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ON THE NIL-CLEAN MATRIX OVER A UFD

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Abstract. In this paper we characterize all 2×2 idempotent and nilpotent matrices over an integral domain and then we characterize all 2×2 strongly nil-clean matrices over a PID. Also, we determine when a 2×2 matrix over a UFD is nil-clean.

1. Introduction and Preliminaries

Throughout this paper, all rings are associative with identity. A clean ring is one in which every element is the sum of an idempotent and a unit, and this definition dates back to a paper by Nicholson [6] in 1977, where he investigated exchange rings (a ring R is an exchange ring if ${}_RR$ has the exchange property) . A strongly clean ring is a ring in which every element is the sum of an idempotent and a unit that commute. Local rings are obviously strongly clean. An element r of a ring R is called nil-clean if it is the sum of an idempotent and a nilpotent. The element r is further called strongly nil clean if it is the sum of an idempotent and a nilpotent and these commute. R is called nil-clean (strongly nil-clean)

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if each of its elements is nil-clean (strongly nil-clean). It is clear that every nilpotent, idempotent and unipotent element is strongly nil-clean (recall that an element is called unipotent if it can be written as $1+b$ for some nilpotent b). Nil-clean rings were investigated by Diesl in [2] and [3]. He has shown that every nil-clean ring is clean and any strongly nil-clean element is strongly clean. S. Breaz et al. showed in [1] that if R is any commutative local nil-clean ring then $\mathbb{M}_n(R)$, the $n \times n$ matrix ring over R, is a nil-clean ring. A recent work on nil-clean matrix ring, due to M. T. Koşan et al. [5], generalizes a Theorem of S. Breaz et al. [1].

2. Strongly nil-clean matrix over a PID

In what follows the rank of A, denoted by rank(A), is the largest integer j such that there exists at least one nonzero minor of size j of the matrix A.

Lemma 2.1. *Let R be an integral domain and $0, I \neq A \in \mathbb{M}_2(R)$. Then A is idempotent if and only if $rank(A) = 1$ and $\mathrm{tr}(A) = 1$.*

Proof. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an idempotent matrix. We have

$$A^2 = \begin{pmatrix} a^2+bc & ab+bd \\ ca+dc & cb+d^2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Thus

$$a(1-a) = bc \tag{1}$$

$$b(1-a-d) = 0 \tag{2}$$

$$c(1-a-d) = 0 \tag{3}$$

$$d(1-d) = bc \tag{4}$$

First let b is not zero. Since R is an integral domain, by 2, we have $a+d=1$, so $\mathrm{tr}(A)=1$ and $a = 1-d$. Thus by 4, $ad = bc$ and $rank(A) = 1$. Now assume that $b=0$. By 1, $a(1-a)=0$. Thus $a=0$ or $a=1$. Also by 4, we have $d=0$ or $d=1$. If $a=d=1$, then by 3, $c=0$ and if $a=d=0$, then by 3, $c=0$. Since $A \neq 0$ and $A \neq I$, hence $A = \begin{pmatrix} 1 & 0 \\ c & 0 \end{pmatrix}$ or $A = \begin{pmatrix} 0 & 0 \\ c & 1 \end{pmatrix}$. Therefore $rank(A) = 1$ and $\mathrm{tr}(A) = 1$, in this case. Conversely, let $rank(A) = 1$ and $\mathrm{tr}(A) = 1$. Hence $A = \begin{pmatrix} a & b \\ c & 1-a \end{pmatrix}$, where $a(1-a) = bc$. Easy computation show that $A^2 = A$. ■

Lemma 2.2. *Let R be an integral domain and $A \in \mathbb{M}_2(R)$. The following conditions are equivalent:*

(1) $A^2 = 0$.

(2) *A is nilpotent.*

(3) *$rank(A) = 1$ and* $\mathrm{tr}(A) = 0$.

Proof. (1) $\Rightarrow$ (2) It is clear.

(2) $\Rightarrow$ (3) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a nonzero nilpotent matrix. Thus there exists some $n \in \mathbb{N}$ such that $A^n = 0$. Thus $\mathrm{adj}(A)A^n = 0$. Hence $det(A)A^{n-1} = 0$. So $det(A)\,\mathrm{adj}(A)A^{n-1} = 0$. Therefore $(det(A))^2A^{n-2} = 0$. Continuing this procces we have $(det(A))^{n-1}A = 0$. Since R is an integral domain and $A \neq 0$, hence $det(A) = 0$. So $rank(A) = 1$ and $ad = bc$. We have

$$A^2 = \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & cb+d^2 \end{pmatrix} = \begin{pmatrix} a^2+ad & ab+bd \\ ac+cd & ad+d^2 \end{pmatrix}$$

$$= \begin{pmatrix} a(a+d) & b(a+d) \\ c(a+d) & d(a+d) \end{pmatrix} = (a+d)A.$$

Therefore $A^n = (a+d)^{n-1}A = 0$. Since A is nonzero, hence $a + d = 0$. This means $\mathrm{tr}(A) = 0$.

