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THE STRUCTURE OF A PAIR OF NILPOTENT LIE ALGEBRAS

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Abstract. Assume that (N, L), is a pair of finite dimensional nilpotent Lie algebras, in which L is non-abelian and N is an ideal in L and also $\mathcal{M}(N, L)$ is the Schur multiplier of the pair (N, L). Motivated by characterization of the pairs (N, L) of finite dimensional nilpotent Lie algebras by their Schur multipliers (Arabyani, et al. 2014) we prove some properties of a pair of nilpotent Lie algebras and generalize results for a pair of non-abelian nilpotent Lie algebras.

1. Introduction

The first important research about nilpotent Lie algebras is due to K. Umlauf in the 19th century. In the 40s and 50s, Morozov and Dixmier began with the systematical study of this class of algebras. In 1958, Morozov [16] gave a classification of six dimensional nilpotent Lie algebras over a field of characteristic zero. Recently, Cical, de Graaf, and Schneider in 2010 [7] presented a full classification of six dimensional nilpotent Lie algebras over arbitrary fields.

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Let L be a Lie algebra over a fixed field Λ and $[,]$ denotes the Lie bracket. Also $Z_n(L)$ and L^{n+1} denote the n-th terms of the upper and lower central series of a Lie algebra L, respectively, defined inductively by $L^1 = L$ and $L^{n+1} = [L^n, L]$, for $n \geq 1$ and also $Z_1(L) = Z(L)$ and $Z_{n+1}(L)/Z_n(L) = Z(L/Z_n(L))$, for $n \geq 1$. We recall that a Lie algebra L is *nilpotent* if $L^s = 0$ for some non-negative integer s. Let L be a Lie algebra with a free presentation

$$0 \longrightarrow R \longrightarrow F \longrightarrow L \longrightarrow 0,$$

in which F is a free Lie algebra. The Schur multiplier of L is denoted by $\mathcal{M}(L)$ and defined as

$$\mathcal{M}(L) = \frac{R \cap [F, F]}{[R, F]}.$$

One can easily verify that the multiplier of a Lie algebra L is abelian and that it is independent of the choice of free presentations (see [3, 5, 4, 6, 11, 12, 24, 26] for more information). In 1993, Batten showed that for a finite dimensional Lie algebra L, $\mathcal{M}(L)$ is isomorphic to $H^2(L, \Lambda)$, the second cohomology Lie algebra of L, where Λ is considered as a trivial L-module. Classification of finite dimensional Lie algebras in terms of Schur multiplier has been the center of attention for many authors. In 1994, Moneyhun [15] proves that for a Lie algebra L of dimension n, we always have

$$\dim \mathcal{M}(L) = \frac{1}{2}n(n-1) - t(L),$$

for some non-negative integer $t(L)$. He et al. in [4] have shown that $t(L) = 0$ if and only if $L \cong A(n)$ is the abelian n-dimensional Lie algebra, $t(L) = 1$ if and only if $L \cong H(1)$, and $t(L) = 2$ if and only if $L \cong H(1) \oplus A(1)$. Remind that a Lie algebra L is called Heisenberg provided that $L^2 = Z(L)$ and $\dim L^2 = 1$. Heisenberg Lie algebra has odd dimension with a basis $e, e_1, \ldots, e_{2m}$ subject to the relations $[e_{2i-1}, e_{2i}] = e$ for $i = 1, \ldots, m$. The Heisenberg Lie algebra of dimension $2m+1$ is denoted by $H(m)$. A classification of nilpotent Lie algebras for which $t(L)$ takes small values is also obtained in [11, 12].

