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ULTRA AND INVOLUTION IDEALS IN BCK-ALGEBRAS

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Abstract. In this paper, we define the notions of ultra and involution ideals in BCK-algebras. Then we get the relation among them and other ideals as (positive) implicative, associative, commutative and prime ideals. Specially, we show that in a bounded implicative BCK-algebra, any involution ideal is a positive implicative ideal and in a bounded positive implicative lower BCK-semilattice, the notions of prime ideals and ultra ideals are coincide.

1. Introduction

The notion of BCK-algebra was formulated first in 1966 by Imai and Iséki [4]. This notion is originated from two different ways. One of the motivations is based on set theory. In set theory, there are three most elementary and fundamental operations among various operations including the general analytical operation introduced by L. Kantorovic and E. Livenson

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to make a new set from old sets. Another motivation is from classical and non-classical propositional calculus. There are some systems which contain the only implication functor among the logical functors. As is well known, there is close relationship between the notion of the set difference in set theory and the implication functor in logical systems. Then the following problems arise from this relationship. What is the most essential and fundamental common properties? Can we establish a good theory of general algebra? To give an answer this problems, Y. Imai and K. Iséki introduced a notion of a new class of general algebras which is called a BCK-algebra. This name is taken from BCK-system of C. A. Meredith. Since 1966, many studies were performed about this subject and many researchers started working on BCK-algebra. Recently, many papers have been written in this field, too [7, 9]. The notion of ideal in BCK-algebra was introduced by K. Iséki in 1975 [5]. The ideal theory plays a fundamental role for the general development of BCK-algebras. In this paper, for completion of study of ideals in BCK-algebras, we introduce the notions of ultra and involution ideals and we investigate the relationship between them and some ideals that have been existed. Now, we state some definitions of BCK-algebras and ideals in BCK-algebras and we review related lemmas and theorems that we use in the next sections.

2. Preliminaries

Definition 2.1. [8] A BCK-algebra is a structure $X = (X, *, 0)$ of type $(2, 0)$ such that:
$(BCK1)$ $((x * y) * (x * z)) * (z * y) = 0$,
$(BCK2)$ $(x * (x * y)) * y = 0$,
$(BCK3)$ $x * x = 0$,
$(BCK4)$ $0 * x = 0$,
$(BCK5)$ $x * y = y * x = 0$ implies that $x = y$, for all $x, y, z \in X$.
The relation $x \leq y$ which is defined by $x * y = 0$ is a partial order on X with 0 as least element.
In BCK-algebra X, for any $x, y, z \in X$, we have
$(BCK6)$ $(x * y) * z = (x * z) * y$,
$(BCK7)$ $x \leq y$ implies $z * y \leq z * x$,
$(BCK8)$ $x \leq y$ implies $x * z \leq y * z$.

Let $(X, *, 0)$ be a BCK-algebra. Subset $\emptyset \neq I \subseteq X$ is called an *ideal* of X, if $0 \in I$ and for any $x, y \in X$, $x * y \in I$ and $y \in I$, implies that $x \in I$. X is called *bounded*, if there exists $1 \in X$ such that $x \leq 1$, for every $x \in X$ and in this case, we let $Nx = 1 * x$. X is said to be a *lower BCK-semilattice* if X is lower semilattice with respect to BCK- order " $\leq$ ". X is called to be *positive implicative* if $(x * z) * (y * z) = (x * y) * z$, for any $x, y, z \in X$. We can prove that X is positive implicative if and only if $x * y = (x * y) * y$, for every $x, y \in X$. X is said to be *commutative*, if $y*(y*x)=x*(x*y)$, for all $x,y\in X$. X is said to be *implicative* if $x*(y*x)=x$, for all $x,y\in X$. In lower BCK-semilattice X, a proper ideal I, is called a *prime ideal* of X, if $x\wedge y\in I$ implies that $x\in I$ or $y\in I$, for any $x,y\in X$. A nonempty subset I of X is said to be a *positive implicative ideal* if $0\in I$ and $(x*y)*z$, $y*z\in I$ implies that $x*z\in I$, for any $x,y,z\in X$. I is said to be an *implicative ideal* of X if $0\in I$ and $(x*(y*x))*z$, $z\in I$ implies that $x\in I$, for any $x,y,z\in X$. I is said to be a *commutative ideal* of X if $0\in I$ and $(x*y)*z$, $z\in I$ implies that $x*(y*(y*x))\in I$, for any $x,y,z\in X$. Furthermore, any positive implicative ideal (implicative ideal, commutative ideal) must be an ideal. In a BCK-algebra X, we let $x\wedge y=y*(y*x)$ and in a bounded BCK-algebra X, we let $x\vee y=N(Nx\wedge Ny)$, for all $x,y\in X$. In bounded commutative BCK-algebra X, for any $x,y\in X$, $x\vee y$ is the least upper bound and $x\wedge y$ is the grate lower bound of x,y and so $(L,\vee,\wedge)$ is a bounded lattice.