(3) $\Rightarrow$ (1) Assume that $rank(A) = 1$ and $\mathrm{tr}(A) = 0$. Then $a + d = 0$ and $ad - bc = 0$. Hence $A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, where $-a^2 = bc$. Easy computation show that $A^2 = 0$. ■

Corollary 2.3. *Let R be an integral domain and $A \in \mathbb{M}_2(R)$ be a nil-clean matrix. Then* $\mathrm{tr}(A) = 0$ *or* $\mathrm{tr}(A) = 1$ *or* $\mathrm{tr}(A) = 2$.

Proof. Since A is a nil-clean matrix, hence $A = B + N$, for some idempotent matrix B and nilpotent matrix N. By Lemma 2.1, $\mathrm{tr}(B) = 0$ or $\mathrm{tr}(B) = 1$ or $\mathrm{tr}(B) = 2$. Also by Lemma 2.2, $\mathrm{tr}(N) = 0$. Hence $\mathrm{tr}(A) = 0$ or $\mathrm{tr}(A) = 1$ or $\mathrm{tr}(A) = 2$. ■

We recall from [4, Proposition VII-2-11], the following Proposition.

Proposition 2.4. *If A is an $n \times m$ matrix of rank $r > 0$ over a principal ideal domain R, then A is equivalent to a matrix of the form $\begin{pmatrix} L_r & 0 \\ 0 & 0 \end{pmatrix}$, where L_r is an $r \times r$ diagonal matrix with nonzero diagonal entries $d_1, ..., d_r$ such that $d_1 \mid ... \mid d_r$. The ideals $(d_1), ..., (d_r)$ in R are uniquely determined by the equivalence class of A.*

Further we use the following lemma.

Lemma 2.5. *Let A be an $n \times n$ idempotent matrix over a ring R. If A is equivalent to a diagonal matrix, then A is similar to a diagonal matrix.*

Proof. [7, Corollary 5]. ■

In the next Theorem, we characterize all 2×2 strongly nil-clean matrices over a PID.

Theorem 2.6. *Let R be a PID and $A \in M_2(R)$ be a strongly nil-clean matrix. Then exactly one of the following holds:*

(1) $A^2 = 0$.
(2) $(A - I)^2 = 0$.
(3) $A^2 = A$.

Proof. Let $A \in M_2(R)$ be a strongly nil-clean matrix. Thus there exists some idempotent matrix B and a nilpotent matrix N such that $A = B + N$. If $B = 0$, then A is a nilpotent matrix. So by Lemma 2.2, $A^2 = 0$. If $B = I$, then $A - I$ is a nilpotent matrix. Again, by Lemma 2.2, $(A - I)^2 = 0$. Now assume that $0, I \neq B$. By Proposition 2.4 and Lemma 2.5, B is similar to a diagonal matrix. Let B be similar to B'. By Lemma 2.1, $\mathrm{tr}(B') = 1$ and $\det(B') = 0$. Since B' is a diagonal matrix and R is an integral domain, it is easily seen that $B' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Thus there exists an invertible matrix $U \in \mathbb{M}_2(R)$ such that $B = U \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} U^{-1}$. On the other hand $N' = UNU^{-1}$ is nilpotent, so by Lemma 2.2, $N' = \begin{pmatrix} a & c \\ b & -a \end{pmatrix}$, for some $a, b, c \in R$ with $-a^2 = bc$. Since A is strongly nil-clean, so $UAU^{-1} = B' + N'$ is strongly nil-clean. Hence $B'N' = N'B'$. Thus

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & c \\ b & -a \end{pmatrix} = \begin{pmatrix} a & c \\ b & -a \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

This implies that $b = c = 0$. Consequently $a = 0$. Therefore $UAU^{-1} = B'$. This means that A is an idempotent matrix. ■

3. Nil-clean matrix over a UFD

Let R be a commutative ring. Elements a, b of R are said to be associates if $a \mid b$ and $b \mid a$. A nonunit and nonzero element $p \in R$ is called an irreducible element, if $p = ab$ implies that either a or b is a unit element of R. Recall that an integral domain R is a unique factorization domain (UFD) provided every nonzero nonunit element of R can be written $a = p_1 ... p_n$, with $p_1, ..., p_n$ irreducible and if $a = q_1 ... q_m$ (q_i irreducible) then $n = m$ and for some permutation σ of $\{1, ..., n\}$, p_i and $q_{\sigma(i)}$ are associates for every i. Note that in a unique factorization domain (UFD), a greatest common divisor (GCD) of any collection of elements always exists. Also, for every a, b, c in a UFD, if $a \mid bc$ and a, b are relatively prime, then $a \mid c$.

