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WHEN DOES THE COMPLEMENT OF THE ANNIHILATING-IDEAL GRAPH OF A COMMUTATIVE RING ADMIT A CUT VERTEX?

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Abstract. The rings considered in this article are commutative with identity which admit at least two nonzero annihilating ideals. Let R be a ring. Let $\mathbb{A}(R)$ denote the set of all annihilating ideals of R and let $\mathbb{A}(R)^* = \mathbb{A}(R)\backslash\{(0)\}$. The annihilating-ideal graph of R, denoted by $\mathbb{AG}(R)$ is an undirected simple graph whose vertex set is $\mathbb{A}(R)^*$ and distinct vertices I, J are joined by an edge in this graph if and only if $IJ = (0)$. The aim of this article is to classify rings R such that $(\mathbb{AG}(R))^c$ (that is, the complement of $\mathbb{AG}(R)$) is connected and admits a cut vertex.

1. Introduction

The rings considered in this article are nonzero commutative with identity and which are not integral domains. Inspired by the work of I. Beck in [8], several researchers have investigated the interplay between ring theoretic properies of a ring R with the graph theoretic properties

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of a graph associated with it. Let R be a ring. Recall from [4] that the *zero-divisor graph* of R, denoted by $\Gamma(R)$ is an undirected simple graph whose vertex set is the set of all nonzero zero-divisors of R, and distinct vertices x, y are joined by an edge in this graph if and only if $xy = 0$. We denote by $Z(R)$, the set of all zero-divisors of R and we denote $Z(R)\backslash\{0\}$ by $Z(R)^*$. Several algebraists have contributed to the area of zero-divisor graphs of commutative rings. For an excellent and inspiring survey of the results proved on zero-divisor graphs of commutative rings and for an extensive bibilography on the subject, the reader is referred to [3].

Let R be a ring. Recall from [9] that an ideal I of R is said to be an *annihilating ideal* if $Ir = (0)$ for some $r \in R\backslash\{0\}$. As in [9], we denote by $\mathbb{A}(R)$, the set of all annihilating ideals of R and we denote the set of all nonzero annihilating ideals of R by $\mathbb{A}(R)^*$. The concept of the *annihilating-ideal graph* of R, denoted by $\mathbb{AG}(R)$ was introduced by M. Behboodi and Z. Rakeei in [9]. Recall from [9] that $\mathbb{AG}(R)$ is an undirected simple graph whose vertex set is $\mathbb{A}(R)^*$ and distinct vertices I, J are joined by an edge in this graph if and only if $IJ = (0)$. Several inspiring and interesting theorems were proved in [9, 10] on $\mathbb{AG}(R)$. Several mathematicians have done research on the annihilating-ideal graph of a ring, to mention a few, refer [1, 2, 15, 17].

The graphs considered in this article are undirected. Let $G = (V, E)$ be a simple graph. Recall from [7, Definition 1.1.13] that the *complement of* G, denoted by G^c is a graph whose vertex set is V and two distinct vertices x, y are joined by an edge in G^c if and only if there is no edge joining x and y in G. In [18, 19], we investigated the interplay between the ring theoretic properties of a ring R and the graph theoretic properties of $(\Gamma(R))^c$. In [20], we studied $(\mathbb{AG}(R))^c$.

Let $G = (V, E)$ be a graph. A subgraph H of G is said to be an *induced subgraph* of G if each edge of G having its ends in $V(H)$ is also an edge of H. The induced subgraph of G with vertex set $S \subseteq V$ is called the *subgraph of G induced by S* and is denoted by $G[S]$ [7, Definition 1.2.1]. Let S be a proper subset of V. The subgraph $G[V\backslash S]$ is said to be obtained from G by the *deletion* of S. This subgraph is denoted by $G - S$. If $S = \{v\}$, then $G - S$ is simply denoted by $G - v$ [7, Definition 1.2.3].

Let $G = (V, E)$ be a connected graph. Recall from [7, Definition 3.1.1] that a subset S of V is said to be a *vertex cut* of G if $G - S$ is disconnected. A vertex v of G is said to be a *cut vertex* of G if $\{v\}$ is a vertex cut of G.

The authors of [6] studied regarding the cut vertices of the zero-divisor graphs of finite commutative rings with identity. The authors of [11] investigated the cut sets of zero-divisor graphs of finite commutative rings with identity. The authors of [17] explored the presence of cut vertices and cut sets in $\mathbb{AG}(R)$, where R is an Artinian ring. Moreover, the authors of [17] provide a classification of Artinian rings R such that $\mathbb{AG}(R)$ admits cut vertices (respectively, cut sets). In [21], we classified commutative rings R such that $Z(R)^*$ contains at least two elements, $(\Gamma(R))^c$ is connected and admits at least one cut vertex. Motivated by the above mentioned investigations, in this article, we consider commutative rings R such that $\mathbb{A}(R)^*$ contains at least two members and try to classify such rings R such that $(\mathbb{AG}(R))^c$ is connected and has at least one cut vertex.

It is useful to recall the following results from commutative ring theory that we make use in this article. Let I be a proper ideal of a ring R. Recall from [14] that a prime ideal $\mathfrak{p}$ of R is said to be a *maximal N-prime* of I, if $\mathfrak{p}$ is maximal with respect to the property of being contained in $Z_R(R/I) = \{r \in R | rx \in I$ for some $x \in R \backslash I\}$. Let $x \in Z(R)$. Let $S = R \backslash Z(R)$. Note that S is a multiplicatively closed subset of R and $Rx \cap S = \emptyset$. Hence, it follows from Zorn's lemma and [16, Theorem 1] that there exists a maximal N-prime $\mathfrak{p}$ of (0) in R that $x \in \mathfrak{p}$. Therefore, it follows that $Z(R) = \cup_{\alpha \in \Lambda} \mathfrak{p}_\alpha$, where $\{\mathfrak{p}_\alpha\}_{\alpha \in \Lambda}$ is the set of all maximal N-primes of (0) in R. Recall from [13] that prime ideal $\mathfrak{p}$ of R is said to be an *associated prime of I in the sense of Bourbaki* if $\mathfrak{p} = (I :_R x)$ for some $x \in R$. In this case, we simply say that $\mathfrak{p}$ is a B-prime of I.

Let R be a ring which has at least two nonzero annihilating ideals. It was shown in [20, Proposition 6.1] that $(\mathbb{AG}(R))^c$ is connected if and only if exactly one of the following holds.

(a) R has exactly one maximal N-prime $\mathfrak{p}$ of (0) and $\mathfrak{p}$ is not a B-prime of (0).

(b) R has exactly two maximal N-primes $\mathfrak{p}_1, \mathfrak{p}_2$ of (0) and $\mathfrak{p}_1 \cap \mathfrak{p}_2 \neq (0)$.

(c) R has more than two maximal N-primes of (0).

