)$ are evident by Theorem 2.9 in [7].

$(d \Rightarrow a)$ Given $gh \in I$, consider the function $|g| - |h|$. By our hypothesis, there must be a $Z^\circ(f) \in Z^\circ[I]$ on which $|g| - |h|$ does not change sign. By Proposition 2.2 we have $f \in I$. Now without loss of generality, let us assume that $|g| - |h|$ is nonnegative on $Z^\circ(f)$. Hence $Z^\circ(f) \cap Z(g) \subseteq Z^\circ(f) \cap Z(h)$. We claim that $Z^\circ(f) \cap Z^\circ(h) = Z^\circ(f) \cap Z^\circ(gh)$. It is clear that $Z^\circ(f) \cap Z^\circ(h) \subseteq Z^\circ(f) \cap Z^\circ(gh)$. Now let $x \in Z^\circ(f) \cap Z^\circ(gh)$, so there exists an open set G in X such that $x \in G \subseteq Z(g) \cup Z(h)$. Hence $x \in G \cap Z^\circ(f) \subseteq Z^\circ(f) \cap (Z(g) \cup Z(h)) = (Z^\circ(f) \cap Z(g)) \cup (Z^\circ(f) \cap Z(h)) = Z^\circ(f) \cap Z(h)$. This implies that $x \in (Z^\circ(f) \cap Z(h))^\circ = Z^\circ(f) \cap Z^\circ(h)$. Thus $Z^\circ(f^2 + h^2) = Z^\circ(f^2 + g^2h^2)$ and our claim is settled. Since I is a z° -ideal and $f^2 + g^2h^2 \in I$ we have $f^2 + h^2 \in I$, and so $h \in I$. ■

Proposition 2.8. *For a z° -filter \mathcal{F} on X , the following statements are equivalent:*

- a) \mathcal{F} is strongly prime.
- b) Of any $Z_1^\circ, Z_2^\circ \in Z^\circ(X)$ with $(Z_1 \cup Z_2)^\circ = X$, at least one is in \mathcal{F} .
- c) Given $Z_1, Z_2 \in Z(X)$, there exists $Z^\circ \in \mathcal{F}$ such that one of $Z^\circ \cap Z_1, Z^\circ \cap Z_2$ contains the other.

Proof. The implication $(a \Rightarrow b)$ is clear.

$(b \Rightarrow c)$ Clearly, (b) is equivalent to part (c) of Proposition 2.7 which implies part (d) of the same proposition. Let $Z_1, Z_2 \in Z(X)$, where $Z_1 = Z(g)$ and $Z_2 = Z(h)$ for $g, h \in C(X)$. Now consider the function $|g| - |h|$. Hence by part (d) of Proposition 2.7 there exists $Z^\circ \in \mathcal{F}$ on which $|g| - |h|$ does not change sign, without loss of generality, let us assume that it is nonnegative on Z° . Then on Z° , whenever g vanishes, h must also vanish. In other words $Z^\circ \cap Z_1 \subseteq Z^\circ \cap Z_2$.

$(c \Rightarrow a)$ Suppose that for some $Z_1, Z_2 \in Z(X)$ we have $(Z_1 \cup Z_2)^\circ \in \mathcal{F}$. Then we can find a $Z^\circ \in \mathcal{F}$ such that, $Z^\circ \cap Z_1 \subseteq Z^\circ \cap Z_2$, say. This implies that $Z^\circ \cap (Z_1 \cup Z_2) = Z^\circ \cap Z_2$ and hence $Z^\circ \cap (Z_1 \cup Z_2)^\circ = Z^\circ \cap Z_2^\circ$. But \mathcal{F} is a z° -filter, hence $Z^\circ \cap (Z_1 \cup Z_2)^\circ \in \mathcal{F}$ implies that $Z^\circ \cap Z_2^\circ \in \mathcal{F}$, and therefore $Z_2^\circ \in \mathcal{F}$, i.e., \mathcal{F} is a strongly prime z° -filter. ■

If in the previous proposition we replace the condition “strongly prime” by “prime”, then we have the following result.