Proposition 3.1. *Let R be a UFD and $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ be a nonzero matrix of rank one. Let $x = GCD(a,b)$ and $y = GCD(c,d)$, then $A = \begin{pmatrix} 0 & c \\ 0 & d \end{pmatrix}$ or $A = \begin{pmatrix} a & \frac{a}{x}y \\ b & \frac{b}{x}y \end{pmatrix}$.*

Proof. Let $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ be a matrix of rank one and $a \neq 0$ or $b \neq 0$. We have $ad = bc$. So $\frac{a}{x}d = \frac{b}{x}c$. Since $GCD(\frac{a}{x}, \frac{b}{x}) = 1$, hence $\frac{b}{x} \mid d$. Therefore there exists an element $t \in R$ such that $d = \frac{b}{x}t$. We have $\frac{a}{x}\frac{b}{x}t = \frac{b}{x}c$. Thus $c = \frac{a}{x}t$. In fact $t = GCD(b,d) = y$. Hence $A = \begin{pmatrix} a & \frac{a}{x}y \\ b & \frac{b}{x}y \end{pmatrix}$. ■

Theorem 3.2. *Let R be a UFD and $A \in \mathbb{M}_2(R)$. If A is nil-clean matrix which use a nontrivial idempotent and nilpotent in it's nil-clean decomposition then exactly one of the following statements holds:*

(1) $A = \begin{pmatrix} \frac{a}{x_1} & c \\ \frac{b}{x_1} & 1 \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ 0 & 1 \end{pmatrix}$, *for some $a, b, c, x_1, x_2 \in R$, where $a + \frac{b}{x_1}x_2 = 0$.*

(2) $A = \begin{pmatrix} \frac{a}{x_1} & 1 \\ \frac{b}{x_1} & 0 \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ 0 & 1 \end{pmatrix}$, *for some $a, b, x_1, x_2 \in R$, where $a + \frac{b}{x_1}x_2 = 1$.*

(3) $A = \begin{pmatrix} \frac{a}{x_1} & \frac{c}{y_1} \\ \frac{b}{x_1} & \frac{d}{y_1} \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$, *for some $a, b, c, d, x_1, x_2, y_1, y_2 \in R$, where $a + \frac{b}{x_1}x_2 = 1$ and $c + \frac{d}{y_1}y_2 = 0$.*

Proof. Let A be a nil-clean matrix. Thus $A = B + N$, for some idempotent matrix B and nilpotent matrix N. Assume that $0, I \neq B$. Thus by Proposition 3.1 and Lemma 2.1, $B = \begin{pmatrix} 0 & e \\ 0 & 1 \end{pmatrix}$ or $B = \begin{pmatrix} a & \frac{a}{x_1}x_2 \\ b & \frac{b}{x_1}x_2 \end{pmatrix}$, for some $e, a, b, x_1, x_2 \in R$ in which $a + \frac{b}{x_1}x_2 = 1$. Since N is a nilpotent matrix, then by Lemma 2.2, $rank(N) = 1$ and $\mathrm{tr}(N) = 0$. So by Proposition 3.1, $N = \begin{pmatrix} 0 & e' \\ 0 & 0 \end{pmatrix}$ or $N = \begin{pmatrix} c & \frac{c}{y_1}y_2 \\ d & \frac{d}{y_1}y_2 \end{pmatrix}$, for some elements $e', c, d, y_1, y_2 \in R$ in which $c + \frac{d}{y_1}y_2 = 0$. We consider four cases:

Case 1) Let $B = \begin{pmatrix} 0 & e \\ 0 & 1 \end{pmatrix}$ and $N = \begin{pmatrix} 0 & e' \\ 0 & 0 \end{pmatrix}$. Then $A = B + N = \begin{pmatrix} 0 & e+e' \\ 0 & 1 \end{pmatrix}$, which is an idempotent matrix.