A ring R with only one maximal ideal is referred to as a *quasilocal* ring. A Noetherian quasilocal ring is referrred to as a *local* ring. Recall that a principal ideal ring R is said to be a *special principal ideal ring* (SPIR) if R has a unique prime ideal. If $\mathfrak{m}$ is the unique prime ideal of a SPIR R, then $\mathfrak{m}$ is necessarily nilpotent. Suppose that $\mathfrak{m} \neq (0)$. If $n \geq 2$ is the least integer with the property that $\mathfrak{m}^n = (0)$, then it follows from $(iii) \Rightarrow (i)$ of [5, Proposition 8.8] that $\{\mathfrak{m}, \ldots, \mathfrak{m}^{n-1}\}$ is the set of all nonzero proper ideals of R. If R is a SPIR with $\mathfrak{m}$ as its unique prime ideal, then we denote this using the notation $(R, \mathfrak{m})$ is a SPIR. A ring R is said to be *reduced* if R has no nonzero nilpotent element. Whenever a set A is a subset of a set B and $A \neq B$, we denote it using the notation $A \subset B$. The cardinality of a set A is denoted using the notation $|A|$. Let R be a ring with $|\mathbb{A}(R)^*| \geq 2$ such that $(\mathbb{AG}(R))^c$ is connected. In Section 2 of this article, we derive some necessary conditions on R in order that $(\mathbb{AG}(R))^c$ to admit at least one cut vertex. It is proved in Section 2 that if $(\mathbb{AG}(R))^c$ admits a cut vertex, then the number of maximal N-primes of (0) in R must be exactly two (See Lemmas 2.1 and 2.3.). The main result of this article is proved in Section 3. Let R be a ring with exactly two maximal N-primes of (0). Then $(\mathbb{AG}(R))^c$ admits a cut vertex if and only if either $R \cong F \times T$

as rings, where F is a field and $(T, Z(T))$ is a SPIR with $Z(T) \neq (0)$ but $(Z(T))^2 = (0)$ or $R \cong D \times T$ as rings, where D is an integral domain and T is quasilocal with $Z(T) \neq (0)$ as its unique maximal ideal and is a B-prime of (0) in T (See Lemma 3.1, Theorems 3.4 and 3.5.). We illustrate Theorems 3.4 and 3.5 with the help of some examples in Example 3.6.

2. Some preliminaries

Unless otherwise specified, the rings R considered in this article are such that $|\mathbb{A}(R)^*| \geq 2$ and $(\mathbb{AG}(R))^c$ is connected. In this section, we prove some results which provide necessary conditions in order that $(\mathbb{AG}(R))^c$ to admit a cut vertex.

Lemma 2.1. *Let R be a ring such that R has $\mathfrak{p}$ as its unique maximal N-prime of (0) (that is, equivalently $Z(R)$ is an ideal of R). If $|\mathbb{A}(R)^*| \geq 2$ and $(\mathbb{AG}(R))^c$ is connected, then for any finite nonempty subcollection $\mathcal{S}$ of $\mathbb{A}(R)^*$, $(\mathbb{AG}(R))^c - \mathcal{S}$ is connected. In particular, $(\mathbb{AG}(R))^c$ does not admit any cut vertex.*

Proof. We adapt an argument found in the proof of [21, Lemma 2.1]. We are assuming that $|\mathbb{A}(R)^*| \geq 2$ and $(\mathbb{AG}(R))^c$ is connected. Since R has $\mathfrak{p}$ as its unique maximal N-prime of (0), it follows from [20, Proposition 6.1(a)] that $\mathfrak{p}$ is not a B-prime of (0). Hence, it follows from [5, Proposiition 7.17] that R is not Noetherian. Therefore, we obtain from [9, Theorem 1.1] that $\mathbb{A}(R)^*$ is infinite. Since $\mathcal{S}$ is a finite collection, it follows that $\mathbb{A}(R)^* \backslash \mathcal{S}$ is infinite. We now prove that $(\mathbb{AG}(R))^c - \mathcal{S}$ is connected. Let $I, J \in \mathbb{A}(R)^* \backslash \mathcal{S}$ with $I \neq J$. We show that there exists a path in $(\mathbb{AG}(R))^c - \mathcal{S}$ between I and J. If $IJ \neq (0)$, then $I - J$ is a path in $(\mathbb{AG}(R))^c - \mathcal{S}$ between I and J. Suppose that $IJ = (0)$. Since $\mathfrak{p}$ is not a B-prime of (0) in R, it follows from [20, Lemma 2.1] that $\mathfrak{p} \not\subseteq ((0) :_R I) \cup ((0) :_R J)$. Hence, there exists $p_1 \in \mathfrak{p}$ such that $p_1 I \neq (0)$ and $p_1 J \neq (0)$. Let us denote Rp_1 by I_1. It is clear that $I_1 \in \mathbb{A}(R)^*$, $II_1 \neq (0)$, and $JI_1 \neq (0)$. If $I_1 \notin \mathcal{S}$, then $I - I_1 - J$ is a path in $(\mathbb{AG}(R))^c - \mathcal{S}$ between I and J. Hence, we may assume that $I_1 \in \mathcal{S}$. It follows from $II_1 \neq (0), JI_1 \neq (0)$, and $\mathfrak{p}$ is not a B-prime of (0) in R that $\mathfrak{p} \not\subseteq ((0) :_R II_1) \cup ((0) :_R JI_1)$. Therefore, there exists $p_2 \in \mathfrak{p}$ such that $p_2 II_1 \neq (0)$ and $p_2 JI_1 \neq (0)$. Let $I_2 = Rp_1p_2 = I_1p_2$. Observe that $I_2 \in \mathbb{A}(R)^*, II_2 \neq (0)$, and $JI_2 \neq (0)$. It follows from $Z(R) = \mathfrak{p}$, $1 - p \notin \mathfrak{p}$ for any $p \in \mathfrak{p}$ that $Rp_1 \neq Rp_1p_2$. That is, $I_1 \neq I_2$. If $I_2 \notin \mathcal{S}$, then $I - I_2 - J$ is a path in $(\mathbb{AG}(R))^c - \mathcal{S}$ between I and J. Suppose that $I_2 \in \mathcal{S}$. Using the fact that $\mathfrak{p}$ is not a B-prime of (0) in R, it follows that there exists $p_3 \in \mathfrak{p}$ such that $II_2p_3 \neq (0)$ and $JI_2p_3 \neq (0)$. Let $I_3 = Rp_1p_2p_3$. It is clear that $I_3 \notin \{I_1, I_2\}$ and $I - I_3 - J$ is a path in $(\mathbb{AG}(R))^c$. Since $\mathcal{S}$ is finite, it follows that in at most $|\mathcal{S}|$ steps, we obtain that there exists $p \in \mathfrak{p}$ such that $Rp \notin \mathcal{S}$ and $I - Rp - J$ is a path in $(\mathbb{AG}(R))^c - \mathcal{S}$ between I and J. This proves that $(\mathbb{AG}(R))^c - \mathcal{S}$ is connected. Let $A \in \mathbb{A}(R)^*$ and let $\mathcal{S} = \{A\}$. It follows that $(\mathbb{AG}(R))^c - A$ is connected. This shows that $(\mathbb{AG}(R))^c$ does not admit any cut vertex.$\square$

In Example 2.2, we illustrate Lemma 2.1.

Example 2.2. (i) Let $(V, \mathfrak{m})$ be a rank one valuation domain which is not discrete. Let $m \in \mathfrak{m}, m \neq 0$. Let $R = V/mV$ and $\mathfrak{p} = \mathfrak{m}/mV$. It is clear that $Z(R) = \mathfrak{p}$ and hence, $\mathfrak{p}$ is the unique maximal N-prime of the zero ideal in R. It was shown in [18, Example 3.1(ii)] that $\mathfrak{p}$ is not a B-prime of the zero ideal in R. It follows from [20, Proposition 6.1(a)] that $(\mathbb{AG}(R))^c$ is connected and moreover, we obtain from Lemma 2.1 that if $\mathcal{S}$ is any finite nonempty subcollection of $\mathbb{A}(R)^*$, then $(\mathbb{AG}(R))^c - \mathcal{S}$ is connected.