Theorem 2.2. [2, 8] (i) *A bounded implicative BCK-algebra X is a distributive lattice.*
(ii) *In a bounded commutative BCK-algebra X, $NNx=x$ and $x*Ny=y*Nx$, for any $x,y\in X$.*
(iii) *Let X be a bounded implicative BCK-algebra . Then*
(1) $x\wedge y=x*Ny$, $x*y=Ny\wedge x$, $NNx=x$, $Nx*Ny\leq y*x$, $x=x*Nx$, $Nx=Nx*x$,
(2) $x*(x\wedge y)=x*y$,
(3) $x\wedge(y*z)=(x\wedge y)*(x\wedge z)$, *for any* $x,y,z\in X$.

Theorem 2.3. [5, 6, 8] (i) *A BCK-algebra is implicative if and only if it is both commutative and positive implicative.*
(ii) *In a positive implicative BCK-algebra, all ideals are positive implicative.*
(iii) *In a commutative BCK-algebra, all ideals are commutative.*
(iv) *An ideal I of BCK-algebra X is commutative if and only if $x*y\in I$ implies that $x*(y*(y*x))\in I$, for any $x,y\in X$.*
(v) *An ideal I of BCK-algebra X is implicative if and only if, for any $x,y\in X$, $x*(y*x)\in I$ implies that $x\in I$. Moreover, an implicative ideal of X must be a positive implicative ideal.*

Definition 2.4. [8] Let X be a BCK-algebra and A be a subset of X. The ideal I generated by A is the intersection of all ideals of X, which contain A and it is denoted by $I=(A]$. Moreover, if A is a finite subset of X, then I is called finitely generated.

Theorem 2.5. [8] *Let X be a BCK-algebra and A be a nonempty subset of X. Then*

$$(A]=\{x\in X:\exists a_1,\cdots,a_n\in A \text{ such that } (...(x*a_1)*...)*a_n=0\},\ (a]=\{x:x*a=0\}.$$

Moreover, if X is bounded implicative, then $(A]=\{x\in X:\exists a_1,\cdots,a_n\in A$ such that $x\leq a_1\vee...\vee a_n\}$.

Theorem 2.6. [8] *Let X be a bounded BCK-algebra, P be a proper ideal of X and exactly one of x or Nx belongs to P. Then P is a prime ideal of X.*

Note: From now on, in this paper, we let X be a BCK-algebra.

3. Ultra ideals

In this section, in bounded BCK-algebras, we define the notion of ultra ideals and we show that in an special case (in bounded BCK-algebras) maximal ideals and ultra ideals are coincide, i.e., maximal ideals will be defined in other shape that we name ultra ideals. Also, we obtain relationship between ultra ideals and prime (positive implicative) ideals.

Definition 3.1. Let X be bounded. Then ideal A of X is called an *ultra* ideal of X if, $x \in A$ if and only if $Nx \notin A$, for any $x \in X$.

Example 3.2. Let $X = \{0, 1, 2, 3, 4\}$ and the operation "$*$" on X is defined as follows:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	2	0	2	0
3	3	3	3	0	0
4	4	4	3	2	0

Then $(X, *, 0)$ is a bounded BCK-algebra. $I = \{0, 1, 2\}$, $J = \{0, 1, 3\}$ and $K = \{0, 1\}$ are ideals of X. It is easy to show that I and J are ultra ideals of X. Since $N2 = 3 \notin K$ and $2 \notin K$, K is not an ultra ideal of X.