Let (N, L) be a pair of Lie algebras, in which N is an ideal of L. Then we define a series of ideals of N as follows:

$$N = [N,_0 L] \supseteq [N, L] \supseteq [N, L, L] \supseteq \cdots \supseteq [N,_n L] \supseteq \cdots$$

where $\gamma_{n+1}(N, L) = [N,_n L]$ ($n-$times) for all $n > 0$. We call such a series the lower central of N in L. We say that a pair (N, L) of Lie algebras is nilpotent if it has a finite lower central series. The shortest length of such series is called the class of nilpotency of the pair (N, L). Similarly we may define the upper central series of N in L as follows:

$$0 = Z_0(N, L) \subseteq Z_1(N, L) \subseteq \cdots \subseteq Z_m(N, L) \subseteq \cdots,$$

where $Z_m(N, L) = \{n \in N | [n, l_1, \cdots, l_m] = 0, \text{ for all } l_1, \cdots, l_m \in L\}$. It can be easily checked that a pair (N, L) of Lie algebras is nilpotent of class at most c if and only if $Z_c(N, L) = N$.

The Schur multiplier of nilpotent Lie algebras is generalized for a pair of nilpotent Lie algebras.

Let (N, L) be a pair of Lie algebras, where N is an ideal in L. Then we define the Schur multiplier of the pair (N, L) to be the abelian Lie algebra $\mathcal{M}(N, L)$ appearing in the following natural exact sequence of Lie algebras

$$H_3(L) \to H_3(\frac{L}{N}) \to \mathcal{M}(N,L) \to \mathcal{M}(L) \to \mathcal{M}(\frac{L}{N}) \to \frac{L}{[N,L]} \to \frac{L}{L^2} \to \frac{L}{(L^2+N)} \to 0$$

where $\mathcal{M}(-)$ and $H_3(-)$ denote the Schur multiplier and the third homology of a Lie algebra, respectively. This is analogous to the definition of a pair given by Ellis [9] (see also [13, 14]). In [9], it is proved that $\mathcal{M}(N, L) \cong ker(N \wedge L \to L)$, in which $N \wedge L$ denotes the non-abelian exterior product of Lie algebras. Also using the above sequence, one may easily observe that if the ideal N possess a complement in L, then $\mathcal{M}(L) = \mathcal{M}(N, L) \oplus \mathcal{M}(L/N)$. In this case, for each free presentation $0 \to R \to F \to L \to 0$ of L, $\mathcal{M}(N, L)$ is isomorphic to the factor Lie algebra $(R \cap [S, F])/[R, F]$, where S is an ideal in F such that $S/R \cong N$. In particular, if $N = L$, then the Schur multiplier of (N, L) will be $\mathcal{M}(L) = (R \cap F^2)/[R, F]$. Saeedi et al. [22] proved that in the pair (N, L) of finite dimensional nilpotent Lie algebras, if N admits a complement K say, in L with dim $N = n$ and dim $K = m$, then dim $\mathcal{M}(N, L) = \frac{1}{2}n(n + 2m - 1) - t(N, L)$, for a non-negative integer $t(N, L)$. Authors in [2] characterized the pairs (N, L) of finite dimesional nilpotent Lie algebras.

In 2011, Niroomand and Russo [18] gave an upper bound for the dimension of the Schur multiplier of an n-dimensional non-abelian nilpotent Lie algebra L as follows:

$$\dim\mathcal{M}(L) = \frac{1}{2}(n-1)(n-2) + 1 - s(L), \tag{1}$$

for some non-negative integer $s(L)$. They also classified the structure of L when $s(L) = 0$. Also, Niroomand [17] classified all nilpotent Lie algebras L satisfying $s(L) = 1$ or 2. Recently authors in [23] generalized these results for $s(L) = 3$.

In this paper, we will prove some properties of a pair of nilpotent Lie algebras. Also in the sequel by using (1) we characterize the pairs (N, L), in which L is non-abelian nilpotent Lie algebra.

2. **Some properties of a pair of nilpotent Lie algebras**

In this section, we give some elementary results of a pair of nilpotent Lie algebras.

Definition 2.1. Put $(N, L)^{(1)} = [N, L]$, and assume that for $i \geq 1$, $(N, L)^{(i)}$ is defined inductively. Then $(N, L)^{(i+1)}$ is defined to be the commutator subalgebra $[(N, L)^{(i)}, (N, L)^{(i)}]$, for all $i \geq 1$. Assuming $(N, L)^{(0)} = N$, we have the ideal series

$$N \supseteq (N,L)^{(1)} \supseteq (N,L)^{(2)} \cdots \supseteq (N,L)^{(i)} \supseteq \cdots ,$$

and we call it the derived series of the pair (N, L).