(ii) We mention an example given in [12, Example, p.16] to illustrate Lemma 2.1. Let $\{X_i\}_{i \in \mathbb{N}}$ be a set of indeterminates over a field K. Let $D = \cup_{n=1}^{\infty} K[[X_1, \ldots, X_n]]$. Let I be the ideal of D generated by $\{X_iX_j | i, j \in \mathbb{N}, i \neq j\}$. Let $R = D/I$. For each $i \in \mathbb{N}$, let us denote $X_i + I$ by x_i. Let $\mathfrak{m}$ be the ideal of R generated by $\{x_i | i \in \mathbb{N}\}$. It was noted in [12] that R is quasilocal with $\mathfrak{m}$ as its unique maximal ideal and R is reduced. It was verified in [18, Example 3.4(i)] that $Z(R) = \mathfrak{m}$. Thus R has $\mathfrak{m}$ as its unique maximal N-prime of its zero ideal. Since R is reduced, we obtain that $\mathfrak{m}$ is not a B-prime of the zero ideal in R. Therefore, as argued in (i) above, we obtain that $(\mathbb{AG}(R))^c$ is connected and moreover, for any finite nonempty subcollection $\mathcal{S}$ of $\mathbb{A}(R)^*$, $(\mathbb{AG}(R))^c - \mathcal{S}$ is connected. $\square$

Lemma 2.3. *Let R be a ring such that R admits at least n maximal N-primes of* (0) *with* $n \geq 3$. *If $\mathcal{S}$ is a finite nonempty subcollection of $\mathbb{A}(R)^*$ such that $|\mathcal{S}| \leq n-2$, then $(\mathbb{AG}(R))^c - \mathcal{S}$ is connected. In particular, $(\mathbb{AG}(R))^c$ does not admit any cut vertex.*

Proof. The proof of this lemma follows closely an argument given in [21, Lemma 4.1]. Let $\mathcal{P}$ denote the collection of all maximal N-primes of (0) in R. By hypothesis, $|\mathcal{P}| \geq n$ with $n \geq 3$. We know from [20, Proposition 6.1(c)] that $(\mathbb{AG}(R))^c$ is connected. Let $\mathcal{S}$ be a finite nonempty subcollection of $\mathbb{A}(R)^*$ such that $|\mathcal{S}| \leq n-2$. Let $\mathcal{S} = \{A_1, \ldots, A_k\}$. Note that $k \leq n-2$. We now show that $(\mathbb{AG}(R))^c - \mathcal{S}$ is connected. Observe that in order to show that $(\mathbb{AG}(R))^c - \mathcal{S}$ is connected, it is enough to prove the following: If $I - A - J$ is a path in $(\mathbb{AG}(R))^c$ with $I \notin \mathcal{S}$, $A \in \mathcal{S}$, $J \in \mathbb{A}(R)^*$ (J may be possibly in $\mathcal{S}$) with $IJ = (0)$, then there exists a path in $(\mathbb{AG}(R))^c$ between I and J whose vertices except possibly J are not in $\mathcal{S}$. Note that if $B \in \mathbb{A}(R)^*$, then $B \subseteq Z(R)$. Hence, $B \cap (R \backslash Z(R)) = \emptyset$. Therefore, it follows from Zorn's lemma and [16, Theorem 1] that there exists a maximal N-prime $\mathfrak{p}$ of (0) in R such that $B \subseteq \mathfrak{p}$. Note that there exists a subcollection $\mathcal{W} = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_t\}$ of $\mathcal{P}$ such that given $A_i \in \mathcal{S}$, then A_i is contained in at least one member of $\mathcal{W}$. It is clear that $t \leq k$ and so $t \leq n-2$. Since $|\mathcal{P}| \geq n$, it is possible to find at least two distinct members say $\mathfrak{p}_{t+1}, \mathfrak{p}_{t+2}$ such that $\mathfrak{p}_{t+1}, \mathfrak{p}_{t+2} \in \mathcal{P} \backslash \mathcal{W}$. We need to consider two cases.

Case 1: $\mathfrak{p}_j \not\subseteq ((0) :_R I) \cup ((0) :_R J)$ for some $j \in \{t+1, t+2\}$.

Without loss of generality, we can assume that $\mathfrak{p}_{t+1} \not\subseteq ((0):_R I) \cup ((0):_R J)$. It follows from prime avoidance lemma [16, Theorem 81] that there exists $x \in \mathfrak{p}_{t+1}$ such that $x \notin \cup_{i=1}^{t} \mathfrak{p}_i$, $Ix \neq (0)$, and $Jx \neq (0)$. Note that $Rx \in \mathbb{A}(R)^* \backslash \mathcal{S}$ and $I - Rx - J$ is a path in $(\mathbb{AG}(R))^c$.

Case 2: $\mathfrak{p}_j \subseteq ((0):_R I) \cup ((0):_R J)$ for each $j \in \{t+1, t+2\}$

Either $\mathfrak{p}_{t+1} \subseteq ((0):_R I)$ or $\mathfrak{p}_{t+1} \subseteq ((0):_R J)$. Without loss of generality, we can assume that $\mathfrak{p}_{t+1} \subseteq ((0):_R I)$. Since $((0):_R I) \subseteq Z(R)$ and $\mathfrak{p}_{t+1}$ is maximal with respect to the property of being contained in $Z(R)$, it follows that $\mathfrak{p}_{t+1} = ((0):_R I)$. Using a similar reasoing, we obtain that $\mathfrak{p}_{t+2} = ((0):_R J)$. It follows from prime avoidance lemma [16, Theorem 81] that there exist $x_1 \in \mathfrak{p}_{t+2} \backslash (\cup_{i=1}^{t+1} \mathfrak{p}_i)$ and $y_1 \in \mathfrak{p}_{t+1} \backslash (\mathfrak{p}_{t+2} \cup (\cup_{i=1}^{t} \mathfrak{p}_i))$. Let $I_1 = Rx_1$ and $I_2 = Ry_1$. It is clear from the choice of the elements x_1, y_1 that $I_1, I_2 \in \mathbb{A}(R)^* \backslash \mathcal{S}, II_1 \neq (0), JI_2 \neq (0)$, and $I_1 \neq I_2$. As $x_1, y_1 \notin \mathfrak{p}_1$, it follows that $x_1 y_1 \neq 0$ and so $I_1 I_2 \neq (0)$. We claim that $I_2 \notin \{I, J\}$. From $IJ = (0)$, whereas $I_2 J \neq (0)$, it follows that $I_2 \neq I$. It follows from $I_1 J = (0)$, whereas $I_1 I_2 \neq (0)$ that $I_2 \neq J$. If $I_1 \notin \{I, J\}$, then it is clear that $I - I_1 - I_2 - J$ is a path in $(\mathbb{AG}(R))^c$ between I and J with $I_1, I_2 \notin \mathcal{S}$. Suppose that $I_1 \in \{I, J\}$. As $IJ = (0)$, whereas $II_1 \neq (0)$, it follows that $I_1 = I$. In such a case, $I = I_1 - I_2 - J$ is a path in $(\mathbb{AG}(R))^c$ between I and J with $I_2 \notin \mathcal{S}$. This proves that $(\mathbb{AG}(R))^c - \mathcal{S}$ is connected. Let $A \in \mathbb{A}(R)^*$. With $\mathcal{S} = \{A\}$, it follows that $(\mathbb{AG}(R))^c - A$ is connected. Therefore, $(\mathbb{AG}(R))^c$ does not admit any cut vertex. $\square$

As an immediate consequence of Lemma 2.3, we have the following Corollary 2.4.

Corollary 2.4. *Let R be a ring which admits at least three maximal N-primes of (0). Let $\mathcal{P}$ be the collection of all maximal N-primes of (0) in R and let $\mathcal{S}$ be a finite nonempty subcollection of $\mathcal{P}$. If $(\mathbb{AG}(R))^c - \mathcal{S}$ is not connected, then $|\mathcal{P}| \leq |\mathcal{S}| + 1$.*

Proof. Let $|\mathcal{S}| = m$. If $|\mathcal{P}| \geq m + 2$, then it follows from Lemma 2.3 that $(\mathbb{AG}(R))^c - \mathcal{S}$ is connected. This is in contradiction to the hypothesis that $(\mathbb{AG}(R))^c - \mathcal{S}$ is not connected. Therefore, $|\mathcal{P}| \leq |\mathcal{S}| + 1$.