Corollary 2.9. *For a z° -filter \mathcal{F} on X , the following statements are equivalent:*

- a) \mathcal{F} is prime.
- b) Of any $Z_1^\circ, Z_2^\circ \in Z^\circ(X)$ with $Z_1^\circ \cup Z_2^\circ = X$, at least one is in \mathcal{F} .
- c) Given $Z_1, Z_2 \in Z(X)$, there exists $Z^\circ \in \mathcal{F}$ such that one of $Z^\circ \cap Z_1^\circ, Z^\circ \cap Z_2^\circ$ contains the other.

The remainder of this section devotes to the outline of the theory of convergence of z° -filters.

Definition 2.10. a) A point $p \in X$ is called a cluster point of a z° -filter \mathcal{F} if p belongs to the closure of every member of \mathcal{F} .

b) The z° -filter \mathcal{F} is said to be convergent to the limit $p \in X$ and we write $\mathcal{F} \rightarrow p$, if every neighborhood of p contains a member of \mathcal{F} .

By the above definition, p is a cluster point of a z° -filter \mathcal{F} if and only if every neighborhood of p intersects all members of \mathcal{F} . The set of all cluster points of a z° -filter \mathcal{F} is denoted by $\text{cl}(\mathcal{F})$. Hence $p \in \text{cl}(\mathcal{F})$ if and only if $p \in \bigcap_{Z^\circ \in \mathcal{F}} \overline{Z^\circ}$. Also, if $\mathcal{F} \rightarrow p$, then $p \in \text{cl}(\mathcal{F})$, in fact in this case we have $\text{cl}(\mathcal{F}) = \{p\}$, the converse is true for prime z° -filter. Furthermore, $\mathcal{F} \rightarrow p$ if and only if $Z^\circ[O_p] \subseteq \mathcal{F}$. It is obvious that $Z^\circ[O_p] \rightarrow p$. To see this if G is a neighborhood of p , then by completely regularity of X , there exists $Z^\circ \in Z^\circ(X)$ such that $p \in Z^\circ \subseteq G$.

Example 2.11. Let $\emptyset \neq A \subseteq X$. It is clear that $\mathcal{F} = \{Z^\circ : A \subseteq Z^\circ\}$ is a z° -filter on X . Furthermore, $\text{cl}(\mathcal{F}) = \overline{A}$. To see this suppose that $Z^\circ \in \mathcal{F}$, then $A \subseteq Z^\circ$ and hence $\overline{A} \subseteq \overline{Z^\circ}$, consequently $\overline{A} \subseteq \text{cl}(\mathcal{F})$. Now let there exists $p \in \text{cl}(\mathcal{F})$ which $p \notin \overline{A}$. Therefore there exists $f \in C(X)$ such that $\overline{A} \subseteq Z^\circ(f)$ and $f(p) \neq 0$. Hence $Z^\circ(f) \in \mathcal{F}$ and $p \in \text{cl}(\mathcal{F})$ implies that $p \in \overline{Z^\circ(f)} \subseteq Z(f)$, which is a contradiction.

The next proposition is the counterpart of Theorem 3.17 in [7].

Proposition 2.12. *Let $p \in X$ and \mathcal{F} be a prime z° -filter on X . Then the following statements are equivalent:*

- a) $p \in \text{cl}(\mathcal{F})$.
- b) $\mathcal{F} \rightarrow p$.
- c) $\text{cl}(\mathcal{F}) = \{p\}$.