Some properties of such central series are displayed in the next theorem.

Theorem 2.2. *Let $0 = N_0 \leq N_1 \leq ... \leq N_n = N$ be a central series of a nilpotent pair of Lie algebras (N, L). Then*

(i) $\gamma_i(N, L) \leq N_{n-i+1}$, for $1 \leq i \leq n+1$ and hence $\gamma_{n+1}(N, L) = 0$.

(ii) $N_i \leq Z_i(N, L)$, so that $Z_n(N, L) = N$,

(iii)The nilpotent class of (N, L) is equal to the length of the upper and the lower central series of L.

Proof. (i) We proceed by induction on i. The case $i = 1$ is clear. Assume the result is true for $i < n$. By the assumption

$$N_{n-i+1}/N_{n-i} \subseteq Z(N/N_{n-i}).$$

Since $[N,_{n-i+1} L] \leq N_{n-i}$, then by induction hypothesis, we have

$$\gamma_{i+1}(N, L) = [\gamma_i(N, L), L] \leq [N,_{n-i+1} L] \leq N_{n-i}.$$

parts (ii) and (iii) can be seen easily.□

Theorem 2.3. *Let (N, L) be a pair of Lie algebras and i and j be positive integers. Then*

(i) $[\gamma_i(N, L), \gamma_j(N, L)] \leq \gamma_{i+j}(N, L)$,

(ii) $\gamma_i(\gamma_j(N, L)) \leq \gamma_{ij}(N, L)$,

(iii) $[\gamma_i(N, L), Z_j(N, L)] \leq Z_{j-i}(N, L)$ if $j \geq i$,

(iv) $Z_i(L/Z_j(N, L), N/Z_j(N, L)) = Z_{i+j}(N, L)/Z_j(N, L)$.

Proof. (i) We proceed by induction on j. for $j = 1$, the result follows by the definition. Now assume the result holds for j, i.e,

$$[\gamma_i(N, L), \gamma_j(N, L)] \subseteq \gamma_{i+j}(N, L).$$

We prove the result for $j+1$. By Jacobian identity, we have

$$\begin{aligned}[\gamma_i(N,L),\gamma_{j+1}(N,L)] &= [\gamma_j(N,L),N,\gamma_i(N,L)]\\ &= [N,\gamma_i(N,L),\gamma_j(N,L)]+[\gamma_i(N,L),\gamma_j(N,L),N]\\ &\subseteq \gamma_{i+j+1}(N,L).\end{aligned}$$

(ii) Use induction on i, the case $i=1$ being obvious. Assume the result holds for i, i.e, $\gamma_i(\gamma_j(N,L))\subseteq\gamma_{ij}(N,L)$. Then by part (i), we have

$$\begin{aligned}\gamma_{i+1}(\gamma_j(N,L)) &= [\gamma_i(\gamma_j(N,L)),\gamma_j(N,L)]\\ &\subseteq [\gamma_{ij}(N,L),\gamma_j(N,L)]\\ &\subseteq \gamma_{(i+1)j}(N,L).\end{aligned}$$

(iii) We use induction on i. The case $i=1$ is clear. So assume the result holds for i, i.e, $[\gamma_i(N,L),Z_j(N,L)]\subseteq Z_{j-i}(N,L)$. Now by Jacobian identity , we have

$$\begin{aligned}[\gamma_{i+1}(N,L),Z_j(N,L)] &= [\gamma_i(N,L),N,Z_j(N,L)]\\ &\subseteq [N,Z_j(N,L),\gamma_i(N,L)]+[Z_j(N,L),\gamma_i(N,L),N]\\ &\subseteq [Z_{j-1}(N,L),\gamma_i(N,L)]+[Z_{j-i}(N,L),N]\\ &\subseteq Z_{j-(i+1)}(N,L).\end{aligned}$$