Case 2) Let $B = \begin{pmatrix} 0 & e \\ 0 & 1 \end{pmatrix}$ and $N = \begin{pmatrix} c & \frac{c}{y_1}y_2 \\ d & \frac{d}{y_1}y_2 \end{pmatrix}$. So $A = B + N = \begin{pmatrix} \frac{c}{y_1} & e \\ \frac{d}{y_1} & 1 \end{pmatrix}\begin{pmatrix} y_1 & y_2 \\ 0 & 1 \end{pmatrix}$, where $c + \frac{d}{y_1}y_2 = 0$.

Case 3) Let $B = \begin{pmatrix} a & \frac{a}{x_1}x_2 \\ b & \frac{b}{x_1}x_2 \end{pmatrix}$ and $N = \begin{pmatrix} 0 & e' \\ 0 & 0 \end{pmatrix}$. Therefore $A = B + N = \begin{pmatrix} \frac{a}{x_1} & e' \\ \frac{b}{x_1} & 0 \end{pmatrix}\begin{pmatrix} x_1 & x_2 \\ 0 & 1 \end{pmatrix}$, where $a + \frac{b}{x_1}x_2 = 1$.

Case 4) Let $B = \begin{pmatrix} a & \frac{a}{x_1}x_2 \\ b & \frac{b}{x_1}x_2 \end{pmatrix}$ and $N = \begin{pmatrix} c & \frac{c}{y_1}y_2 \\ d & \frac{d}{y_1}y_2 \end{pmatrix}$. So $A = B+N = \begin{pmatrix} \frac{a}{x_1} & \frac{c}{y_1} \\ \frac{b}{x_1} & \frac{d}{y_1} \end{pmatrix}\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$, where $a + \frac{b}{x_1}x_2 = 1$ and $c + \frac{d}{y_1}y_2 = 0$. ■

Theorem 3.2, asserts that every matrix $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ with $x_1 = GCD(a,b)$ and $y_1 = GCD(c,d)$ in which $b \mid x_1(1-a)$ and $d \mid y_1 c$ gives us a nil-clean matrix and conversely. See the following example:

Example 3.3. Let k be a field. It is well known that $k[x,y]$, the ring of polynomials in two indeterminates over k, is a UFD. Consider the matrix $\begin{pmatrix} x+1 & x+y \\ x & x+y \end{pmatrix}$. We have $GCD(x+1,x) = 1$ and $GCD(x+y,x+y) = x+y$. Therefore the matrix

$$A = \begin{pmatrix} x+1 & 1 \\ x & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ x+y & -(x+y) \end{pmatrix} = \begin{pmatrix} 2x+y+1 & 1-y \\ 2x+y & y \end{pmatrix}$$

is a nil-clean matrix.

Example 3.4. Let R be a PID and $0, I \neq A \in \mathbb{M}_2(R)$ be a nil-clean matrix. Thus there exists some idempotent matrix B and a nilpotent matrix N such that $A = B + N$. Assume that $0, I \neq B$. By Proposition 2.4 and Lemma 2.5, B is similar to a diagonal matrix B'. By Lemma 2.1, $\mathrm{tr}(B') = 1$ and $\det(B') = 0$. Since B' is a diagonal matrix and R is an integral domain, it is easily seen that $B' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Thus there exists an invertible matrix $U \in \mathbb{M}_2(R)$ such that $B = U^{-1}B'U = U^{-1}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}U$. On the other hand $N' = UNU^{-1}$ is nilpotent, so by Lemma 2.2, $N' = \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}$

or $N' = \begin{pmatrix} a & \frac{a}{x_1}x_2 \\ b & \frac{b}{x_1}x_2 \end{pmatrix}$, for some $a, b, c, x_1, x_2 \in R$ with $x_1 = GCD(a, b)$ and $a + \frac{b}{x_1}x_2 = 0$. Hence $UAU^{-1} = B' + N' = \begin{pmatrix} 1 & c \\ 0 & 0 \end{pmatrix}$, which is an idempotent matrix, or

$$UAU^{-1} = B' + N' = \begin{pmatrix} 1+a & \frac{a}{x_1}x_2 \\ b & \frac{b}{x_1}x_2 \end{pmatrix} = \begin{pmatrix} 1 & \frac{a}{x_1} \\ 0 & \frac{b}{x_1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x_1 & x_2 \end{pmatrix}.$$

Therefore if R is a PID, then every nil-clean matrix in $M_2(R)$, which use a nontrivial idempotent and nilpotent in their nil-clean decomposition, is similar to a matrix of the following set:

$$\{\begin{pmatrix} 1+a & \frac{a}{x_1}x_2 \\ b & -a \end{pmatrix} \mid a, b, x_1, x_2 \in R\}.$$

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