Proof. (a \Rightarrow b) It suffices to show that $Z^\circ[O_p] \subseteq \mathcal{F}$. Let $p \in Z^\circ$, since X is completely regular, there exist $Z_1, Z_2 \in Z(X)$ such that $p \in Z_1^\circ$ and $X \setminus Z^\circ \subseteq Z_2^\circ$ with $Z_1^\circ \cap Z_2^\circ = \emptyset$. Clearly $X = Z^\circ \cup Z_2^\circ \in \mathcal{F}$. Therefore either Z° or Z_2° belongs to the prime z° -filter \mathcal{F} . But $Z_2^\circ \notin \mathcal{F}$, because $p \notin \overline{Z_2^\circ}$, so $Z^\circ \in \mathcal{F}$.

(b \Rightarrow c) Let $Z^\circ \in \mathcal{F}$ and $p \notin \overline{Z^\circ}$. Then there exist $Z_1, Z_2 \in Z(X)$ such that $p \in Z_1^\circ$ and $\overline{Z^\circ} \subseteq Z_2^\circ$ with $Z_1^\circ \cap Z_2^\circ = \emptyset$. By our hypothesis, there exists $Z_3^\circ \in \mathcal{F}$ such that $Z_3^\circ \subseteq Z_1^\circ$. Since $Z^\circ \in \mathcal{F}$ we have $Z_2^\circ \in \mathcal{F}$. But, $Z_3^\circ \cap Z_2^\circ = \emptyset$, a contradiction. Now suppose that $q \in \text{cl}(\mathcal{F})$ and $q \neq p$. Then there exist $F_1, F_2 \in Z(X)$ such that $p \in F_1^\circ$ and $q \in F_2^\circ$ with $F_1^\circ \cap F_2^\circ = \emptyset$. Obviously, $q \notin \overline{F_1^\circ}$ and since $\mathcal{F} \rightarrow p$, there exists $F_3^\circ \in \mathcal{F}$ such that $F_3^\circ \subseteq F_1^\circ$. Therefore $q \notin \overline{F_3^\circ}$. This implies that $q \notin \text{cl}(\mathcal{F})$ which is a contradiction.

(c \Rightarrow a) Evident. ■

3. z° -ULTRAFILTERS AND CI-FREE z° -FILTERS

Recall that a z -filter \mathcal{F} on X is free (resp., fixed) if $\bigcap_{Z \in \mathcal{F}} Z = \emptyset$ (resp., $\bigcap_{Z \in \mathcal{F}} Z \neq \emptyset$). Also an ideal I in $C(X)$ is free (resp., fixed) if $\bigcap_{f \in I} Z(f) = \emptyset$ (resp., $\bigcap_{f \in I} Z(f) \neq \emptyset$). We say that a z° -filter \mathcal{F}

is i -free (resp., i -fixed) if $\bigcap_{Z^\circ \in \mathcal{F}} Z^\circ = \emptyset$ (resp., $\bigcap_{Z^\circ \in \mathcal{F}} Z^\circ \neq \emptyset$). Also a z° -filter \mathcal{F} is called a ci -free (resp., ci -fixed) if $\bigcap_{Z^\circ \in \mathcal{F}} \overline{Z^\circ} = \emptyset$ (resp., $\bigcap_{Z^\circ \in \mathcal{F}} \overline{Z^\circ} \neq \emptyset$). Similarly an ideal I in $C(X)$ is said to be i -free (resp., i -fixed) if $\bigcap_{f \in I} Z^\circ(f) = \emptyset$ (resp., $\bigcap_{f \in I} Z^\circ(f) \neq \emptyset$). Also I is said to be ci -free (resp., ci -fixed) if $\bigcap_{f \in I} \overline{Z^\circ(f)} = \emptyset$ (resp., $\bigcap_{f \in I} \overline{Z^\circ(f)} \neq \emptyset$). We show in this section that X is a compact space if and only if every z° -ultrafilter on X is ci -fixed. It is also shown that an arbitrary product of i -compact (i.e., every z° -filter on X is i -fixed) spaces is an i -compact space. We begin by the following definition.

Definition 3.1. By a z° -ultrafilter on X we mean a maximal z° -filter, i.e., one not contained in any other z° -filter.