(iv) This part can be also proved, by using induction on i. □

Theorem 2.4. *If (N,L) is a pair of Lie algebras, then $(N,L)^{(i)}\le\gamma_{2^i}(N,L)$ also if (N,L) is nilpotent of class at most $c\ge 1$, then the length of the derived series of (N,L) is at most $[\log_2^c]+1$.*

Proof. We proceed by induction on i, the case $i=0$ being obvious. Assume the result holds for i, i.e, $(N,L)^{(i)}\le\gamma_{2^i}(N,L)$. Then by Theorem 2.3 (ii), we have

$$(N,L)^{(i+1)}=[(N,L)^{(i)},(N,L)^{(i)}]=\gamma_2((N,L)^{(i)})\le\gamma_2(\gamma_{2^i}(N,L))\le\gamma_{2^{i+1}}((N,L)).$$

Let d be the length of the derived series of (N,L). Then we have

$$(N,L)^{(i)}\le\gamma_{2^i}(N,L)\le\gamma_{c+1}(N,L)=0 \text{ if and only if } 2^i\ge c+1.$$

Then the smallest such i is $[\log_2^c]+1$, whence $d\le[\log_2^c]+1$. □

Theorem 2.5. *If (N,L) is a pair of nilpotent Lie algebras and M is a nontrivial ideal of L such that $M\cap N$ is nontrivial, then $M\cap Z(N,L)\neq 0$.*

Proof. Since (N, L) is nilpotent, then there exists a positive integer c such that $N = Z_c(N, L)$.Let i be the least integer such that $M \cap Z_i(N, L) \neq 0$.Now, $[M \cap Z_i(N, L), L] \leq M \cap Z_{i-1}(N, L) = 0$ and $M \cap Z_i(N, L) \leq M \cap Z_1(N, L)$. Hence $M \cap Z_1(N, L) = M \cap Z_i(L, N) \neq 0$. □

Corollary 2.6. *If (N, L) is a pair of nilpotent Lie algebras with $N \neq 0$, then $Z(N, L) \neq 0$.*

3. On characterizing pairs of non-abelian nilpotent Lie algebras

In what follows we state some lemmas and theorems that will be used in the proof of Theorem 3.5.

Lemma 3.1. *(See [3] Example 3, See [15] Theorem 24) Let $A(n)$ be an abelian Lie algebra and $H(n)$ be a Hiesenberg Lie algebra. Then*

(i) $dim\ \mathcal{M}(A(n)) = \frac{1}{2}n(n-1)$;

(ii) $dim\ \mathcal{M}(H(1)) = 2$;

(iii) *For* $n \geq 2$ $dim\ \mathcal{M}(H(n)) = 2n^2 - n - 1$.

Lemma 3.2. *(See [4] Theorem 2) Let $L = A \oplus B$. Then*

$$dim\ \mathcal{M}(L) = dim\ \mathcal{M}(A) + dim\ \mathcal{M}(B) + dim\ (A/A^2 \otimes B/B^2).$$

Theorem 3.3. *(See [4, 11, 12])Let L be an n-dimensional nilpotent Lie algebra and* $\dim\mathcal{M}(L) = \frac{1}{2}n(n-1) - t(L)$ *for some non-negative integer $t(L)$. Then*

(a) $t(L) = 0$ *if and only if L is abelian;*

(b) $t(L) = 1$ *if and only if* $L \cong H(1)$;

(c) $t(L) = 2$ *if and only if* $L \cong H(1) \oplus A(1)$;

(d) $t(L) = 3$ *if and only if* $L \cong H(1) \oplus A(2)$;

(e) $t(L) = 4$ *if and only if* $L \cong H(1) \oplus A(3),\ L(3, 4, 1, 4)$ *or* $L(4, 5, 2, 4)$;

(f) $t(L) = 5$ *if and only if* $L \cong H(1) \oplus A(4)$ *or* $H(2)$;

(g) $t(L) = 6$ *if and only if* $L \cong H(1) \oplus A(5), H(2) \oplus A(1),\ L(4, 5, 1, 6),\ L(3, 4, 1, 4) \oplus A(1)$ *or* $L(4, 5, 2, 4) \oplus A(1)$;