Theorem 3.3. *Let I be an ultra ideal of bounded BCK-algebra X, J be a proper ideal of X and $I \subseteq J$. Then J is an ultra ideal of X, too.*

Proof. Let $x \in J$. If $Nx \in J$, then $1 \in J$, which is a contradiction. Now, let $Nx \notin J$. If $x \notin J$, then $x \notin I$ and so $Nx \in I \subseteq J$, which is a contradiction. □

Theorem 3.4. *Every ultra ideal of a bounded commutative BCK-algebra X is a positive implicative ideal of X.*

Proof. Let J be an ultra ideal of X and $(z * y) * x$, $y * x \in J$, for $x, y, z \in X$. If $z * x \notin J$, then $N(z * x) \in J$. By $(BCK1)$ and $(BCK6)$, $((z * x) * (1 * x)) * (z * 1) = ((z * x) * (z * 1)) * (1 * x) = 0 \in J$. Since $z * 1 = 0 \in J$, $(z * x) * (1 * x) \in J$. On the other hand, by Theorem 2.2(ii), $x * N(z * x) = (z * x) * Nx = (z * x) * (1 * x) \in J$. Since J is an ideal of X and $N(z * x) \in J$, $x \in J$. Now, since $y * x$ and $x \in J$, $y \in J$. On the other hand, $(z * x) * y = (z * y) * x \in J$

and since J is an ideal of X, $z * x \in J$, which is a contradiction. Therefore, J is a positive implicative ideal of X. $\square$

The converse of Theorem 3.4, is not correct, in general.

Example 3.5. By Theorem 2.3(i) and (ii), in any bounded implicative BCK-algebra X with $|X| \geq 3$, $I = \{0\}$ is a positive implicative ideal of X, but it is not an ultra ideal of X.

Definition 3.6. Let X be bounded. Then $A \subseteq X$ is said to have *the finite intersection* property if $((\cdots(1 * a_1) * a_2) * \cdots) * a_n > 0$, for any $a_1, a_2, \cdots, a_n \in A$ and $a_i \neq 1$, where $1 \leq i \leq n$. Moreover, $A \subseteq X$ is said to have *the finite union* property if $a_1 \vee a_2 \vee \cdots \vee a_n \neq 1$ for any $a_1, \cdots, a_n \in A$ and $a_i \neq 1$, where $1 \leq i \leq n$.

Theorem 3.7. *Let X be bounded, $A \subseteq X$ and $1 \notin A$. Then $(A]$ is a proper ideal of X if and only if A has the finite intersection property.*

Proof. ($\Rightarrow$) Let $(A]$ be a proper ideal of X and A has not the finite intersection property. Then there exist $a_1, \cdots, a_n \in A$ such that $((\cdots(1 * a_1) * a_2) * \cdots) * a_n = 0$ and so by Theorem 2.5, $1 \in (A]$. It means that $(A] = X$, which is a contradiction.
($\Leftarrow$) Let A has the finite intersection property and $(A] = X$. It follows that $1 \in (A]$ and so by Theorem 2.5, there exists $a_1, \cdots, a_n \in A$ such that $((\cdots(1 * a_1) * a_2) * \cdots) * a_n = 0$, which is a contradiction. $\square$

Theorem 3.8. *Let X be bounded implicative, $A \subseteq X$ and $1 \notin A$. Then $(A]$ is a proper ideal of X if and only if A has the finite union property.*

Proof. ($\Rightarrow$) Let $(A]$ be a proper ideal of X and A has not the finite union property. Then there exist $a_1, \cdots, a_n \in A$ such that $a_1 \vee a_2 \vee \cdots \vee a_n = 1$. By Theorem 2.5, $1 \in (A]$ and so for any $x \in X$, $x * 1 = 0 \in (A]$ implies that $x \in (A]$. It means that $(A] = X$, which is a contradiction.
($\Leftarrow$) Let A has the finite union property and $(A] = X$. It follows that $1 \in (A]$ and so by Theorem 2.5, there exist $a_1, \cdots, a_n \in A$ such that $a_1 \vee a_2 \vee \cdots \vee a_n \geq 1$, which is a contradiction.$\square$

Corollary 3.9. *Let X be bounded implicative, $A \subseteq X$ and $1 \notin A$. Then A has the finite intersection property if and only if A has the finite union property.*

Proof. By Theorems 3.7 and 3.8, the proof is clear.$\square$

Lemma 3.10. *Let X be bounded (implicative), $x \in X$ and A be an ideal of X such that A has the finite intersection property (finite union property). If $x \notin A$ and $Nx \notin A$, then $A \cup \{x\}$ has the finite intersection property (finite union property).*