$\square$

In view of Lemmas 2.1 and 2.3, in the rest of this section we assume that R is a ring which has exactly two maximal N-primes of (0) such that $(\mathbb{AG}(R))^c$ is connected and try to find some necessary conditions in order that $(\mathbb{AG}(R))^c$ to admit at least one cut vertex.

Lemma 2.5. *Let R be a ring such that $\{\mathfrak{p}_1, \mathfrak{p}_2\}$ is the set of all maximal N-primes of (0) in R. Suppose that $(\mathbb{AG}(R))^c$ is connected. If $\mathcal{S}$ be a nonempty subcollection of $\mathbb{A}(R)^*$ such that $I \subseteq \mathfrak{p}_1 \cap \mathfrak{p}_2$ for each $I \in \mathcal{S}$, then $\mathcal{S}$ is not a vertex cut of $(\mathbb{AG}(R))^c$. In particular, if $I \in \mathbb{A}(R)^*$ is such that $I \subseteq \mathfrak{p}_1 \cap \mathfrak{p}_2$, then I is not a cut vertex of $(\mathbb{AG}(R))^c$.*

Proof. We know from [20, Proposition 6.1(b)] that $\mathfrak{p}_1 \cap \mathfrak{p}_2 \neq (0)$. As $\mathfrak{p}_1 \cap \mathfrak{p}_2 \subseteq \mathfrak{p}_1 \cup \mathfrak{p}_2 = Z(R)$, it follows that for any nonzero $x \in \mathfrak{p}_1 \cap \mathfrak{p}_2, Rx \in \mathbb{A}(R)^*$. Let $\mathcal{S}$ be as in the statement of this lemma. We now prove that $(\mathbb{AG}(R))^c - \mathcal{S}$ is connected. The proof of this assertion follows closely the argument given in [21, Lemma 3.2]. Let $A, B \in \mathbb{A}(R)^* \backslash \mathcal{S}$ with $A \neq B$. We prove that there exists a path in $(\mathbb{AG}(R))^c - \mathcal{S}$ between A and B. This is clear if $AB \neq (0)$. Suppose that $AB = (0)$. We know from the proof of [18, Proposition 1.7(i)] that there exist $a \in \mathfrak{p}_1 \backslash \mathfrak{p}_2, b \in \mathfrak{p}_2 \backslash \mathfrak{p}_1$ such that $ab \neq 0$. Since $Z(R) = \mathfrak{p}_1 \cup \mathfrak{p}_2$, it is clear that $a + b \notin Z(R)$. Note that for any nonzero ideal J of R, either $Ja \neq (0)$ or $Jb \neq (0)$. Observe that $Ra, Rb \in \mathbb{A}(R)^* \backslash \mathcal{S}$. If $Aa \neq (0)$ and $Ba \neq (0)$, then $A - Ra - B$ is a path in $(\mathbb{AG}(R))^c - \mathcal{S}$. Similarly, if $Ab \neq (0)$ and $Bb \neq (0)$, then $A - Rb - B$ is a path between A and B in $(\mathbb{AG}(R))^c - \mathcal{S}$. Suppose that $Aa \neq (0), Bb \neq (0)$, whereas $Ab = Ba = (0)$. In such a case, $A - Ra - Rb - B$ is a path in $(\mathbb{AG}(R))^c - \mathcal{S}$. This proves that $(\mathbb{AG}(R))^c - \mathcal{S}$ is connected. Therefore, $\mathcal{S}$ is not a vertex cut of $(\mathbb{AG}(R))^c$. Let $I \in \mathbb{A}(R)^*$ be such that $I \subseteq \mathfrak{p}_1 \cap \mathfrak{p}_2$. Then with $\mathcal{S} = \{I\}$, we obtain that $(\mathbb{AG}(R))^c - I$ is connected. Hence, I is not a cut vertex of $(\mathbb{AG}(R))^c$. □

Lemma 2.6. *Let $R, \mathfrak{p}_1, \mathfrak{p}_2$ be as in the statement of Lemma 2.5. Suppose that $(\mathbb{AG}(R))^c$ is connected. If $\mathfrak{p}_1$ is not a B-prime of (0) in R, then the following statements hold:*
(i) If $\mathcal{S}$ is any nonempty subcollection of $\mathbb{A}(R)^*$ such that $I \subseteq \mathfrak{p}_2$ for each $I \in \mathcal{S}$, then S is not a vertex cut of $(\mathbb{AG}(R))^c$.
(ii) Let $I \in \mathbb{A}(R)^*$, Then I is not a cut vertex of $(\mathbb{AG}(R))^c$.

Proof. (i) We prove that $(\mathbb{AG}(R))^c - \mathcal{S}$ is connected. Let $A, B \in \mathbb{A}(R)^* \backslash \mathcal{S}$ with $A \neq B$. We adapt an argument found in the proof of [21, Lemma 3.3(i)] to prove that there exists a path between A and B in $(\mathbb{AG}(R))^c - \mathcal{S}$. This is clear if $AB \neq (0)$. Hence, we may assume that $AB = (0)$. By hypothesis, $\mathfrak{p}_1$ is not a B-prime of (0) in R. Therefore, it follows from [20, Lemma 2.1] that $\mathfrak{p}_1 \not\subseteq ((0) :_R A) \cup ((0) :_R B)$. Moreover, it is clear that $\mathfrak{p}_1 \not\subseteq \mathfrak{p}_2$. Hence, it follows from the prime avoidance lemma [16, Theorem 81] that $\mathfrak{p}_1 \not\subseteq \mathfrak{p}_2 \cup ((0) :_R A) \cup ((0) :_R B)$. Thus there exists $x \in \mathfrak{p}_1$ such that $x \notin \mathfrak{p}_2$, $Ax \neq (0)$, and $Bx \neq (0)$. Let $J = Rx$. Since $J \not\subseteq \mathfrak{p}_2$, it follows that $J \notin \mathcal{S}$. Observe that $A - J - B$ is a path in $(\mathbb{AG}(R))^c - \mathcal{S}$. This proves that $(\mathbb{AG}(R))^c - \mathcal{S}$ is connected. Therefore, $\mathcal{S}$ is not a vertex cut of $(\mathbb{AG}(R))^c$.

(ii) Let $A, B \in \mathbb{A}(R)^* \backslash \{I\}$ be such that $A \neq B$. We show using an argument found in the proof of [21, Lemma 3.3(ii)] that there exists a path between A and B in $(\mathbb{AG}(R))^c - I$. This is clear if $AB \neq (0)$. Hence, we may assume that $AB = (0)$. Note that $I \subseteq Z(R) = \mathfrak{p}_1 \cup \mathfrak{p}_2$. So, either $I \subseteq \mathfrak{p}_1$ or $I \subseteq \mathfrak{p}_2$. If $I \subseteq \mathfrak{p}_2$, then we know from ($i$) that $(\mathbb{AG}(R))^c - I$ is connected. Therefore, we may assume that $I \subseteq \mathfrak{p}_1$ but $I \not\subseteq \mathfrak{p}_2$. We consider two cases.

Case(a): $\mathfrak{p}_2 \not\subseteq ((0) :_R A) \cup ((0) :_R B)$

In this case, there exists $x \in \mathfrak{p}_2$ such that $Ax \neq (0)$ and $Bx \neq (0)$. As $Rx \subseteq \mathfrak{p}_2$, whereas $I \not\subseteq \mathfrak{p}_2$, it is clear that $Rx \neq I$. Moreover, observe that $A - Rx - B$ is a path in $(\mathbb{AG}(R))^c - I$.