(h) $t(L) = 7$ *if and only if* $L \cong H(1) \oplus A(6),\ H(2) \oplus A(2),\ H(3),\ L(7, 5, 2, 7),\ L(7, 5, 1, 7),\ L'(7, 5, 1, 7),\ L(7, 6, 2, 7)$ *or* $L(7, 6, 2, 7, \beta_1, \beta_2)$;

(i) $t(L) = 8$ *if and only if* $L \cong H(1) \oplus A(7),\ H(2) \oplus A(3),\ H(3) \oplus A(1),\ L(4, 5, 1, 6) \oplus A(1),\ L(3, 4, 1, 4) \oplus A(2)$ *or* $L(4, 5, 2, 4) \oplus A(2)$.

Here $H(m)$ denotes the Heisenberg Lie algebra of dimension $2m+1$, $A(n)$ is an n-dimensional abelian Lie algebra and $L(a, b, c, d)$ will denote the algebra discovered during $t(L) = a$ case, where $b = dim\ L$, $c = dim\ Z(L)$ and $d = t(L)$ (see Table I).

Theorem 3.4. *(see [17, 18]) Let L be an n-dimensional non-abelian nilpotent Lie algebra and* $\dim\mathcal{M}(L) = \frac{1}{2}(n-1)(n-2) + 1 - s(L)$ *for some non-negative integer $s(L)$. Then*

(i) $s(L) = 0$ *if and only if* $L \cong H(1) \oplus A(n-3)$.

(ii) $s(L) = 1$ *if and only if* $L \cong L(4, 5, 2, 4)$.

(iii) $s(L) = 2$ *if and only if L isomorphic to one of the following Lie algebras*

$$L(3, 4, 1, 4), \quad L(4, 5, 2, 4) \oplus A(1), \quad H(m) \oplus A(n-2m-1), \quad m \geq 2$$

Now, in the following theorem we will characterize the pairs (N, L), for which $0 \leq t(K) \leq s(L) \leq 2$.

Theorem 3.5. *Let (N, L) be a pair of finite dimensional nilpotent Lie algebras, in which L is non-abelian and N is a non-trivial ideal in L. Also, assume that K is a subalgebra of L which is complement of N in L such that* $\dim K = m$ *and* $\dim L = n$. *Then* $\dim\mathcal{M}(N, L) = \frac{1}{2}(n^2 + 2nm - 3n - 2m + 2) + 1 - (s(L) - t(K))$, *where* $t(K) = \frac{1}{2}m(m-1) - \dim\mathcal{M}(K)$ *and* $s(L) = \frac{1}{2}(n-1)(n-2) + 1 - \dim\mathcal{M}(L)$. *Moreover,*

(a) $(s(L), t(K)) = (0, 0)$ *if and only if* $(N, L) \cong (H(1) \oplus A(j), H(1) \oplus A(n-3)), 0 \leq j \leq n-4$.

(b) There is not any pair (N, L), for $(s(L), t(K))$ with $s(L) = 1$ and $t(K) = 0, 1$.

(c) $(s(L), t(K)) = (2, 0)$ *if and only if (N, L) is isomorphic to one of the following,*

$(L(4, 5, 2, 4), L(4, 5, 2, 4) \oplus A(1))$, $(A(4), L(4, 5, 2, 4) \oplus A(1))$, $(A(3), L(4, 5, 2, 4) \oplus A(1))$,

$(H(m) \oplus A(j), H(m) \oplus A(n-2m-1))$ *such that* $0 \leqslant j \leqslant n-2m-2, \quad m \geqslant 2$.

(d) There is not any pair (N, L), for $(s(L), t(K))$ with $s(L) = 2, t(K) = 1, 2$.

Proof. By the assumptions an the fact that $\mathcal{M}(L) = \mathcal{M}(N, L) \oplus \mathcal{M}(L/N)$, we have $\dim\mathcal{M}(N, L) = \frac{1}{2}(n^2 + 2nm - 3n - 2m + 2) + 1 - (s(L) - t(K))$.