Remark 3.2. a) If M is a nonregular maximal ideal in $C(X)$, then $Z^\circ[M]$ is a z° -ultrafilter on X . Furthermore, if $Z^\circ(f)$ meets every member of $Z^\circ[M]$, then $f \in M$.

b) If \mathcal{A} is a z° -ultrafilter on X , then $Z^{\circ-1}[\mathcal{A}]$ is a maximal z° -ideal (maximal in the realm of z° -ideals) in $C(X)$. Furthermore, if Z° meets every member of \mathcal{A} , then $Z^\circ \in \mathcal{A}$.

c) Every z° -ultrafilter is a prime z° -filter.

Remark 3.3. a) Every ci -free z° -filter is i -free and every i -fixed z° -filter is ci -fixed.

b) Every free ideal is ci -free and every ci -free ideal is i -free.

c) Every i -fixed ideal is ci -fixed and every ci -fixed ideal is fixed.

d) If X is a P -space, then the set of free ideals, the set of i -free ideals, and the set of ci -free ideals in $C(X)$ coincide.

e) If X is an almost P -space, then an ideal I is free if and only if it is ci -free.

f) If X is a basically disconnected space, then an ideal I is i -free if and only if it is ci -free.

An i -free z° -filter may not be ci -free. In other words a ci -fixed z° -filter may not be i -fixed, see the next example.

Example 3.4. Let $X = [0, 1]$. We consider the ideal $I = \{f \in C(X) : (0, r) \subseteq Z^\circ(f) \text{ for some } 0 < r < 1\}$ in $C(X)$. Clearly, I is a z° -ideal and so $Z^\circ[I]$ is a z° -filter. Now take $f_r \in C(X)$ such that $Z(f_r) = [0, r]$, for each $0 < r < 1$. It is obvious that $f_r \in I$ and $\bigcap_{0 < r < 1} Z^\circ(f_r) = \emptyset$. This implies that $\bigcap_{Z^\circ \in Z^\circ[I]} Z^\circ = \emptyset$, i.e., $Z^\circ[I]$ is an i -free z° -filter. On the other hand, $\bigcap_{Z^\circ \in Z^\circ[I]} \overline{Z^\circ} \neq \emptyset$, for $0 \in \overline{Z^\circ}$, for each $Z^\circ \in Z^\circ[I]$. Hence $Z^\circ[I]$ is not a ci -free z° -filter.

Proposition 3.5. Let I be a z° -ideal in $C(X)$. Then the following statements are equivalent:

a) I is free.

- b) I is ci-free.
- c) $Z^\circ[I]$ is ci-free.

Proof. The implications $(a \Rightarrow b \Rightarrow c)$ are evident.

$(c \Rightarrow a)$ Assume that $Z^\circ[I]$ is a ci-free z° -filter and on the contrary suppose $x \in \bigcap_{f \in I} Z(f)$. We claim that $x \in \bigcap_{f \in I} \overline{Z^\circ(f)}$. Otherwise there exists $f \in I$ such that $x \notin \overline{Z^\circ(f)}$. Now we can find a function $g \in C(X)$ such that $g(x) = 1$ and $\overline{Z^\circ(f)} \subseteq Z(g)$. Hence, $Z^\circ(f) \subseteq Z^\circ(g)$. Since I is a z° -ideal we have $g \in I$, which is a contradiction, for $x \notin Z(g)$. ■

Remark 3.6. Let $p \in X$ be a P -point. Then

- a) p is a cluster point of a z° -filter \mathcal{F} if and only if $\mathcal{F} \subseteq Z^\circ[O_p]$.
- b) p is a cluster point of a prime z° -filter \mathcal{F} if and only if $\mathcal{F} = Z^\circ[O_p]$.
- c) $Z^\circ[O_p]$ is the unique z° -ultrafilter converging to p .
- d) if \mathcal{F} is a z° -filter converging to p , then $Z^\circ[O_p]$ is the unique z° -ultrafilter containing \mathcal{F} .
- e) $Z^\circ[O_p]$ is a strongly prime z° -filter.