Case(b)**:** $\mathfrak{p}_2 \subseteq ((0) :_R A) \cup ((0) :_R B)$

In this case, either $\mathfrak{p}_2 \subseteq ((0) :_R A)$ or $\mathfrak{p}_2 \subseteq ((0) :_R B)$. Without loss of generality, we can assume that $\mathfrak{p}_2 \subseteq ((0) :_R A)$. As $((0) :_R A) \subseteq Z(R)$ and $\mathfrak{p}_2$ is maximal with respect to the property of being contained in $Z(R)$, we obtain that $\mathfrak{p}_2 = ((0) :_R A)$. By hypothesis, $\mathfrak{p}_1$ is not a B-prime of (0) in R. As $I \in \mathbb{A}(R)^*$, it follows that $\mathfrak{p}_1 \not\subseteq I \cup ((0) :_R B)$. Since $\mathfrak{p}_1 \not\subseteq \mathfrak{p}_2$, it follows from the prime avoidance lemma [16, Lemma 81] that $\mathfrak{p}_1 \not\subseteq \mathfrak{p}_2 \cup I \cup ((0) :_R B)$. Hence, there exists $x \in \mathfrak{p}_1$ such that $x \notin I$, $Ax \neq (0)$, and $Bx \neq (0)$. It is clear that $A - Rx - B$ is a path in $(\mathbb{AG}(R))^c - I$.

This proves that $(\mathbb{AG}(R))^c - I$ is connected. Therefore, I is not a cut vertex of $(\mathbb{AG}(R))^c$. □

Remark 2.7. *Let $R, \mathfrak{p}_1, \mathfrak{p}_2$ be as in the statement of Lemma 2.5. Suppose that $(\mathbb{AG}(R))^c$ is connected. It follows from Lemma 2.6(ii) that if $(\mathbb{AG}(R))^c$ has a cut vertex, then both $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are B-primes of (0) in R.* □

Lemma 2.8. *Let D be an integral domain and let $(T, \mathfrak{m})$ be a quasilocal ring which is not a field. Suppose that $\mathfrak{m} = Z(T)$ is a B-prime of (0) in T. Let $R = D \times T$. Then $(\mathbb{AG}(R))^c$ is connected and moreover, $(0) \times T$ is a cut vertex of $(\mathbb{AG}(R))^c$.*

Proof. Note that $\{\mathfrak{p}_1 = (0) \times T, \mathfrak{p}_2 = D \times \mathfrak{m}\}$ is the set of all maximal N-primes of $(0) \times (0)$ in R. Since $\mathfrak{p}_1 \cap \mathfrak{p}_2 = (0) \times \mathfrak{m} \neq (0) \times (0)$, we obtain from [20, Proposition 6.1(b)] that $(\mathbb{AG}(R))^c$ is connected. It is clear that $\mathbb{A}(R)^*$ contains at least four elements. By hypothesis, there exists $x \in T \backslash \{0\}$ such that $\mathfrak{m} = ((0) :_T x)$. We claim that $(0) \times Tx$ is a vertex of degree 1 in $(\mathbb{AG}(R))^c$. Note that $((0) \times Tx)((0) \times T) \neq (0) \times (0)$. Let $A \in \mathbb{A}(R)^*$ be such that A is adjacent to $(0) \times Tx$ in $(\mathbb{AG}(R))^c$. Observe that $A = I_1 \times I_2$ for some ideal I_1 of D and an ideal I_2 of T. From $(I_1 \times I_2)((0) \times Tx) \neq (0) \times (0)$, it follows that $I_2(Tx) \neq (0)$. Since $\mathfrak{m} = ((0) :_T x)$, we obtain that $I_2 = T$. As D is an integral domain and $A = I_1 \times T \in \mathbb{A}(R)^*$, it follows that $I_1 = (0)$. Hence, $A = (0) \times T$. This shows that $(0) \times T$ is the only vertex that is adjacent to $(0) \times Tx$ in $(\mathbb{AG}(R))^c$. Therefore, $(\mathbb{AG}(R))^c - A$ with $A = (0) \times T$ is disconnected. Hence, $(0) \times T$ is a cut vertex of $(\mathbb{AG}(R))^c$. □

We illustrate in Example 2.10 that Lemma 2.6(i) can fail to hold if $\mathfrak{p}_1$ is a B-prime of (0).

Lemma 2.9. *Let $(R_1, \mathfrak{m}_1)$ be a quasilocal ring which is not a field with $Z(R_1) = \mathfrak{m}_1$ is a B-prime of (0) in R_1 and let $(R_2, \mathfrak{m}_2)$ be a quasilocal ring with $Z(R_2) = \mathfrak{m}_2$. Let $R = R_1 \times R_2$.*

Then $(\mathbb{AG}(R))^c$ *is connected and moreover,* $\mathcal{S} = \{R_1 \times J | J \in \mathbb{A}(R_2)\}$ *is a vertex cut of* $(\mathbb{AG}(R))^c$.

Proof. Note that R has exactly two maximal N-primes of $(0) \times (0)$ and they are given by $\mathfrak{p}_1 = \mathfrak{m}_1 \times R_2$ and $\mathfrak{p}_2 = R_1 \times \mathfrak{m}_2$. Since $\mathfrak{p}_1 \cap \mathfrak{p}_2 = \mathfrak{m}_1 \times \mathfrak{m}_2 \neq (0) \times (0)$, we obtain from [20, Proposition 6.1(b)] that $(\mathbb{AG}(R))^c$ is connected. We claim that $(\mathbb{AG}(R))^c - \mathcal{S}$ is not connected. By hypothesis, there exists $x \in \mathfrak{m}_1 \backslash \{0\}$ such that $\mathfrak{m}_1 = ((0) :_{R_1} x)$. Let $A = \mathfrak{p}_1$ and $B = R_1 x \times (0)$. It is clear that $A, B \in \mathbb{A}(R)^* \backslash \mathcal{S}$, $A \neq B$, and $AB = (0) \times (0)$. Let $A - I_1 - \cdots - I_k - B$ be any path in $(\mathbb{AG}(R))^c$. Note that $I_k = C \times D$ for some ideal C of R_1 and an ideal D of R_2. From $I_k B \neq (0) \times (0)$, it follows that $C(R_1 x) \neq (0)$. Therefore, we obtain that $C = R_1$. Hence, $D \in \mathbb{A}(R_2)$ and so $I_k \in \mathcal{S}$. This proves that $(\mathbb{AG}(R))^c - \mathcal{S}$ is not connected. Hence, $\mathcal{S}$ is a vertex cut of $(\mathbb{AG}(R))^c$. $\square$

Example 2.10. *Let* $R_1 = R_2 = \mathbb{Z}/4\mathbb{Z}$. *Let* $R = R_1 \times R_2$. *Then* $(\mathbb{AG}(R))^c$ *is connected and moreover,* $\mathcal{S} = \{R_1 \times (0 + 4\mathbb{Z}), R_1 \times 2\mathbb{Z}/4\mathbb{Z}\}$ *is a vertex cut of* $(\mathbb{AG}(R))^c$.

