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CHARACTERIZATION AND AXIOMATIZATION OF ALL SEMIGROUPS WHOSE SQUARE IS GROUP

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Abstract. In this paper we give a characterization for all semigroups whose square is a group. Moreover, we axiomatize such semigroups and study some relations between the class of these semigroups and Grouplikes, introduced by the author. Also, we observe that this paper characterizes and axiomatizes a class of Homogroups (semigroups containing an ideal subgroup). Finally, several equivalent conditions for a semigroup S with $S^2 \leq S$ (the square-group property) will be considered.

1. Introduction and Preliminaries

The term "Homogroup" was introduced by G. Thierrin on 1961 [10], but earlier it was studied by A. H. Clifford, and D. D. Miller under the title "Semigroups having zeroid elements" [1]. A semigroup S is called homogroup, if it contains an ideal subgroup or equivalently the set of its ziroid elements is nonempty , i.e.,

$$\{t \in S | \forall a \in S\ \exists x, y \in S \text{ such that } t = xa = ay\} \neq \emptyset.$$

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In fact the above set is the same ideal subgroup in the notion of homogroup. It is shown that the ideal subgroup is the least ideal and also a maximal subgroup of S and its identity is a central idempotent element of S. Many properties of homogroups can be seen in [1, 8, 2, 10, 9]. On the other hand, in [3], we have introduced and studied the algebraic structure "Grouplike" that is something between semigroup and group.

The first idea of grouplikes comes from b-parts, b-addition of real numbers, b-bounded groups (defined as follows) and their generalization, namely decomposer functions (see [6, 4]).
For every real number a denote by $[a]$ the largest integer not exceeding a and put $(a) = \{a\} = a - [a]$ (the decimal or fractional part of a). Now let b be a nonzero constant real number. For any real number a put

$$[a]_b = b[\frac{a}{b}] \ \ , \ \ (a)_b = b(\frac{a}{b}).$$

The notation $[a]_b$ is called *b-integer part of a*, and $(a)_b$ *b-decimal part of a*. Now for a fixed real number $b \neq 0$ and every $x, y \in \mathbb{R}$, we set

$$x +_b y = (x + y)_b.$$

It is shown that $(\mathbb{R}, +_b)$ is a semigroup and $\mathbb{R} +_b \mathbb{R}$ is its subgroup (that is referred as the reference b-bounded group).
According to [3], a semigroup Γ is a *grouplike* if and only if it enjoys the following additional properties:
(1) There exists $\varepsilon \in \Gamma$ such that

$$\varepsilon x = \varepsilon^2 x = x\varepsilon^2 = x\varepsilon \ : \ \ \forall x \in \Gamma,$$

(2) For every ε satisfying (1) and every $x \in \Gamma$, there exists $y \in \Gamma$ such that

$$xy = yx = \varepsilon^2.$$

Every $\varepsilon \in \Gamma$ satisfying the axioms (1) and (2) is called an *identity-like.*

The semigroup $(\mathbb{R}, +_b)$ is a grouplike (namely, real b-grouplike) with identity-likes $b\mathbb{Z}$. In 2013, Hooshmand and Rahimian studied regular semigroups that are grouplikes [7].
It has been shown that every grouplike contains a unique idempotent identity-like element e. Also, there are the following axioms for grouplikes that is very similar to the four groups axioms:
(i) Closure,
(ii) Associativity,
(iii) There exists a unique idempotent element $e \in \Gamma$ such that $ex = xe$, for all $x \in X$.
(iv) For every $x \in \Gamma$, there exists $y \in \Gamma$ such that $xy = yx = e$.

Finally, it is interesting to know that a semigroup S is grouplike if and only it is a homogroup containing a unique central idempotent e.

2. A fundamental theorem

Through this section, the main one, we will characterize semigroups whose square is a group. In order to do this, let us recall some notion and notations from [3]. A class group $\mathcal{G}$ is a group whose elements are nonempty disjoint sets. A map $\Psi : \cup\mathcal{G} \to \mathcal{G}$ is called a class function if $x \in \Psi(x)$, for every $x \in \cup\mathcal{G}$. Because of our assumption for $\mathcal{G}$, the surjective class function Ψ exists and is always unique. We use the notation $\Psi(x) = A_x$, when $A \in \mathcal{G}$ and $x \in A = \Psi(x)$. Note that if φ is a choice function from $\mathcal{G}$ to $\cup\mathcal{G}$ ($\varphi(A) \in A$), then it is injective and $\Psi = \Psi\varphi\Psi$ or equivalently $A_x = A_{\varphi(A_x)}$, for every $x \in \cup\mathcal{G}$.
The main result of this section, as well as this paper is the following theorem.