Proof. Let X be bounded such that for $x \in X$, $x \notin A$ and $Nx \notin A$. We show that $B = A \cup \{x\}$ has the finite intersection property. Let $b_1, \cdots, b_n \in B$ and $b_i \neq 1$, for any $1 \leq i \leq n$. If $b_1, \cdots, b_n \in A$, then it is clear that $((\cdots(1 * b_1) * b_2) * \cdots) * b_n > 0$. Let W. O. L. G, $b_1 = x$. Then $((\cdots(1 * x) * b_2) * \cdots) * b_n > 0$. Because, if $((\cdots(1 * x) * b_2) * \cdots) * b_n = 0 \in A$, then $Nx = 1 * x \in A$, which is a contradiction. Hence, B has the finite intersection property. Now, let X be bounded implicative such that for $x \in X$, $x \notin A$ and $Nx \notin A$. We show that $b_1 \vee b_2 \cdots \vee b_n \neq 1$, for any $b_1, \cdots, b_n \in B$ and $b_i \neq 1$. If $b_1, \cdots, b_n \in A$, then the proof is clear. If W. O. L. G, $b_1 = x$, then $x \vee b_2 \cdots \vee b_n = 1$ and so $N(Nx \wedge N(b_2 \vee \cdots \vee b_n)) = 1$. Hence, $NN(Nx \wedge N(b_2 \vee \cdots \vee b_n)) = 0$. By Theorem 2.2 (iii), $(Nx \wedge N(b_2 \vee \cdots \vee b_n)) = 0$ and so

$$Nx * (Nx * (N(b_2 \vee \cdots \vee b_n)) = N(b_2 \vee \cdots \vee b_n) \wedge Nx = 0 \in A. \; (1)$$

On the other hand, by Theorem 2.2 (iii), $Nx * N(b_2 \vee \cdots \vee b_n) \leq (b_2 \vee \cdots \vee b_n) * x \leq b_2 \vee \cdots, b_n$. Hence, by Theorem 2.5, $Nx * N(b_2 \vee \cdots \vee b_n) \in (A] = A$ and so by (1), $Nx \in A$, which is a contradiction. Therefore, $A \cup \{x\}$ has the finite union property.$\square$

Theorem 3.11. *Let X be bounded and $A \subseteq X$. Then the following statements are equivalent: (i) A is an ultra ideal. (ii) A is a maximal ideal.*

Proof. $(i) \Rightarrow (ii)$ Let A be an ultra ideal. Since $0 \in A$, $1 = N0 \notin A$ and so A is a proper ideal of X. Now, let A is not maximal. Then there exists a proper ideal B of X such that $A \subset B$ and so there exists $x \in B$ such that $x \notin A$. By (i), $1 * x = Nx \in A \subset B$. Since B is an ideal of X and $x \in B$, $1 \in B$ and so $B = X$, which is a contradiction.
$(ii) \Rightarrow (i)$ Let A be a maximal ideal of X. If $x \in A$ and $Nx \in A$, for some $x \in X$, then $1 \in A$, which is a contradiction. Now, let there exists $x \in X$ such that $Nx \notin A$ and $x \notin A$. Consider $B = A \cup \{x\}$. Then by Lemma 3.10, B has the finite intersection property. Hence, by Theorem 3.7, $(B]$ is a proper ideal of X, which is a contradiction. Because, $A \subset (B] \subset X$ and A is a maximal ideal of X. Therefore, if $Nx \notin A$, then $x \in A$ and so A is an ultra ideal of X.$\square$

Example 3.12. Let $X = \{0, 1, 2, 3, 4\}$ and the operation "$*$" is defined as follows:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	1	0	1	1
3	3	3	3	0	3
4	4	4	4	4	0

Then $(X, *, 0)$ is a BCK-algebra and $I = \{0, 1, 2, 3\}$, $J = \{0, 1, 2, 3\}$ are maximal ideals of X. Since X is not bounded, we have not any ultra ideal in X.