Proof. Observe that $R_1 = R_2$ is a finite local ring with $Z(R_1) = 2\mathbb{Z}/4\mathbb{Z}$ as its unique maximal ideal. Note that $\mathbb{A}(R_1) = \mathbb{A}(R_2) = \{(0 + 4\mathbb{Z}), 2\mathbb{Z}/4\mathbb{Z}\}$. Hence, we obtain from Lemma 2.9 that $(\mathbb{AG}(R))^c$ is connected and moreover, $\mathcal{S} = \{R_1 \times (0 + 4\mathbb{Z}), R_1 \times 2\mathbb{Z}/4\mathbb{Z}\}$ is a vertex cut of $(\mathbb{AG}(R))^c$. $\square$

3. Main Result

In this section, we try to classify the rings R such that $(\mathbb{AG}(R))^c$ admits at least one cut vertex.

Lemma 3.1. *Let R be a ring such that $(\mathbb{AG}(R))^c$ is connected. If $I \in \mathbb{A}(R)^*$ is a cut vertex of $(\mathbb{AG}(R))^c$, then the following hold:*
R has exactly two maximal N-primes $\mathfrak{p}_1, \mathfrak{p}_2$ of (0) such that $\mathfrak{p}_1 \cap \mathfrak{p}_2 \neq (0)$, both $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are B-primes of (0) in R, $I \in \{\mathfrak{p}_1, \mathfrak{p}_2\}$, and moreover, I is principal.

Proof. Assume that $(\mathbb{AG}(R))^c$ is connected and $I \in \mathbb{A}(R)^*$ is a cut vertex of $(\mathbb{AG}(R))^c$. It follows from Lemmas 2.1 and 2.3 that R has exactly two maximal N-primes of (0). Let $\{\mathfrak{p}_1, \mathfrak{p}_2\}$ denote the set of all maximal N-primes of (0) in R. It follows from [20, Proposition 6.1(b)] that $\mathfrak{p}_1 \cap \mathfrak{p}_2 \neq (0)$. Moreover, we obtain from Remark 2.7 that both $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are B-primes of (0) in R. Furthermore, we obtain from Lemma 2.5 that $I \not\subseteq \mathfrak{p}_1 \cap \mathfrak{p}_2$. Hence, I is contained in exactly one between $\mathfrak{p}_1$ and $\mathfrak{p}_2$. Without loss of generality, we can assume that $I \subseteq \mathfrak{p}_1$. Since I is a cut vertex of $(\mathbb{AG}(R))^c$, it ffollows from [7, Theorem 3.1.6] that there exist $A, B \in \mathbb{A}(R)^* \backslash \{I\}$ such that I is in every path between A and B in $(\mathbb{AG}(R))^c$. Let $A - \cdots - I_1 - I - I_2 - \cdots - B$ be any path of shortest length in $(\mathbb{AG}(R))^c$ between A and B. Note that $I_1 I_2 = (0)$. We claim that I is principal. From $II_1 \neq (0), II_2 \neq (0)$ we obtain that $I \not\subseteq ((0) :_R I_1) \cup ((0) :_R I_2)$. Hence, there exists $a \in I$ such that $I_1 a \neq (0)$ and $I_2 a \neq (0)$. If $I \neq Ra$, then we can replace I by Ra in the above path and thereby obtain a path between A and B in $(\mathbb{AG}(R))^c$ that does not pass through I. This is impossible. Therefore, $I = Ra$ is principal. Note that $\mathfrak{p}_1 I_1 \neq (0)$ and $\mathfrak{p}_1 I_2 \neq (0)$. If $I \neq \mathfrak{p}_1$, then we can replace I by $\mathfrak{p}_1$ in the above path and arrive at a path in $(\mathbb{AG}(R))^c$ between A and B that does not pass through I. This is a contradiction. Therefore, $I = \mathfrak{p}_1$ and moreover, I is principal. $\square$

Lemma 3.2. *Let R be a ring with exactly two maximal N-primes of (0) such that $(\mathbb{AG}(R))^c$ is connected. Let $\{\mathfrak{p}_1, \mathfrak{p}_2\}$ denote the set of all maximal N-primes of (0) in R. If $\mathfrak{p}_1$ is a cut vertex of $(\mathbb{AG}(R))^c$, then $\mathfrak{p}_2$ is a maximal ideal of R. Moreover, if $\mathfrak{p}_1 \neq \mathfrak{p}_1^2$, then $\mathfrak{p}_1^2\mathfrak{p}_2 = (0)$.*

Proof. We are assuming that $\mathfrak{p}_1$ is a cut vertex of $(\mathbb{AG}(R))^c$. Therefore, it follows from [7, Theorem 3.6.1] that there exist distinct $A, B \in \mathbb{A}(R)^* \backslash \mathfrak{p}_1$ such that any path in $(\mathbb{AG}(R))^c$ between A and B passes through $\mathfrak{p}_1$. Let $A - \cdots - I_1 - \mathfrak{p}_1 - I_2 - \cdots - B$ be any path of shortest length between A and B in $(\mathbb{AG}(R))^c$. Note that $I_1 I_2 = (0)$. It follows as in the proof of Lemma 3.1 that there exists $p \in \mathfrak{p}_1$ such that $I_1 p \neq (0), I_2 p \neq (0)$, and moreover, $\mathfrak{p}_1 = Rp$ for any $p \in \mathfrak{p}_1$ such that $I_1 p \neq (0)$ and $I_2 p \neq (0)$. We now verify that $\mathfrak{p}_2$ is a maximal ideal of R. Suppose that $\mathfrak{p}_2$ is not a maximal ideal of R. Let $\mathfrak{m}$ be a maximal ideal of R such that $\mathfrak{p}_2 \subset \mathfrak{m}$. Observe that $\mathfrak{m} \not\subseteq \mathfrak{p}_1 \cup \mathfrak{p}_2 = Z(R)$. Hence, there exists $m \in \mathfrak{m}$ such that $m \notin Z(R)$. Note that $pm \in \mathfrak{p}_1$ is such that $I_1 pm \neq (0)$ and $I_2 pm \neq (0)$. Therefore, $\mathfrak{p}_1 = Rp = Rpm$. This implies that $p = rpm$ for some $r \in R$. Hence, $p(1 - rm) = 0 \in \mathfrak{p}_2$. Since $p \notin \mathfrak{p}_2$, we obtain that $1 - rm \in \mathfrak{p}_2 \subset \mathfrak{m}$. Therefore, $1 = rm + 1 - rm \in \mathfrak{m}$. This is impossible and so $\mathfrak{p}_2$ is a maximal ideal of R. Hence, $\mathfrak{p}_1 + \mathfrak{p}_2 = R$. If $\mathfrak{p}_1 \neq \mathfrak{p}_1^2$, then we verify that $\mathfrak{p}_1^2\mathfrak{p}_2 = (0)$. We claim that either $I_1\mathfrak{p}_2 = (0)$ or $I_2\mathfrak{p}_2 = (0)$. Suppose that $I_1\mathfrak{p}_2 \neq (0)$ and $I_2\mathfrak{p}_2 \neq (0)$. Then as $\mathfrak{p}_2 \not\subseteq \mathfrak{p}_1$, we obtain from the prime avoidance lemma [16, Lemma 81] that $\mathfrak{p}_2 \not\subseteq \mathfrak{p}_1 \cup ((0) :_R I_1) \cup ((0) :_R I_2)$. Therefore, there exists $x \in \mathfrak{p}_2$ such that $x \notin \mathfrak{p}_1, I_1 x \neq (0)$, and $I_2 x \neq (0)$. Note that $Rx \in \mathbb{A}(R)^*$ is such that $Rx \neq \mathfrak{p}_1$. Hence, on replacing $\mathfrak{p}_1$ by Rx in the above path, we get a path in $(\mathbb{AG}(R))^c$ between A and B that does not pass through $\mathfrak{p}_1$. This is a contradiction. Thus either $I_1\mathfrak{p}_2 = (0)$ or $I_2\mathfrak{p}_2 = (0)$. Without loss of generality, we can assume that $I_1\mathfrak{p}_2 = (0)$. Since $\mathfrak{p}_1^2 + \mathfrak{p}_2 \not\subseteq \mathfrak{p}_1 \cup \mathfrak{p}_2 = Z(R)$ (indeed, $\mathfrak{p}_1^2 + \mathfrak{p}_2 = R$), $I_1\mathfrak{p}_2 = (0)$, we obtain that $I_1\mathfrak{p}_1^2 \neq (0)$. If $I_2\mathfrak{p}_1^2 \neq (0)$, then on replacing $\mathfrak{p}_1$ by $\mathfrak{p}_1^2$, in the above path, we arrive at a path in $(\mathbb{AG}(R))^c$ between A and B that does not pass through $\mathfrak{p}_1$. This is impossible. Hence, $I_2\mathfrak{p}_1^2 = (0)$. Therefore, $I_2\mathfrak{p}_2 \neq (0)$. Suppose that $\mathfrak{p}_1^2\mathfrak{p}_2 \neq (0)$. It is clear that $\mathfrak{p}_1^2, \mathfrak{p}_2 \notin \{I_1, I_2\}$, and indeed, it can be easily verified that $\mathfrak{p}_1^2, \mathfrak{p}_2$ are not in the above path. Observe that $A - \cdots - I_1 - \mathfrak{p}_1^2 - \mathfrak{p}_2 - I_2 - \cdots - B$ is a path in $(\mathbb{AG}(R))^c$ between A and B that does not pass through $\mathfrak{p}_1$. This is a contradiction. Therefore, $\mathfrak{p}_1^2\mathfrak{p}_2 = (0)$. $\square$

