



CHARACTERIZATION AND AXIOMATIZATION OF ALL SEMIGROUPS WHOSE SQUARE IS GROUP

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ABSTRACT. In this paper we give a characterization for all semigroups whose square is a group. Moreover, we axiomatize such semigroups and study some relations between the class of these semigroups and Grouplikes, introduced by the author. Also, we observe that this paper characterizes and axiomatizes a class of Homogroups (semigroups containing an ideal subgroup). Finally, several equivalent conditions for a semigroup S with $S^2 \leq S$ (the square-group property) will be considered.

1. INTRODUCTION AND PRELIMINARIES

The term "Homogroup" was introduced by G. Thierrin on 1961 [10], but earlier it was studied by A. H. Clifford, and D. D. Miller under the title "Semigroups having zero elements" [1]. A semigroup S is called homogroup, if it contains an ideal subgroup or equivalently the set of its zero elements is nonempty, i.e.,

$$\{t \in S \mid \forall a \in S \exists x, y \in S \text{ such that } t = xa = ay\} \neq \emptyset.$$

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In fact the above set is the same ideal subgroup in the notion of homogroup. It is shown that the ideal subgroup is the least ideal and also a maximal subgroup of S and its identity is a central idempotent element of S . Many properties of homogroups can be seen in [1, 8, 2, 10, 9]. On the other hand, in [3], we have introduced and studied the algebraic structure "Grouplike" that is something between semigroup and group.

The first idea of grouplikes comes from b -parts, b -addition of real numbers, b -bounded groups (defined as follows) and their generalization, namely decomposer functions (see [6, 4]).

For every real number a denote by $[a]$ the largest integer not exceeding a and put $\{a\} = a - [a]$ (the decimal or fractional part of a). Now let b be a nonzero constant real number. For any real number a put

$$[a]_b = b\left[\frac{a}{b}\right], \quad (a)_b = b\left(\frac{a}{b}\right).$$

The notation $[a]_b$ is called b -integer part of a , and $(a)_b$ b -decimal part of a . Now for a fixed real number $b \neq 0$ and every $x, y \in \mathbb{R}$, we set

$$x +_b y = (x + y)_b.$$

It is shown that $(\mathbb{R}, +_b)$ is a semigroup and $\mathbb{R} +_b \mathbb{R}$ is its subgroup (that is referred as the reference b -bounded group).

According to [3], a semigroup Γ is a *grouplike* if and only if it enjoys the following additional properties:

(1) There exists $\varepsilon \in \Gamma$ such that

$$\varepsilon x = \varepsilon^2 x = x \varepsilon^2 = x \varepsilon : \quad \forall x \in \Gamma,$$

(2) For every ε satisfying (1) and every $x \in \Gamma$, there exists $y \in \Gamma$ such that

$$xy = yx = \varepsilon^2.$$

Every $\varepsilon \in \Gamma$ satisfying the axioms (1) and (2) is called an *identity-like*.

The semigroup $(\mathbb{R}, +_b)$ is a grouplike (namely, real b -grouplike) with identity-likes $b\mathbb{Z}$. In 2013, Hooshmand and Rahimian studied regular semigroups that are grouplikes [7].

It has been shown that every grouplike contains a unique idempotent identity-like element e . Also, there are the following axioms for grouplikes that is very similar to the four groups axioms:

(i) Closure,

(ii) Associativity,

(iii) There exists a unique idempotent element $e \in \Gamma$ such that $ex = xe$, for all $x \in X$.

(iv) For every $x \in \Gamma$, there exists $y \in \Gamma$ such that $xy = yx = e$.

Finally, it is interesting to know that a semigroup S is grouplike if and only it is a homogroup containing a unique central idempotent e .

2. A FUNDAMENTAL THEOREM

Through this section, the main one, we will characterize semigroups whose square is a group. In order to do this, let us recall some notion and notations from [3]. A class group \mathcal{G} is a group whose elements are nonempty disjoint sets. A map $\Psi : \cup\mathcal{G} \rightarrow \mathcal{G}$ is called a class function if $x \in \Psi(x)$, for every $x \in \cup\mathcal{G}$. Because of our assumption for \mathcal{G} , the surjective class function Ψ exists and is always unique. We use the notation $\Psi(x) = A_x$, when $A \in \mathcal{G}$ and $x \in A = \Psi(x)$. Note that if φ is a choice function from \mathcal{G} to $\cup\mathcal{G}$ ($\varphi(A) \in A$), then it is injective and $\Psi = \Psi\varphi\Psi$ or equivalently $A_x = A_{\varphi(A_x)}$, for every $x \in \cup\mathcal{G}$.