Theorem 2.1. *A binary system $(S, \cdot)$ is a semigroup with $S^2 \leq S$ if and only if there exist a class group $(\mathcal{G}, \circ)$ and a choice function $\varphi : \mathcal{G} \to \cup\mathcal{G}$ such that $(S, \cdot) = (\cup\mathcal{G}, \centerdot^{\varphi})$, where*

$$x \centerdot^{\varphi} y = \varphi(\Psi(x) \circ \Psi(y)) \;\; : \;\; \forall x, y \in \cup\mathcal{G},$$

and Ψ is the unique (surjective) class function.

Proof. If a semigroup $(S, \cdot)$ satisfies $S^2 \leq S$, then there exists an idempotent element e in S^2 such that

$$exy = xye = xy \; : \; \forall x, y \in S.$$

Putting $x = e$ and $y = e$ (separately) in the above identity jointly imply e is a central element of S. Now for every $x, y \in S$ define

$$x \sim_e y \;\Leftrightarrow\; ex = ey.$$

Also let $\overline{S}$ denotes the set of all equivalent classes induced by $\sim_e$. We define the binary operation $\circ$ in $\overline{S}$, by $\overline{x} \circ \overline{y} = \overline{xy}$. It is well-defined, for if $\overline{x_1} = \overline{y_1}$ and $\overline{x_2} = \overline{y_2}$, then $ex_1 = ex_2$ and $ey_1 = ey_2$ so $(ex_1)(ey_1) = (ex_2)(ey_2)$ thus $ex_1y_1 = ex_2y_2$ and so $\overline{x_1y_1} = \overline{x_2y_2}$. Since $\sim_e$ is a semigroup congruence and $\overline{S} = S/\sim_e$ is the quotient semigroup, then $(\overline{S}, \circ)$ is a semigroup. Also $\overline{e}$ is its identity and $\overline{(ex)^{-1}}$ is the inverse of $\overline{x}$ for a given $\overline{x} \in \overline{S}$ ($(ex)^{-1}$ is the inverse of ex in the group S^2). Therefore $(\overline{S}, \circ)$ is a class group. Putting $(\mathcal{G}, \circ) = (\overline{S}, \circ)$ and $\varphi(\overline{x}) = ex$ we have $\cup\mathcal{G} = S$, φ is a choice function from $\mathcal{G}$ to $\cup\mathcal{G}$ and $\centerdot^{\varphi} = \cdot$, because

$$x \centerdot^{\varphi} y = \varphi(\overline{x} \circ \overline{y}) = \varphi(\overline{xy}) = exy = x \cdot y,$$

for every $x, y \in \cup\mathcal{G} = S$.
Conversely, let $(\mathcal{G}, \circ)$ be a class group, E the identity element of $\mathcal{G}$ and put $(S, \cdot) := (\cup\mathcal{G}, \cdot^{\varphi})$ where

$$x \cdot^{\varphi} y = \varphi(\Psi(x) \circ \Psi(y)) = \varphi(A_x \circ A_y) \quad : \quad \forall x, y \in \cup\mathcal{G},$$

Ψ is the unique class function and φ is an arbitrary choice function from $\mathcal{G}$ to $\cup\mathcal{G}$. Then, $(\cup\mathcal{G}, \cdot^{\varphi})$ is a semigroup (in fact it is a class united grouplike, see [3]). Also, we have

$$e \cdot x \cdot y = \varphi(A_e \circ A_x \circ A_y) = \varphi(E \circ A_x \circ A_y) = \varphi(A_x \circ A_y) = x \cdot y,$$

for every $x, y \in S$. Therefore e is the unite element of S^2.
On the other hand if $x \in S$, then putting $y = \varphi(B)$, where B is the inverse element of $A = A_x$ in $\mathcal{G}$, then

$$x \cdot y = \varphi(A_x \circ B) = \varphi(E) = e = y \cdot x.$$

So, for every $x_1, x_2 \in S$ there exist $y_2, y_1 \in S$ such that

$$\begin{aligned}(y_2 \cdot y_1) \cdot (x_1 \cdot x_2) &= y_2 \cdot (y_1 \cdot x_1) \cdot x_2 = y_2 \cdot e \cdot x_2 = e \cdot (y_2 \cdot x_2) \\ &= e^2 = e = (x_1 \cdot x_2) \cdot (y_2 \cdot y_1).\end{aligned}$$

Therefore $S^2 \leq S$. $\square$

We say a semigroup S has *square-group property* if $S^2 \leq S$.