Lemma 3.3. *Let $R, \mathfrak{p}_1, \mathfrak{p}_2$ be as in the statement of Lemma 3.2. Suppose that $(\mathbb{AG}(R))^c$ is connected. If $\mathfrak{p}_1$ is a cut vertex of $(\mathbb{AG}(R))^c$ with $\mathfrak{p}_1 \neq \mathfrak{p}_1^2$, then $R \cong F \times T$ as rings, where F is a field, $(T, Z(T))$ is a SPIR $Z(T) \neq (0)$ but $(Z(T))^2 = (0)$.*

Proof. We know from Lemma 3.1 that $\mathfrak{p}_1$ is principal. Also, we know from Lemma 3.2 that $\mathfrak{p}_2$ is a maximal ideal of R and $\mathfrak{p}_1^2\mathfrak{p}_2 = (0)$. Note that $\mathfrak{p}_1^2 + \mathfrak{p}_2 = R$. Therefore, we obtain from the Chinese remainder theorem [5, Proposition 1.10 (ii) and (iii)] that the mapping $f : R \to R/\mathfrak{p}_2 \times R/\mathfrak{p}_1^2$ given by $f(r) = (r + \mathfrak{p}_2, r + \mathfrak{p}_1^2)$ is an isomorphism of rings. Let $F = R/\mathfrak{p}_2$ and $T = R/\mathfrak{p}_1^2$. It is clear that F is a field, $Z(T) = \mathfrak{p}_1/\mathfrak{p}_1^2$ is a nonzero principal ideal of T with $(Z(T))^2 =$ the zero-ideal of T. This proves that $R \cong F \times T$ as rings. Observe that $(\mathbb{AG}(R))^c$ is isomorphic to $(\mathbb{AG}(F \times T))^c$. Since $\mathfrak{p}_1$ is mapped onto $F \times Z(T)$, under the isomorphism f, it follows that $F \times Z(T)$ is a cut vertex of $(\mathbb{AG}(F \times T))^c$. Therefore, there exist $A, B \in \mathbb{A}(F \times T)^*$ such that any path in $(\mathbb{AG}(F \times T))^c$ between A and B passes through $F \times Z(T)$. Let $A - \cdots - I_1 - F \times Z(T) - I_2 - \cdots - B$ be any path of shortest length in $(\mathbb{AG}(F \times T))^c$ between A and B. Note that $I_1I_2 = (0) \times (0)$. Let $I_1 = I_{11} \times I_{12}$ and $I_2 = I_{21} \times I_{22}$. Observe that either $I_{11} = (0)$ or $I_{21} = (0)$. Without loss of generality, we can assume that $I_{11} = (0)$. It follows from $(Z(T))^2 = (0)$ and $I_{12}Z(T) \neq (0)$ that $I_{12} \not\subseteq Z(T)$. From $I_{12}I_{22} = (0)$, it follows that $I_{22} = (0)$. Since $I_2 \neq (0) \times (0)$, we get that $I_{21} = F$. Note that $Z(T) = Tt$ for some $t \in T\backslash\{0\}$. Let $s \in T\backslash Z(T)$. If $Z(T) = Tt \neq T(ts)$, then on replacing $F \times Z(T)$ in the above path by $F \times T(ts)$, we arrive at a path in $(\mathbb{AG}(F \times T))^c$ between A and B that does not pass through $F \times Z(T)$. This is a contradiction. Therefore, for any $s \in T\backslash Z(T)$, $Tt = T(ts)$. This implies that $t(1 - sy) = 0$ for some $y \in T$. Hence, $1 - sy \in Z(T)$. and so $Z(T) + Ts = T$. This shows that $Z(T)$ is a maximal ideal of T. As $(Z(T))^2 = (0)$, it follows that $Z(T)$ is the only maximal ideal of T. It follows from $Z(T) = Tt$ with $t^2 = 0$ that $Z(T)$ is the only nonzero proper ideal of T. This proves that $(T, Z(T))$ is a SPIR with $Z(T) \neq (0)$ but $(Z(T))^2 = (0)$. □

Theorem 3.4. *Let $R, \mathfrak{p}_1, \mathfrak{p}_2$ be as in the statement of Lemma 3.2. Suppose that $(\mathbb{AG}(R))^c$ is connected and $\mathfrak{p}_1 \neq \mathfrak{p}_1^2$. Then the following statements are equivalent:*
(i) $\mathfrak{p}_1$ is a cut vertex of $(\mathbb{AG}(R))^c$.
(ii) $R \cong F \times T$ as rings, where F is a field and $(T, Z(T))$ is a SPIR with $Z(T) \neq (0)$ but $(Z(T))^2 = (0)$.
(iii) $(\mathbb{AG}(R))^c$ is a path of length 3. Both $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are cut vertices of $(\mathbb{AG}(R))^c$.

Proof. $(i) \Rightarrow (ii)$ This is an immediate consequence of Lemma 3.3.

$(ii) \Rightarrow (iii)$ Let $S = F \times T$. Note that the set of all nonzero proper ideals of S equals $\mathbb{A}(S)^* = \{(0) \times T, (0) \times Z(T), F \times (0), F \times Z(T)\}$. Observe that $(\mathbb{AG}(S))^c$ is the path $F \times (0) - F \times Z(T) - (0) \times T - (0) \times Z(T)$. Observe that $\{F \times Z(T), (0) \times T\}$ is the set of all maximal N-primes of $(0) \times (0)$ in S and both are cut vertices of $(\mathbb{AG}(S))^c$. As $R \cong S$ as rings, it follows that $(\mathbb{AG}(R))^c$ is a path of length 3 and moreover, both $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are cut vertices of $(\mathbb{AG}(R))^c$.

$(iii) \Rightarrow (i)$ This is clear. □

Theorem 3.5. *Let $R, \mathfrak{p}_1, \mathfrak{p}_2$ be as in the staement of Lemma 3.2. Suppose that $(\mathbb{AG}(R))^c$ is connected and $\mathfrak{p}_1 = \mathfrak{p}_1^2$. Then the following statements are equivalent:*

(i) $\mathfrak{p}_1$ is a cut vertex of $(\mathbb{AG}(R))^c$.