The main result of this section, as well as this paper is the following theorem.

Theorem 2.1. *A binary system (S, \cdot) is a semigroup with $S^2 \leq S$ if and only if there exist a class group (\mathcal{G}, \circ) and a choice function $\varphi : \mathcal{G} \rightarrow \cup\mathcal{G}$ such that $(S, \cdot) = (\cup\mathcal{G}, \cdot^\varphi)$, where*

$$x \cdot^\varphi y = \varphi(\Psi(x) \circ \Psi(y)) \quad : \quad \forall x, y \in \cup\mathcal{G},$$

and Ψ is the unique (surjective) class function.

Proof. If a semigroup (S, \cdot) satisfies $S^2 \leq S$, then there exists an idempotent element e in S^2 such that

$$exy = xye = xy \quad : \quad \forall x, y \in S.$$

Putting $x = e$ and $y = e$ (separately) in the above identity jointly imply e is a central element of S . Now for every $x, y \in S$ define

$$x \sim_e y \Leftrightarrow ex = ey.$$

Also let \bar{S} denotes the set of all equivalent classes induced by \sim_e . We define the binary operation \circ in \bar{S} , by $\bar{x} \circ \bar{y} = \overline{xy}$. It is well-defined, for if $\bar{x}_1 = \bar{y}_1$ and $\bar{x}_2 = \bar{y}_2$, then $ex_1 = ex_2$ and $ey_1 = ey_2$ so $(ex_1)(ey_1) = (ex_2)(ey_2)$ thus $ex_1y_1 = ex_2y_2$ and so $\overline{x_1y_1} = \overline{x_2y_2}$. Since \sim_e is a semigroup congruence and $\bar{S} = S / \sim_e$ is the quotient semigroup, then (\bar{S}, \circ) is a semigroup. Also \bar{e} is its identity and $\overline{(ex)^{-1}}$ is the inverse of \bar{x} for a given $\bar{x} \in \bar{S}$ ($(ex)^{-1}$ is the inverse of ex in the group S^2). Therefore (\bar{S}, \circ) is a class group. Putting $(\mathcal{G}, \circ) = (\bar{S}, \circ)$ and $\varphi(\bar{x}) = ex$ we have $\cup\mathcal{G} = S$, φ is a choice function from \mathcal{G} to $\cup\mathcal{G}$ and $\cdot^\varphi = \cdot$, because

$$x \cdot^\varphi y = \varphi(\bar{x} \circ \bar{y}) = \varphi(\overline{xy}) = exy = x \cdot y,$$

for every $x, y \in \cup \mathcal{G} = S$.

Conversely, let (\mathcal{G}, \circ) be a class group, E the identity element of \mathcal{G} and put $(S, \cdot) := (\cup \mathcal{G}, \cdot^\varphi)$ where

$$x \cdot^\varphi y = \varphi(\Psi(x) \circ \Psi(y)) = \varphi(A_x \circ A_y) \quad : \quad \forall x, y \in \cup \mathcal{G},$$

Ψ is the unique class function and φ is an arbitrary choice function from \mathcal{G} to $\cup \mathcal{G}$. Then, $(\cup \mathcal{G}, \cdot^\varphi)$ is a semigroup (in fact it is a class united grouplike, see [3]). Also, we have

$$e \cdot x \cdot y = \varphi(A_e \circ A_x \circ A_y) = \varphi(E \circ A_x \circ A_y) = \varphi(A_x \circ A_y) = x \cdot y,$$

for every $x, y \in S$. Therefore e is the unite element of S^2 .

On the other hand if $x \in S$, then putting $y = \varphi(B)$, where B is the inverse element of $A = A_x$ in \mathcal{G} , then

$$x \cdot y = \varphi(A_x \circ B) = \varphi(E) = e = y \cdot x.$$

So, for every $x_1, x_2 \in S$ there exist $y_2, y_1 \in S$ such that

$$\begin{aligned} (y_2 \cdot y_1) \cdot (x_1 \cdot x_2) &= y_2 \cdot (y_1 \cdot x_1) \cdot x_2 = y_2 \cdot e \cdot x_2 = e \cdot (y_2 \cdot x_2) \\ &= e^2 = e = (x_1 \cdot x_2) \cdot (y_2 \cdot y_1). \end{aligned}$$

Therefore $S^2 \leq S$. \square

We say a semigroup S has *square-group property* if $S^2 \leq S$.