Example 2.2. Consider $\Gamma = \{e, a, \eta, \alpha\}$ and define a binary operation "$\cdot$" by the following multiplication table

$\cdot$	e	a	η	α
e	e	a	e	a
a	a	e	a	e
η	e	a	e	a
α	a	e	a	e

It is easy to see that $(\Gamma, \cdot)$ is a semigroup and $\Gamma^2 = \{e, a\} \approxeq \mathbb{Z}_2$.
As an infinite example, $(\mathbb{R}, +_b)$ is a semigroup with the square-group property.

If S has the square-group property, then S^2 is its largest subgroup, least ideal and it contains all its sub-monoids. In fact, every subset of the form S_1S_2, where $S_1, S_2 \subseteq S$, and all left and right lower periodic subsets of S (introduced and studied in [5]) are subsets of the ideal subgroup.

3. Axiomatization of semigroups with the square-group property and some equivalent conditions

In this section we first describe all semigroups with the square-group property ($S^2 \leq S$) by five axioms, then try to minimize them.

Theorem 3.1. *(Axiomatization of semigroups with the square-group property) A pair $(\Gamma, \cdot)$ is a semigroup with the property $\Gamma^2 \leq \Gamma$ if and only if satisfies the following axioms:*
(A_1) *Closure,*
(A_2) *Associativity,*
(A_3) *There exists $\varepsilon \in \Gamma$ such that*

$$\varepsilon^2 x = \varepsilon x = x\varepsilon = x\varepsilon^2 \ : \ \forall x \in \Gamma.$$

(A_4) *For every ε satisfying (A_3) and every $x \in \Gamma$, there exists $x' \in \Gamma$ such that*

$$xx' = x'x = \varepsilon^2.$$

(A_5) *For every ε satisfying (A_3) (and so (A_4)):*

$$\varepsilon xy = xy\varepsilon = xy \ : \ \forall x, y \in \Gamma.$$

Proof. The axioms (A_1), (A_2) are equivalent to the statement: $(\Gamma, \cdot)$ is semigroup and Γ^2 its sub-semigroup, and (A_3), (A_5) imply that ε^2 is the identity of Γ^2. Also $x_2' x_1'$ is the inverse of $x_1 x_2$ in Γ^2, by (A_4). Therefore the axioms follow $\Gamma^2 \leq \Gamma$.
Conversely, let $(\Gamma, \cdot)$ be a semigroup with the property $\Gamma^2 \leq \Gamma$. Then (A_1) and (A_2) hold and if e is the identity element of Γ^2, then it is an idempotent element of Γ such that $exy = xye = xy$, for every $x, y \in \Gamma$, and so e satisfies (A_3). Now let ε satisfies (A_3) and put $x' = \varepsilon^2 y$, where y is the inverse of εx in Γ^2. Then

$$xx' = x'x = \varepsilon^2(yx) = \varepsilon^2 e = \varepsilon^2.$$

Finally if ε satisfies (A_3), then for every $x, y \in \Gamma$

$$xy\varepsilon = \varepsilon xy = \varepsilon(exy) = (\varepsilon e)xy = exy = xy,$$

because if t is the inverse of ε^2 in Γ^2, then $e = t\varepsilon^2 = t\varepsilon^3 = \varepsilon e$. □

Note that in any semigroup S (even any alternative) only the axiom (A_3) is equivalent to the following statements

(i) S contains a central idempotent element (equivalently $Zt(S) := Z(S) \cap It(S) \neq \emptyset$, where $Z(S)$ [resp. $It(S) = E(S)$] is the center of S [resp. the set of all idempotent elements of S]).

$$(ii) \quad \exists \varepsilon \in S \text{ such that } \varepsilon^2 x = \varepsilon x = x\varepsilon \ : \ \forall x \in S.$$

$$(iii) \quad \exists \varepsilon \in S \text{ such that } \varepsilon x = x\varepsilon = x\varepsilon^2 \ : \ \forall x \in S.$$

Remark 3.2. We observe that the axioms (A_1) through (A_4) are the same grouplike axioms. Therefore, every semigroup with the square-group property is a type of grouplike. We call an element ε of a semigroup *neutral-like* [resp. *identity-like*] if it satisfies (A_3) [resp. (A_3) and (A_4)]. If ε is an identity-like of a grouplike, then (A_5) is entitled the *identity-like hypothesis* (the hypothesis (H_1)), in the paper [3], and we show that a grouplike satisfies (H_1) if and only if it is class united grouplike. Therefore, Theorem 2.1 , 3.1 imply that a semigroup Γ has the square-group property if and only if it is a grouplike with the hypothesis (H_1) if and only if it is a class united grouplike.