Case $s(L) = 0$. In this case $t(K) = 0$ and Theorem3.3 and 3.4 imply that $L \cong H(1) \oplus A(n-3)$ and K is an abelian subalgebra of L. According to the structure of L, we have

$$(N, L) \cong (H(1) \oplus A(j), H(1) \oplus A(n-3)), \text{ such that } 0 \leq j \leq n-4.$$

case $s(L) = 1$. In this case $t(K) = 0, 1$ and by Theorem 3.4, $L \cong L(4, 5, 2, 4)$. Let $t(K) = 0$. Thus K is abelian. Now if $\dim N = 4$ and $\dim K = 1$, then

$$N = \langle x_1, x_3, x_4, x_5 | [x_1, x_4] = x_5 \rangle \cong H(1) \oplus A(1),$$

such that $\{x_1, x_2, x_3, x_4, x_5\}$ is a basis for L. Hence Theorem 3.3 implies that $t(N) = 2$ and $\dim\mathcal{M}(N) = 4$. But by lemma 3.2, we have $\dim\mathcal{M}(N) = 3$, which is a contradiction. Also if $\dim N = 3$ and $\dim K = 2$ or $\dim N = 2$ and $\dim K = 3$, then there is not any ideal and subalgebra satisfying in the assumptions of the theorem. If $t(K) = 1$, then $K \cong H(1)$ and there is not any ideal and subalgebra satisfying in the theorem.

Case $s(L) = 2$. In this case $t(K) = 0, 1, 2$. By using Theorem 3.4, we get $L \cong L(3, 4, 1, 4)$, $L(4, 5, 2, 4) \oplus A(1)$ or $H(m) \oplus A(n-2m-1)$ such that $m \geq 2$. First suppose that $L \cong L(3, 4, 1, 4)$ and $t(K) = 0$, then K is abelian. If $\dim N = 3$ and $\dim K = 1$ or $\dim K = \dim N = 2$, then choosing a suitable subalgebra K and an ideal N in L and lemma 3.2 imply that there is no such a pair. If $t(K) = 1$, then $K \cong H(1)$. By choosing a suitable ideal N in L and lemma 3.2, there is no any pair. And if $t(K) = 2$, then ideal N is trivial.

Now, let $L \cong L(4, 5, 2, 4) \oplus A(1)$. If $t(K) = 0$ then K is abelian. Therefore $(N, L) \cong (A(4), L(4, 5, 2, 4) \oplus A(1))$, $(A(3), L(4, 5, 2, 4) \oplus A(1))$ or $(L(4, 5, 2, 4), L(4, 5, 2, 4) \oplus A(1))$. If $t(K) = 1, 2$, then Theorem 3.3, lemma 3.2 and choosing a suitable subalgebra K and an ideal N of L imply that there is not any pair.

Finally, let $L \cong H(m) \oplus A(n - 2m - 1)$, such that $m \geq 2$. If $t(K) = 0$, then similar to the first case, we can easily see that $(N, L) \cong (H(m) \oplus A(j), H(m) \oplus A(n - 2m - 1))$, where $0 \leq j \leq n - 2m - 2$ and $m \geq 2$. If $t(K) = 1$, then there is not any ideal N in L, and if $t(K) = 2$, then by Theorem 3.3 and lemma 3.2, there is no such a pair. □