Example 2.2. Consider $\Gamma = \{e, a, \eta, \alpha\}$ and define a binary operation "·" by the following multiplication table

·	e	a	η	α
e	e	a	e	a
a	a	e	a	e
η	e	a	e	a
α	a	e	a	e

It is easy to see that (Γ, \cdot) is a semigroup and $\Gamma^2 = \{e, a\} \cong \mathbb{Z}_2$.

As an infinite example, $(\mathbb{R}, +_b)$ is a semigroup with the square-group property.

If S has the square-group property, then S^2 is its largest subgroup, least ideal and it contains all its sub-monoids. In fact, every subset of the form $S_1 S_2$, where $S_1, S_2 \subseteq S$, and all left and right lower periodic subsets of S (introduced and studied in [5]) are subsets of the ideal subgroup.

3. AXIOMATIZATION OF SEMIGROUPS WITH THE SQUARE-GROUP PROPERTY AND SOME EQUIVALENT CONDITIONS

In this section we first describe all semigroups with the square-group property ($S^2 \leq S$) by five axioms, then try to minimize them.

Theorem 3.1. *(Axiomatization of semigroups with the square-group property) A pair (Γ, \cdot) is a semigroup with the property $\Gamma^2 \leq \Gamma$ if and only if satisfies the following axioms:*

(A₁) *Closure,*

(A₂) *Associativity,*

(A₃) *There exists $\varepsilon \in \Gamma$ such that*

$$\varepsilon^2 x = \varepsilon x = x \varepsilon = x \varepsilon^2 : \forall x \in \Gamma.$$

(A₄) *For every ε satisfying (A₃) and every $x \in \Gamma$, there exists $x' \in \Gamma$ such that*

$$xx' = x'x = \varepsilon^2.$$

(A₅) *For every ε satisfying (A₃) (and so (A₄)):*

$$\varepsilon xy = xy \varepsilon = xy : \forall x, y \in \Gamma.$$

Proof. The axioms (A₁), (A₂) are equivalent to the statement: (Γ, \cdot) is semigroup and Γ^2 its sub-semigroup, and (A₃), (A₅) imply that ε^2 is the identity of Γ^2 . Also $x'_2 x'_1$ is the inverse of $x_1 x_2$ in Γ^2 , by (A₄). Therefore the axioms follow $\Gamma^2 \leq \Gamma$.

Conversely, let (Γ, \cdot) be a semigroup with the property $\Gamma^2 \leq \Gamma$. Then (A₁) and (A₂) hold and if e is the identity element of Γ^2 , then it is an idempotent element of Γ such that $exy = xye = xy$, for every $x, y \in \Gamma$, and so e satisfies (A₃). Now let ε satisfies (A₃) and put $x' = \varepsilon^2 y$, where y is the inverse of εx in Γ^2 . Then

$$xx' = x'x = \varepsilon^2(yx) = \varepsilon^2 e = \varepsilon^2.$$

Finally if ε satisfies (A₃), then for every $x, y \in \Gamma$

$$xy \varepsilon = \varepsilon xy = \varepsilon(exy) = (\varepsilon e)xy = exy = xy,$$

because if t is the inverse of ε^2 in Γ^2 , then $e = t\varepsilon^2 = t\varepsilon^3 = \varepsilon e$. \square

Note that in any semigroup S (even any alternative) only the axiom (A₃) is equivalent to the following statements

(i) S contains a central idempotent element (equivalently $Zt(S) := Z(S) \cap It(S) \neq \emptyset$, where $Z(S)$ [resp. $It(S) = E(S)$] is the center of S [resp. the set of all idempotent elements of S]).

$$(ii) \quad \exists \varepsilon \in S \text{ such that } \varepsilon^2 x = \varepsilon x = x \varepsilon : \forall x \in S.$$

$$(iii) \quad \exists \varepsilon \in S \text{ such that } \varepsilon x = x \varepsilon = x \varepsilon^2 : \forall x \in S.$$

Remark 3.2. We observe that the axioms (A_1) through (A_4) are the same grouplike axioms. Therefore, every semigroup with the square-group property is a type of grouplike. We call an element ε of a semigroup *neutral-like* [resp. *identity-like*] if it satisfies (A_3) [resp. (A_3) and (A_4)]. If ε is an identity-like of a grouplike, then (A_5) is entitled the *identity-like hypothesis* (the hypothesis (H_1)), in the paper [3], and we show that a grouplike satisfies (H_1) if and only if it is class united grouplike. Therefore, Theorem 2.1 , 3.1 imply that a semigroup Γ has the square-group property if and only if it is a grouplike with the hypothesis (H_1) if and only if it is a class united grouplike.