Lemma 3.13. *Let X be bounded and $A \subseteq X$. If A has the finite intersection property, then there exists an ultra ideal B of X such that $A \subseteq B$.*

Proof. Let $E = \{B : A \subseteq B,\ \textit{where } B \textit{ is a proper ideal of } X\}$. Since A has the finite intersection property, by Theorem 3.7, $(A]$ is a proper ideal of X. Since $A \subseteq (A]$, $(A] \in E$ and so $E \neq \emptyset$. Let $F = \{B_i\}_{i\in\mathbb{N}}$ be a chain in E and $B_1 = \bigcup_{i\in\mathbb{N}} B_i$. Since B_1 is an upper bound of F in E and B_1 is an ideal of X, $B_1 \in E$. Hence, by Zorn's lemma, E has a maximal element B and so by Theorem 3.11, B is an ultra ideal of X such that $A \subseteq B$.$\square$

Theorem 3.14. *Any proper ideal in a bounded BCK-algebra contained at least one ultra ideal.*

Proof. Let X be a bounded BCK-algebra and A be a proper ideal of X. Since $A = (A]$, then by Theorem 3.7, A has the finite intersection property and so by Lemma 3.13, there exists an ultra ideal B of X such that $A \subseteq B$.$\square$

Theorem 3.15. *Let X be a bounded positive implicative lower BCK-semilattice and P be a prime ideal of X. Then P is an ultra ideal of X.*

Proof. Let P is not an ultra ideal of X. Then by Theorem 3.11, P is not a maximal ideal of X and so there exists a proper ideal F of X such that $P \subsetneq F$. It follows that there exists an element $a \in F$ such that $a \notin P$. Let b be an arbitrary element in X. Since X is positive implicative lower BCK-semilattice, $a \wedge (b * a) = (b * a) * ((b * a) * a) = (b * a) * (b * a) = 0 \in P$. Since P is a prime ideal and $a \notin P$, $b * a \in P \subseteq F$ and so $b \in F$. Hence, $X \subseteq F$, which is a contradiction. Therefore, P is an ultra ideal of X.$\square$

Example 3.16. Let $X = \{0, 1, 2, 3\}$ and the operation "$*$" is defined as follows:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	1	0	0
3	3	2	1	0

Then $(X, *, 0)$ is a bounded BCK-algebra. It is easy to see that $I = \{0\}$ is a prime ideal of X, but it is not an ultra ideal of X.

Corollary 3.17. *In a bounded positive implicative lower BCK-semilattice, the concepts of prime ideals and ultra ideals coincide.*

Proof. By Theorems 2.6 and 3.15, the proof is clear.□

4. More results of associative ideals

In this section, we obtain the relationship between associative ideals and commutative (positive implicative) ideals.

Definition 4.1. [3] Let X be a BCK-algebra. An ideal I of X is called an *associative* ideal of X if $(z * y) * x \in I$ and $y * x \in I$ imply that $z \in I$, for any $x, y, z \in X$.

Example 4.2. Let $X_1 = \{0, 1, 2, 3, 4\}$, $X_2 = \{0, 1, 2, 3\}$ and the operations "$*_1$", "$*_2$" are defined as follows:

$*_1$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	1	1
2	2	2	0	2	2
3	3	3	3	0	3
4	4	4	4	4	0

$*_2$	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	2	1	0

Then $(X_1, *_1, 0)$ and $(X_2, *_2, 0)$ are BCK-algebras. It is easy to show that all ideals of X_1 are associative. $J = \{0, 1\}$ is an ideal of X_2, but it is not an associative ideal of X_2. Because, $(3 *_2 2) *_2 2 = 1 \in J$ and $2 *_2 2 = 0 \in J$, but $3 \notin J$.

Theorem 4.3. *Let A be an ideal of X. Then the following statements are equivalent:*
(i) A is an associative ideal.
(ii) $(z * y) * x \in A$ implies that $z * (y * x) \in A$, for any $x, y, z \in X$.
(iii) $(y * x) * x \in A$ implies that $y \in A$, for any $x, y \in X$.

Proof. $(i) \Rightarrow (ii)$ Let A be an associative ideal of X and $(z*y)*x \in A$, for any $x, y, z \in X$. By $(BCK1)$ and $(BCK6)$,

$$\begin{aligned} ((z*(y*x))*(z*y))*x &= ((z*(y*x))*x)*(z*y) = ((z*x)*(y*x))*(z*y) \\ &= ((z*x)*(z*y))*(y*x) = 0 \in A \end{aligned}$$