(ii) There exists a ring isomorphism f from R onto $D \times T$, where D is an integral domain, T is a quasilocal ring with $Z(T) \neq (0)$ as its unique maximal ideal, f maps $\mathfrak{p}_1$ onto $(0) \times T$ and moreover, $Z(T)$ is a B-prime of (0) in T.

Proof. $(i) \Rightarrow (ii)$ We know from Remark 2.7 that both $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are B-primes of (0) in R, and moreover, we know from Lemma 3.1 that $\mathfrak{p}_1$ is principal. Also, we know from Lemma 3.2 that $\mathfrak{p}_2$ is a maximal ideal of R. Since $\mathfrak{p}_1$ is principal, there exists $a \in \mathfrak{p}_1$ such that $\mathfrak{p}_1 = Ra$. By hypothesis, $\mathfrak{p}_1 = \mathfrak{p}_1^2$. Hence, $Ra = Ra^2$. Therefore, there exists an idempotent element $e \in \mathfrak{p}_1 \backslash \{0, 1\}$ such that $\mathfrak{p}_1 = Re$. Note that the mapping $f : R \to R(1-e) \times Re$ given by $f(r) = (r(1-e), re)$ is an isomorphism of rings. Observe that f maps $\mathfrak{p}_1$ onto $(0) \times Re$. Hence, we obtain that $R(1-e)$ is an integral domain. Moreover, f maps $\mathfrak{p}_2$ onto $R(1-e) \times Z(Re)$ and $\mathfrak{p}_2$ is a maximal ideal of R, it follows that $Z(Re)$ is a maximal ideal of Re. As $((0) \times Re) \cap (R(1-e) \times Z(Re)) \neq (0) \times (0)$, it follows that $Z(Re) \neq (0)$. It follows from $\mathfrak{p}_2$ is a B-prime of (0) in R that $Z(Re)$ is a B-prime of (0) in Re. Let us denote $R(1-e)$ by D and Re by T. Then it is clear that $R \cong D \times T$ as rings, D is an integral domain, $Z(T)$ is a nonzero maximal ideal of T and is a B-prime of (0) in T. As $\mathfrak{p}_1$ is mapped onto $(0) \times T$ under f, it follows that $(0) \times T$ is a cut vertex of $(\mathbb{AG}(D \times T))^c$. We next verify that T is quasilocal with $Z(T)$ as its unique maximal ideal. Since $(0) \times Z(T)$ is a cut vertex of $(\mathbb{AG}(D \times T))^c$, there exist distinct $A, B \in \mathbb{A}(D \times T)^*$ such that any path in $(\mathbb{AG}(D \times T))^c$ between A and B passes through $(0) \times T$. Let $A - \cdots I_1 - (0) \times Z(T) - I_2 - \cdots - B$ be any path of shortest length in $(\mathbb{AG}(D \times T))^c$ between A and B. Observe that $I_1 I_2 = (0) \times (0)$. Let $I_1 = I_{11} \times I_{12}$ and let $I_2 = I_{21} \times I_{22}$. It is clear that $I_{12} \neq (0)$ and $I_{22} \neq (0)$. If T admits a maximal ideal $\mathfrak{m}$ different from $Z(T)$, then $\mathfrak{m} \not\subseteq Z(T)$. Hence, $I_{12}\mathfrak{m} \neq (0)$ and $I_{22}\mathfrak{m} \neq (0)$. Therefore, on replacing $(0) \times Z(T)$ by $(0) \times \mathfrak{m}$ in the above path, we obtain a path in $(\mathbb{AG}(D \times T))^c$ between A and B that does not pass through $(0) \times Z(T)$. This is impossible. Therefore, T is quasilocal with $Z(T)$ as its unique maximal ideal.

$(ii) \Rightarrow (i)$ Observe that $(\mathbb{AG}(R))^c$ is isomorphic to $(\mathbb{AG}(D \times T))^c$. It follows from Lemma 2.8 that $(0) \times T$ is a cut vertex of $(\mathbb{AG}(D \times T))^c$. Since the isomorphism f maps $\mathfrak{p}_1$ onto $(0) \times T$, we obtain that $\mathfrak{p}_1$ is a cut vertex of $(\mathbb{AG}(R))^c$. □

We next illustrate Theorems 3.4 and 3.5 with the help of Example 3.6.

Example 3.6. (i) Let F be any field. Let $R = F \times \mathbb{Z}/4\mathbb{Z}$. Observe that R has $\{\mathfrak{p}_1 = F \times 2\mathbb{Z}/4\mathbb{Z}, \mathfrak{p}_2 = (0) \times \mathbb{Z}/4\mathbb{Z}\}$ as its set of all maximal N-primes of the zero ideal in R. It follows from $(ii) \Rightarrow (iii)$ of Theorem 3.4 that both $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are cut vertices of $(\mathbb{AG}(R))^c$.

(ii) Let $R = \mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$. Note that $\{\mathfrak{p}_1 = (0) \times \mathbb{Z}/4\mathbb{Z}, \mathfrak{p}_2 = \mathbb{Z} \times 2\mathbb{Z}/4\mathbb{Z}\}$ is the set of all maximal N-primes of the zero ideal in R. It follows from $(ii) \Rightarrow (i)$ of Theorem 3.5 that $\mathfrak{p}_1$ is a cut vertex of $(\mathbb{AG}(R))^c$. Since $\mathfrak{p}_1$ is not a maximal ideal of R, it follows from Lemma 3.2 that $\mathfrak{p}_2$ is not a cut vertex of $(\mathbb{AG}(R))^c$.

(iii) Let F be any field and let $R = F \times \mathbb{Z}/8\mathbb{Z}$. It is clear that $\{\mathfrak{p}_1 = F \times 2\mathbb{Z}/8\mathbb{Z}, \mathfrak{p}_2 = (0) \times \mathbb{Z}/8\mathbb{Z}\}$ is the set of all maximal N-primes of the zero ideal in R. Note that $\mathfrak{p}_1 \neq \mathfrak{p}_1^2$ and $\mathfrak{p}_1^2\mathfrak{p}_2$ is a nonzero ideal of R. Therefore, it follows from Lemma 3.2 that $\mathfrak{p}_1$ is not a cut vertex of $(\mathbb{AG}(R))^c$. However, we obtain from $(ii) \Rightarrow (i)$ of Theorem 3.5 that $\mathfrak{p}_2$ is a cut vertex of $(\mathbb{AG}(R))^c$.

(iv) Let $R = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$. Note that $\{\mathfrak{p}_1 = 2\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}, \mathfrak{p}_2 = \mathbb{Z}/4\mathbb{Z} \times 2\mathbb{Z}/4\mathbb{Z}\}$ is the set of all maximal N-primes of the zero ideal in R. Observe that $\mathfrak{p}_i \neq \mathfrak{p}_i^2$ for each $i \in \{1, 2\}$. Moreover, both $\mathfrak{p}_1^2\mathfrak{p}_2$ and $\mathfrak{p}_2^2\mathfrak{p}_1$ are nonzero ideals of R. Hence, we obtain from Lemma 3.2 that neither $\mathfrak{p}_1$ nor $\mathfrak{p}_2$ is a cut vertex of $(\mathbb{AG}(R))^c$. Indeed, we obtain from Lemma 3.1 that $(\mathbb{AG}(R))^c$ does not admit any cut vertex. Moreover, it is verified in Example 2.10 that $\{\mathbb{Z}/4\mathbb{Z} \times 4\mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/4\mathbb{Z} \times 2\mathbb{Z}/4\mathbb{Z}\}$ is a vertex cut of $(\mathbb{AG}(R))^c$. $\square$

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