TABLE I

t(L)	dim L	Non Zero Multiplication	Nilpotent Lie algebra
0			Abelian
1	3	$[x_1, x_2] = x_3$	$H(1)$
2	4	$[x_1, x_2] = x_3$	$H(1) \oplus A(1)$
3	5	$[x_1, x_2] = x_3$	$H(1) \oplus A(2)$
4	4	$[x_1, x_2] = x_3,\ [x_1, x_3] = x_4$	$L(3, 4, 1, 4)$
4	5	$[x_1, x_2] = x_3,\ [x_1, x_4] = x_5$	$L(4, 5, 2, 4)$
4	6	$[x_1, x_2] = x_3$	$H(1) \oplus A(3)$
5	5	$[x_1, x_2] = x_5,\ [x_3, x_4] = x_5$	$H(2)$
5	7	$[x_1, x_2] = x_3$	$H(1) \oplus A(4)$
6	5	$[x_1, x_2] = x_3,\ [x_1, x_3] = x_5$	$L(3, 4, 1, 4) \oplus A(1)$
6	5	$[x_1, x_2] = x_3,\ [x_1, x_3] = x_5,\ [x_2, x_4] = x_5$	$L(4, 5, 1, 6)$
6	6	$[x_1, x_2] = x_5,\ [x_1, x_3] = x_5,\ [x_3, x_4] = x_5$	$H(2) \oplus A(1)$
6	6	$[x_1, x_2] = x_3,\ [x_1, x_4] = x_6$	$L(4, 5, 2, 4) \oplus A(1)$
6	8	$[x_1, x_2] = x_3$	$H(1) \oplus A(5)$
7	5	$[x_1, x_2] = x_3,\ [x_1, x_3] = x_4,\ [x_2, x_3] = x_5$	$L(7, 5, 2, 7)$
7	5	$[x_1, x_2] = x_3,\ [x_1, x_3] = x_4,\ [x_1, x_4] = x_5$	$L(7, 5, 1, 7)$
7	5	$[x_1, x_2] = x_3,\ [x_1, x_3] = x_4,\ [x_2, x_3] = x_5,\ [x_1, x_4] = x_5$	$L'(7, 5, 1, 7)$
7	6	$[x_1, x_2] = x_3,\ [x_1, x_4] = x_6,\ [x_2, x_5] = x_6$	$L(5, 6, 2, 7)$
7	6	$[x_1, x_2] = x_3,\ [x_4, x_5] = x_6$	$L'(5, 6, 2, 7)$
7	6	$[x_1, x_2] = x_5,\ [x_3, x_4] = x_6$	$L(7, 6, 2, 7)$
7	6	$[x_1, x_2] = x_5 + \beta_1 x_6,\ [x_3, x_4] = x_5$ $[x_1, x_4] = x_6,\ [x_3, x_2] = \beta_2 x_6$	$L(7, 6, 2, 7, \beta_1, \beta_2)$
7	7	$[x_1, x_2] = x_5,\ [x_3, x_4] = x_5$	$H(2) \oplus A(2)$
7	7	$[x_1, x_2] = x_7,\ [x_3, x_4] = x_7,\ [x_5, x_6] = x_7$	$H(3)$
8	6	$[x_1, x_2] = x_3,\ [x_1, x_3] = x_6$	$L(3, 4, 1, 4) \oplus A(2)$
8	6	$[x_1, x_2] = x_3,\ [x_1, x_3] = x_6,\ [x_2, x_4] = x_6$	$L(4, 5, 1, 6) \oplus A(1)$
8	7	$[x_1, x_2] = x_3,\ [x_1, x_4] = x_7$	$L(4, 5, 2, 4) \oplus A(2)$
8	8	$[x_1, x_2] = x_5,\ [x_3, x_4] = x_5$	$H(2) \oplus A(3)$
8	8	$[x_1, x_2] = x_7,\ [x_3, x_4] = x_7,\ [x_5, x_6] = x_7$	$H(3) \oplus A(1)$
8	10	$[x_1, x_2] = x_3$	$H(1) \oplus A(7)$

TABLE II

s(L)	dim L	Non-zero Lie barckets	Non-abelian nilpotent Lie algebras
0	n	$[x_1, x_2] = x_3$	$H(1) \oplus A(n-3)$
1	5	$[x_1, x_2] = x_3, [x_1, x_4] = x_5$	$L(4,5,2,4)$
2	4	$[x_1, x_2] = x_3, [x_1, x_3] = x_4$	$L(3,4,1,4)$
2	6	$[x_1, x_2] = x_3,\ [x_1, x_4] = x_5$	$L(4,5,2,4) \oplus A(1)$
2	n	$[x_i, x_{i+1}] = y,\ 1 \leq i \leq 2m-1$	$H(m) \oplus A(n-2m-1)$

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