Now we are ready to prove the following result which is a counterpart of Lemma 4.10 in [7].

Theorem 3.7. Let Z be a zero-set, then $\overline{Z^\circ}$ is a compact set if and only if Z° belongs to no ci-free z° -filter.

Proof. Assume that $\overline{Z^\circ}$ is compact and there exists a ci-free z° -filter \mathcal{F} such that $Z^\circ \in \mathcal{F}$. Hence $\bigcap_{F^\circ \in \mathcal{F}} \overline{F^\circ} = \emptyset$ and so $\overline{Z^\circ} \cap (\bigcap_{Z^\circ \neq F^\circ \in \mathcal{F}} \overline{F^\circ}) = \emptyset$, therefore $\overline{Z^\circ} \subseteq \bigcup_{Z^\circ \neq F^\circ \in \mathcal{F}} X \setminus \overline{F^\circ}$. Since $\overline{Z^\circ}$ is compact, there exist $F_{n_1}^\circ, \dots, F_{n_m}^\circ \in \mathcal{F}$ such that $\overline{Z^\circ} \subseteq \bigcup_{i=1}^m X \setminus \overline{F_{n_i}^\circ}$ and hence $\overline{Z^\circ} \cap (\bigcap_{i=1}^m \overline{F_{n_i}^\circ}) = \emptyset$, which is a contradiction. Conversely, let \mathcal{A} be any family of closed subsets of Z with the finite intersection property (note that if $Z^\circ = \emptyset$, clearly $\overline{Z^\circ}$ is compact). Then

$$\mathcal{F} = \{F^\circ : \bigcap_{A \in \mathcal{B}} A \subseteq F^\circ, \text{ for some finite subset } \mathcal{B} \text{ of } \mathcal{A}\}$$

is a z° -filter on X and it is clear that $Z^\circ \in \mathcal{F}$. By our hypothesis, \mathcal{F} is ci-fixed, i.e., $\bigcap_{F^\circ \in \mathcal{F}} \overline{F^\circ} \neq \emptyset$. Now it is enough to show that $\bigcap_{F^\circ \in \mathcal{F}} \overline{F^\circ} \subseteq \bigcap \mathcal{A}$. Let $x \in \bigcap_{F^\circ \in \mathcal{F}} \overline{F^\circ} \setminus \bigcap \mathcal{A}$ and seek a contradiction. Clearly, there exists $A \in \mathcal{A}$ such that $x \notin A$. Since the zero-sets in a completely regular space form a base for the closed sets, we infer that $A = \bigcap_{\alpha \in \Lambda} F_\alpha$ for $F_\alpha \in Z(X)$. Therefore there are $\alpha \in \Lambda$ such that $x \notin F_\alpha$ and hence there exist $F_1, F_2 \in Z(X)$ such that $x \in F_1^\circ$ and $F_\alpha \subseteq F_2^\circ$ with $F_1^\circ \cap F_2^\circ = \emptyset$. Since $A \subseteq F_\alpha$, we have $A \subseteq F_2^\circ$ which implies that $F_2^\circ \in \mathcal{F}$. Since $F_1^\circ \cap F_2^\circ = \emptyset$ and $x \in F_1^\circ$, we infer that $x \notin \overline{F_2^\circ}$ which is a contradiction. ■

Corollary 3.8. For a topological space X , the following statements are equivalent:

- a) X is a compact space.