Since $(z*y)*x \in A$, $z*(y*x) \in A$.
$(ii) \Rightarrow (iii)$ Let $(y*x)*x \in A$. Then by (ii), $y = y*0 = y*(x*x) \in A$.
$(iii) \Rightarrow (ii)$ Let $(z*y)*x \in A$. Then by $(BCK1)$ and $(BCK6)$,

$$((z*x)*(y*x))*(z*y) = ((z*x)*(z*y))*(y*x) = 0.$$

It means that $((z*x)*(y*x)) \leq (z*y)$ and so by $(BCK8)$, $((z*x)*(y*x))*x \leq (z*y)*x$. Since $(z*y)*x \in A$, by $(BCK6)$, $(((z*(y*x))*x)*x) = ((z*x)*(y*x))*x \in A$. Hence, by (iii), $z*(y*x) \in A$.
$(ii) \Rightarrow (i)$ Let $(z*y)*x,\ y*x \in A$. Since $z*(y*x) \in A$ and A is an ideal of A, $z \in A$, for any $x, y, z \in X$. □

Theorem 4.4. *Every associative ideal of X is a positive implicative ideal of X.*

Proof. Let I be an ideal of X and $(x*y)*z,\ y*z \in I$, for any $x, y, z \in X$. By Theorem 4.3 (ii), since $(x*y)*z \in I$, $x*(y*z) \in I$ and so $x \in I$. On the other hand, since $x*z \leq x \in I$, $x*z \in I$. Hence, I is a positive implicative ideal of X. □

The converse of Theorem 4.4, is not correct, in general.

Example 4.5. Let $X = \{0, 1, 2, 3\}$ and operation "$*$" is defined on X as follows:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	2	0	0
3	3	3	3	0

Then $(X, *, 0)$ is a positive implicative BCK-algebra and $I = \{0, 1\}$ is an ideal of X. By Theorem 2.3(ii), I is a positive implicative ideal of X, but I is not an associative ideal of X. Because, $(2*1)*2 = 0 \in I$ and $1*2 = 0 \in I$, but $2 \notin I$.

Theorem 4.6. *Every associative ideal of X is a commutative ideal of X.*

Proof. Let I be an associative ideal of X and $x*y \in I$, for any $x, y \in X$. Since $x*(y*(y*x)) \leq x$, by $(BCK8)$, $(x*(y*(y*x)))*y \leq x*y \in I$ and so $(x*(y*(y*x)))*y \in I$. Hence, by Theorem 4.3 (ii), $x*((y*(y*x)))*y) \in I$. On the other hand, $x = x*0 = x*((y*y)*(y*x)) = x*((y*(y*x))*y) \in I$. Since $x*(y*(y*x)) \leq x \in I$, $x*(y*(y*x)) \in I$ and so by Theorem 2.3(iv), I is a commutative ideal of X. □

The converse of Theorem 4.6, is not correct, in general.

Example 4.7. In Example 4.2, by Theorem 2.3(iii), since X_2 is commutative, all ideals of X_2 are commutative and so J is a commutative ideal of X_2, where J is not an associative ideal of X_2.

5. Involution ideals

We know that the commutative BCK-algebras have interesting properties. In this section, we investigate some results in commutative BCK-algebras. We define the notion of involution ideal and obtain a new different representation of $(A]$, where $A \subseteq X$. Also we state the relationship between involution ideals and positive implicative (implicative) ideals.

Definition 5.1. Let X be commutative and $a \in X$. We say that a is an *atom* of X, if there exist no $z \in X$ such that $0 < z < a$.

Proposition 5.2. *Let X be commutative, $a \in X$ and $F_a = \{x : x \wedge a = 0\}$. Then*
(i) F_a is an ideal of X,
(ii) $F_a \cap (a] = \{0\}$,
(iii) if a is an atom of X, then F_a is a prime ideal of X.

Proof. (i) $0 \in F_a$ is trivial. Let $x*y, y \in F_a$. Then $a*(a*(x*y)) = x*y \wedge a = 0$ and so $a \leq a*(x*y)$. On the other hand, $a*(x*y) \leq a$. Hence, $a*(x*y) = a$. Similarly, $y \wedge a = 0$ implies that $a*y = a$. By $(BCK1)$, we have $((a*y)*(x*y))*(a*x) = ((a*y)*(a*x))*(x*y) = 0$. It results that $((a*y)*(x*y)) \leq (a*x)$. Now, we have $a = a*(x*y) = (a*y)*(x*y) \leq (a*x)$ and so $x \wedge a = a*(a*x) = 0$. Therefore, $x \in F_a$.
(ii) Let $x \in F_a \cap (a]$. Since $x \in F_a$, $x*(x*a) = a \wedge x = x \wedge a = 0$ and so $x \leq x*a$. On the other hand, since $x \in (a]$, $x*a = 0$ and so $x = 0$. Therefore, $F_a \cap (a] = \{0\}$.
(iii) Let $x \wedge y \in F_a$ and $x \notin F_a$, for $x, y \in X$. Then $(x \wedge y) \wedge a = 0$. Since a is an atom, $a < x$ and so $0 = (x \wedge y) \wedge a = (x \wedge a) \wedge y = a \wedge y$. It results that $y \in F_a$. Therefore, P is a prime ideal of X. □