Therefore, we also have characterized and axiomatized all semigroups S with a zeroid element e such that $exy = xye = xy$ for all x, y (i.e. e is a bi-identity). For if δ is an idempotent of S with the peoperty, then $e\delta = e\delta^2 = \delta^2 = \delta$ and $e = \beta\delta = \delta\gamma$ for some $\beta, \gamma \in S$, $e\delta = \beta\delta^2 = \beta\delta = e$. Thus $e = \delta$ that means S is a unipotent semigroup (with the unique idempotent e) such that e is its zeroid and also a bi-identity. This fact implies S is a homogroup with the unique (central) idempotent e and the property (H_1). Therefore, S is a grouplike with the hypotehsis (H_1) and so it is a semigroup with the square-group property (by the above results).

Example 3.3. The semigroup $(\mathbb{R}, +_b)$ satisfies the axioms (A_1) till (A_5) and it is a class united grouplike.

Now we want to minimize the axioms for semigroups with the square-group property.

Theorem 3.4. *For a pair $(\Gamma, \cdot)$, the axioms (A_1) through (A_5) are equivalent to the following axioms:*

(A_1) *Closure,*

(A_2) *Associativity,*

(A'_3) *There exists $\varepsilon \in \Gamma$ such that*

: $\quad A'_3(i) \qquad \varepsilon xy = xy \ : \ \forall x, y \in \Gamma,$

: $\quad A'_3(ii) \qquad \forall x \in \Gamma \ \exists x' \in \Gamma$ *such that* $x'x = \varepsilon^2$.

Proof. If Γ satisfies the axioms, then the above conditions hold, clearly. Conversely, let Γ satisfies (A_1), (A_2) and (A'_3). Then Γ^2 is a sub-semigroup of Γ with a left identity ε^2 (because

$\varepsilon^2 \in \Gamma^2$ and $A'_3(i)$ implies $\varepsilon^2 xy = xy$, for every $x, y \in \Gamma$). Now if $x, y \in \Gamma$, then there exists $t \in \Gamma$ such that $t(xy) = \varepsilon^2$ and so $(\varepsilon t)(xy) = \varepsilon^3 = \varepsilon^2$. Therefore Γ^2 is a group and Theorem 3.1 completes this proof. $\square$

Theorems 3.1, 3.4 lead us to many equivalent axioms for semigroups having the square-group property.

Corollary 3.5. *For every semigroup $(\Gamma, \cdot)$ each of the (I), ... , (V) is equivalent to the axioms $(A_3), (A_4)$ and (A_5).*
(I) There exists $\varepsilon \in \Gamma$ such that
(1) $\varepsilon xy = xy$, for every $x, y \in \Gamma$.
(2) For every $x \in \Gamma$ there exists $x' \in \Gamma$ such that $x'x = \varepsilon^2$.

(II) There exists $\varepsilon \in \Gamma$ such that
(1) $xy\varepsilon = xy$, for every $x, y \in \Gamma$.
(2) For every $x \in \Gamma$ there exists $x' \in \Gamma$ such that $xx' = \varepsilon^2$.

(III) There exists an idempotent element $e \in \Gamma$ such that
(1) $exy = xy$, for every $x, y \in \Gamma$.
(2) For every $x \in \Gamma$ there exists $x' \in \Gamma$ such that $x'x = e$.

(IV) There exists an idempotent element $e \in \Gamma$ such that
(1) $xy = xye$, for every $x, y \in \Gamma$.
(2) For every $x \in \Gamma$ there exists $x' \in \Gamma$ such that $xx' = e$.

(V) There exists a central idempotent element e of Γ such that
(1) $exy = xy$, for every $x, y \in \Gamma$.
(2) For every $x \in \Gamma$ there exists $x' \in \Gamma$ such that $x'x = e$.

It is interesting to know that the following important properties hold, for every semigroup Γ with the square-group property.

Corollary 3.6. *Let Γ be a semigroup with $\Gamma^2 \leq \Gamma$. Then*
(a) e is the unique idempotent [central idempotent] element of Γ (where e is the idempotent element introduced in Corollary 3.5).
(b) Every left and right coset of Γ is its subgroup ($x\Gamma \leq \Gamma$ and $\Gamma x \leq \Gamma$, for every $x \in \Gamma$). In

fact every left and right coset of Γ *is equal to* Γ^2 *(*$x\Gamma = \Gamma^2 = \Gamma x$*, for every* $x \in \Gamma$*).*

(c) Γ *contains a left or right identity if and only if it is monoid if and only if* Γ *is group.*

(d) Γ *contains the zero element if and only if it is a null semigroup.*

(e) Γ *contains the least ideal which is also the largest subgroup.*

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