- b) Every z° -filter on X is ci-fixed.
- c) Every z° -ultrafilter on X is ci-fixed.
- d) Every z° -ideal in $C(X)$ is ci-fixed.
- e) Every z° -ideal in $C(X)$ is fixed.
- f) Every family of the form $\mathcal{A} = \{\overline{Z_\alpha^\circ} : Z_\alpha^\circ \in Z^\circ(X), \alpha \in S\}$ with the finite intersection property has nonempty intersection.
- g) Every open cover of X of the form $\mathcal{A} = \{(X \setminus Z_\alpha^\circ)^\circ : Z_\alpha^\circ \in Z^\circ(X), \alpha \in S\}$ has a finite subcover.

In the previous corollary we can not replace ci-fixed by i-fixed, see our Example 3.4.

Now we are ready to prove a counterpart of similar result to Tychonoff's product theorem. We should also remind the reader that Tychonoff's product theorem is equivalent to the axiom of choice.

Definition 3.9. A space X is said to be i-compact if every z° -filter on X is i-fixed.

Every i-compact space is compact, the converse is not true in general, see our Example 3.4. But, whenever a compact space is basically disconnected, then it is easy to see that it is i-compact. In fact, if X is basically disconnected, then βX is i-compact. Recall that βX is the Stone-Ćech compactification of X .

Proposition 3.10. For a topological space X , the following statements are equivalent:

- a) X is an i-compact space.
- b) Every z° -ultrafilter on X is i-fixed.
- c) Every z° -ideal in $C(X)$ is i-fixed.
- d) Every family of the form $\mathcal{A} = \{Z_\alpha^\circ \in Z^\circ(X) : \alpha \in S\}$ with the finite intersection property has nonempty intersection.
- e) Every closed cover of X of the form $\mathcal{A} = \{X \setminus Z_\alpha^\circ : Z_\alpha^\circ \in Z^\circ(X), \alpha \in S\}$ has a finite subcover.

Proof. Evident. ■

Let τ be a continuous mapping from X to Y , where X, Y are topological spaces. It is clear that if $g \in C(Y)$, then $g \circ \tau \in C(X)$ and $\tau^{-1}(Z(g)) = Z(g \circ \tau)$. Furthermore, if τ is open, then $\tau^{-1}(Z^\circ(g)) = (\tau^{-1}(Z(g)))^\circ$. Consequently, $\tau^{-1}(Z^\circ(g)) = Z^\circ(g \circ \tau)$.

Proposition 3.11. If \mathcal{F} is a z° -filter on X , then $\tau^*\mathcal{F} = \{Z^\circ \in Z^\circ(Y) : \tau^{-1}(Z^\circ) \in \mathcal{F}\}$ is a z° -filter on Y . Furthermore, if \mathcal{F} is a strongly prime (resp., prime) z° -filter, then $\tau^*\mathcal{F}$ is also strongly prime (resp., prime).

Proof. It is evident that conditions (a) and (c) of Definition 2.1 hold. Hence, it suffices to show that condition (b) also holds. To see this let $Z^\circ(g_1), Z^\circ(g_2) \in \tau^*\mathcal{F}$, where $g_1, g_2 \in C(Y)$. Therefore $\tau^{-1}(Z^\circ(g_1)), \tau^{-1}(Z^\circ(g_2)) \in \mathcal{F}$. Since \mathcal{F} is a z° -filter, we have $\tau^{-1}(Z^\circ(g_1)) \cap \tau^{-1}(Z^\circ(g_2)) = \tau^{-1}(Z^\circ(g_1) \cap Z^\circ(g_2)) = \tau^{-1}((Z(g_1) \cap Z(g_2))^\circ) \in \mathcal{F}$. This shows that $(Z(g_1) \cap Z(g_2))^\circ = Z^\circ(g_1) \cap Z^\circ(g_2) \in \tau^*\mathcal{F}$. The proof of the second part of proposition is trivial. ■

In the following proposition we show that an open continuous image of every i-compact space is an i-compact space.