Notation: Let $A \subseteq X$. We define $A^* = \bigcap_{a \in A} F_a$. It is clear that A^* is an ideal of X.

Theorem 5.3. *Let X be commutative and $A, B \subseteq X$. Then*
(i) if $A \subseteq B$, then $B^* \subseteq A^*$,
(ii) $A \subseteq (A^*)^*$,
(iii) $A^* = ((A^*)^*)^*$,
(iv) $(A \cup B)^* = A^* \cap B^*$.

Proof. (i) Let $x \in B^*$. Then $x \wedge b = 0$, for any $b \in B$. Since $A \subseteq B$, for any $a \in A$, we have $a \in B$ and so $x \wedge a = 0$. Hence, $x \in A^*$.
(ii) Let $a \in A$. Then $a \wedge a^* = 0$, for any $a^* \in A^*$. It results that $a \in (A^*)^*$.
(iii) By (ii), $A^* \subseteq ((A^*)^*)^*$. On the other hand, by (i) and (ii), $A \subseteq (A^*)^*$ implies that $((A^*)^*)^* \subseteq A^*$. Then $A^* = ((A^*)^*)^*$
(iv) Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, by (i), $(A \cup B)^* \subseteq A^* \cap B^*$. On the other hand, if $x \in A^* \cap B^*$, then $x \wedge a = 0$, for any $a \in A \cup B$ and so $x \in (A \cup B)^*$. □

Definition 5.4. Let A be an ideal of X. Then A is called an *involution* ideal of X if $A = (A^*)^*$.

Example 5.5. Let $X = \{0, 1, 2, 3, 4\}$ and the operation "$*$" on X is defined as follows:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	1
2	2	1	0	1	2
3	3	1	1	0	3
4	4	4	4	4	0

Then $(X, *, 0)$ is a commutative BCK-algebra and $A = \{0, 1, 2, 3\}$ is an ideal of X. It is easy to show that $F_1 = F_2 = F_3 = \{0, 4\}$ and $F_0 = X$ and so $A^* = \{0, 4\}$. On the other hand, $F_4 = \{0, 1, 2, 3, \}$ and so $(A^*)^* = F_4 \cap F_0 = \{0, 1, 2, 3, \} = A$. Hence, A is an involution ideal of X.

Theorem 5.6. *Let X be commutative and $A \subseteq X$. Then*
(i) $A^* \cap (A] = \{0\}$.
(ii) If X is bounded implicative, then $A^* = (A]^*$.

Proof. (i) Let $x \in A^* \cap (A]$. Then there exist $a_1, \cdots, a_n \in A$ such that $(...(x * a_1) * ...) * a_n = 0$ and $x \wedge a = 0$, for any $a \in A$. Since $x \wedge a = 0$, $x * (x * a) = 0$ and so $x \leq x * a$, for any $a \in A$. On the other hand, $x * a \leq x$, for any $a \in A$. Therefore, $x * a = x$, for any $a \in A$ and so $x = (...(x * a_1) * ...) * a_n = 0$.
(ii) Since $A \subseteq (A]$, by Theorem 5.3 (i), $(A]^* \subseteq A^*$. Conversely, if $x \in A^*$, then $x \wedge a = 0$,

for any $a \in A$. Let $y \in (A]$. Then by Theorem 2.5, there exist $a_1, \cdots, a_n \in A$ such that $y \leq a_1 \vee \cdots \vee a_n$.
By $(BCK7)$, we have $x * y \geq x * (a_1 \vee \cdots \vee a_n)$. Now, by Theorem 2.2 (i),

$$x * (x * (a_1 \vee \cdots \vee a_n)) = x \wedge (a_1 \vee \cdots \vee a_n) = (x \wedge a_1) \vee \cdots (x \wedge a_n) = 0.$$

It means that $x * (a_1 \vee \cdots \vee a_n) \geq x$ and so $x * y \geq x$. Hence, $x * y = x$ and so $x \wedge y = 0$, for any $y \in (A]$. It results that $x \in (A]^*$ and so $A^* \subseteq (A]^*$. □

The converse of the above theorem (ii) may not be true.