Therefore, we also have characterized and axiomatized all semigroups S with a zero element e such that $exy = xye = xy$ for all x, y (i.e. e is a bi-identity). For if δ is an idempotent of S with the property, then $e\delta = e\delta^2 = \delta^2 = \delta$ and $e = \beta\delta = \delta\gamma$ for some $\beta, \gamma \in S$, $e\delta = \beta\delta^2 = \beta\delta = e$. Thus $e = \delta$ that means S is a unipotent semigroup (with the unique idempotent e) such that e is its zero and also a bi-identity. This fact implies S is a homogroup with the unique (central) idempotent e and the property (H_1) . Therefore, S is a grouplike with the hypothesis (H_1) and so it is a semigroup with the square-group property (by the above results).

Example 3.3. The semigroup $(\mathbb{R}, +_b)$ satisfies the axioms (A_1) till (A_5) and it is a class united grouplike.

Now we want to minimize the axioms for semigroups with the square-group property.

Theorem 3.4. For a pair (Γ, \cdot) , the axioms (A_1) through (A_5) are equivalent to the following axioms:

(A_1) Closure,

(A_2) Associativity,

(A'_3) There exists $\varepsilon \in \Gamma$ such that

$$: A'_3(i) \quad \varepsilon xy = xy : \forall x, y \in \Gamma,$$

$$: A'_3(ii) \quad \forall x \in \Gamma \exists x' \in \Gamma \text{ such that } x'x = \varepsilon^2.$$

Proof. If Γ satisfies the axioms, then the above conditions hold, clearly. Conversely, let Γ satisfies (A_1) , (A_2) and (A'_3) . Then Γ^2 is a sub-semigroup of Γ with a left identity ε^2 (because

$\varepsilon^2 \in \Gamma^2$ and $A'_3(i)$ implies $\varepsilon^2 xy = xy$, for every $x, y \in \Gamma$. Now if $x, y \in \Gamma$, then there exists $t \in \Gamma$ such that $t(xy) = \varepsilon^2$ and so $(\varepsilon t)(xy) = \varepsilon^3 = \varepsilon^2$. Therefore Γ^2 is a group and Theorem 3.1 completes this proof. \square

Theorems 3.1, 3.4 lead us to many equivalent axioms for semigroups having the square-group property.

Corollary 3.5. *For every semigroup (Γ, \cdot) each of the (I), ... , (V) is equivalent to the axioms $(A_3), (A_4)$ and (A_5) .*

(I) *There exists $\varepsilon \in \Gamma$ such that*

(1) $\varepsilon xy = xy$, for every $x, y \in \Gamma$.

(2) *For every $x \in \Gamma$ there exists $x' \in \Gamma$ such that $x'x = \varepsilon^2$.*

(II) *There exists $\varepsilon \in \Gamma$ such that*

(1) $xy\varepsilon = xy$, for every $x, y \in \Gamma$.

(2) *For every $x \in \Gamma$ there exists $x' \in \Gamma$ such that $xx' = \varepsilon^2$.*

(III) *There exists an idempotent element $e \in \Gamma$ such that*

(1) $exy = xy$, for every $x, y \in \Gamma$.

(2) *For every $x \in \Gamma$ there exists $x' \in \Gamma$ such that $x'x = e$.*

(IV) *There exists an idempotent element $e \in \Gamma$ such that*

(1) $xy = xye$, for every $x, y \in \Gamma$.

(2) *For every $x \in \Gamma$ there exists $x' \in \Gamma$ such that $xx' = e$.*

(V) *There exists a central idempotent element e of Γ such that*

(1) $exy = xy$, for every $x, y \in \Gamma$.

(2) *For every $x \in \Gamma$ there exists $x' \in \Gamma$ such that $x'x = e$.*

It is interesting to know that the following important properties hold, for every semigroup Γ with the square-group property.

Corollary 3.6. *Let Γ be a semigroup with $\Gamma^2 \leq \Gamma$. Then*

(a) *e is the unique idempotent [central idempotent] element of Γ (where e is the idempotent element introduced in Corollary 3.5).*

(b) *Every left and right coset of Γ is its subgroup ($x\Gamma \leq \Gamma$ and $\Gamma x \leq \Gamma$, for every $x \in \Gamma$). In*

fact every left and right coset of Γ is equal to Γ^2 ($x\Gamma = \Gamma^2 = \Gamma x$, for every $x \in \Gamma$).

(c) Γ contains a left or right identity if and only if it is monoid if and only if Γ is group.

(d) Γ contains the zero element if and only if it is a null semigroup.

(e) Γ contains the least ideal which is also the largest subgroup.

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