Proposition 3.12. *If there exists a continuous and open function $\tau : X \rightarrow Y$ of an i-compact space X onto a space Y , then Y is an i-compact space.*

Proof. Suppose that the family $\mathcal{A} = \{Z^\circ(g_\alpha) \in Z^\circ(Y) : \alpha \in S\}$ has the finite intersection property. The family $\mathcal{B} = \{\tau^{-1}(Z^\circ(g_\alpha)) : Z^\circ(g_\alpha) \in \mathcal{A}\}$ is contained in $Z^\circ(X)$ with the finite intersection property. Since X is an i-compact, we have $\bigcap_{\alpha \in S} \tau^{-1}(Z^\circ(g_\alpha)) \neq \emptyset$. This implies that $\bigcap_{\alpha \in S} (Z^\circ(g_\alpha)) \neq \emptyset$. Thus by part (d) of Proposition 3.10, Y is i-compact. ■

Recall that the map $\pi_\beta : \prod_{\alpha \in S} X_\alpha \rightarrow X_\beta$, defined by $\pi_\beta(x) = x_\beta$ is called the projection map of $\prod_{\alpha \in S} X_\alpha$ onto X_β , where x_β is the β th coordinate of $x = (x_\alpha)_{\alpha \in S} \in \prod_{\alpha \in S} X_\alpha$. It is well-known that π_β is continuous and open map for any $\beta \in S$. The next result is similar to Lemma 4.13 in [7].

Lemma 3.13. *Let \mathcal{A} be a z° -ultrafilter on product space $X = \prod_{\alpha \in S} X_\alpha$. If for any $\alpha \in S$, every z° -filter $\pi_\alpha^*\mathcal{A} := \mathcal{A}_\alpha$ is i-fixed on X_α , then \mathcal{A} is i-fixed on X .*

Proof. For each $\alpha \in S$, take $x_\alpha \in \bigcap \mathcal{A}_\alpha$ and let $x = (x_\alpha)_{\alpha \in S} \in X$. We shall show that $x \in Z^\circ$, for each $Z^\circ \in \mathcal{A}$. If $Z^\circ = \pi_\alpha^{-1}(Z_\alpha^\circ)$, where $Z_\alpha^\circ \in Z^\circ(X_\alpha)$, then clearly $x \in Z^\circ$. If $Z^\circ = \pi_{\alpha_1}^{-1}(Z_{\alpha_1}^\circ) \cup \dots \cup \pi_{\alpha_n}^{-1}(Z_{\alpha_n}^\circ)$, where $n \in \mathbb{N}$ and $Z_{\alpha_k}^\circ \in Z^\circ(X_{\alpha_k})$ for $k = 1, 2, \dots, n$, then there exists $1 \leq k \leq n$ such that $\pi_{\alpha_k}^{-1}(Z_{\alpha_k}^\circ) \in \mathcal{A}$, for \mathcal{A} is prime. This implies that $x \in Z^\circ$. Finally, an arbitrary member of \mathcal{A} is an intersection of sets of the latter form; consequently, x belongs to every $Z^\circ \in \mathcal{A}$. ■

Based on the results of this section, in the following theorem, we obtain a similar result to Tychonoff's product theorem.

Theorem 3.14. *The Cartesian product $\prod_{\alpha \in S} X_\alpha$, where $X_\alpha \neq \emptyset$ for $\alpha \in S$, is i-compact if and only if all spaces X_α are i-compact.*

Proof. Let $X = \prod_{\alpha \in S} X_\alpha$, where X_α is i-compact, for any $\alpha \in S$. Now consider any z° -ultrafilter \mathcal{A} on X . Since each space X_α is i-compact, each z° -filter $\pi_\alpha^*\mathcal{A}$ is i-fixed. By the above lemma, \mathcal{A} is i-fixed and therefore X is i-compact. Conversely, since the projection map $\pi_\alpha : X \rightarrow X_\alpha$ is a continuous and

open function of X onto X_α , for any $\alpha \in S$, and X is an i -compact space, we infer that X_α is an i -compact space, by Proposition 3.12. ■

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