Example 5.7. In Example 5.10, $A^* = (A]^*$, but X is not bounded implicative.

Theorem 5.8. *Let each ideal of X be an involution ideal. Then*
(i) $(A \cap B)^* = (A^* \cup B^*]$,
(ii) $(A] = (A^*)^*$,
(iii) $(A] = \{x \in X : x \wedge y = 0,\ for\ any\ y \in X\ that\ y \wedge a = 0, for\ any\ a \in A\}$, *for every* $A, B \subseteq X$.

Proof. (i) By Theorem 5.3 (i), $A^* \subseteq (A \cap B)^*$ and $B^* \subseteq (A \cap B)^*$ and so $A^* \cup B^* \subseteq (A \cap B)^*$. Since $(A \cap B)^*$ is an ideal of X, $(A^* \cup B^*] \subseteq ((A \cap B)^*] = (A \cap B)^*$. Conversely, since $A^* \subseteq (A^* \cup B^*) \subseteq (A^* \cup B^*]$ and $B^* \subseteq (A^* \cup B^*) \subseteq (A^* \cup B^*]$, $(A^* \cup B^*]^* \subseteq (A^*)^* = A$ and $(A^* \cup B^*]^* \subseteq (B^*)^* = B$. It results that $(A^* \cup B^*]^* \subseteq A \cap B$ and so $(A \cap B)^* \subseteq ((A^* \cup B^*]^*)^* = (A^* \cup B^*]$.
(ii) Since $A \subseteq (A]$, $(A]^* \subseteq A^*$ and so $(A^*)^* \subseteq ((A]^*)^* = (A]$. On the other hand, by Theorem 5.3 (ii), $A \subseteq (A^*)^*$ and so $(A] \subseteq ((A^*)^*] = (A^*)^*$. Because, $(A^*)^*$ is an ideal of X.
(iii) By (ii), the proof is clear. □

Theorem 5.9. *Let X be bounded implicative and A be an involution ideal of X. Then A is both positive implicative and implicative ideal of X.*

Proof. Let $(x * y) * z,\ y * z \in A$, where $x, y, z \in X$. Then $(x * y) * z \wedge a = y * z \wedge a = 0$, for any $a \in A^*$. By Theorems 2.2(iii) and 2.3(i),

$$(x*z)\wedge a = ((x*z)\wedge a)*0 = ((x*z)\wedge a)*((y*z)\wedge a) = ((x*z)*(y*z))\wedge a = ((x*y)*z)\wedge a = 0,$$

for any $a \in A^*$. Then $x * z \in A$ and so A is a positive implicative BCK-algebra. Now, we show that A is an implicative ideal of X. Let $x * (y * x) \in A$, for $x, y \in X$. Since $y * (y * x) \leq x$, by $(BCK8)$, $(y * (y * x)) * (y * x) \leq x * (y * x)$. Since A is an ideal of X, $(y * (y * x)) * (y * x) \in A$. Since $(y * x) * (y * x) = 0 \in A$, $y * (y * x) \in A$. On the other hand, by $(BCK7)$, $y * x \leq y$

implies that $x * y \leq x * (y * x)$ and so $x * y \in A$. Since $x * (x * y) = y * (y * x) \in A$, $x \in A$ and so by Theorem 2.3(v), A is an implicative ideal in X. $\square$

The converse of the above theorem may not be true.

Example 5.10. Let $X = \{0, 1, 2, 3, 4\}$ and the operation "$*$" on X is defined as follows:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	1	0	1	0
3	3	3	3	0	0
4	4	4	4	4	0

Then $(X, *, 0)$ is a BCK-algebra. It is easy to see that $A = \{0, 1, 2, 3\}$ is a positive implicative and implicative ideal of X, but X is not a bounded implicative BCK-algebra.

6. Conclusion

In any algebraic structure, the relationship between types of ideals with each other and preparing conditions for equating them can be interesting and useful. In this paper, we tried that verify this conditions for ideals in BCK-algebras. We investigated some conditions that we can have for example, when ultra and maximal ideals are coincide, prime ideals are ultra, associative ideals are positive implicative and so on .It seems that one can obtain nice conditions for finishing this goal.

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