



Algebraic Structures and Their Applications Vol. 2 No. 1 (2015), pp 59-71.

SIMILARITY DH-ALGEBRAS

FEDERICO GABRIEL ALMIÑANA AND MATHIAS EXEQUIEL PELAYES

Communicated by B. Davvaz

ABSTRACT. In [7], B. Gerla and I. Leuştean introduced the notion of similarity on MV-algebra. A similarity MV-algebra is an MV-algebra endowed with a binary operation S that verifies certain additional properties. Also, Chirteş in [4], study the notion of similarity on Lukasiewicz-Moisil algebras. In particular, strong similarity Lukasiewicz-Moisil algebras were defined. In this paper we define and study the variety of similarity symmetric Heyting algebras (or similarity DH-algebras), i.e. symmetric Heyting algebras endowed with an operation of similarity S . These algebras are a generalization of strong similarity Lukasiewicz-Moisil algebras. In addition, we introduce a propositional calculus and prove this calculus has similarity DH-algebras as algebraic counterpart.

1. INTRODUCTION

Similarities are an extension of equivalence relations to a fuzzy context. Similarities have been used for a wide range of applications as for example in clustering, fuzzy control, fuzzy

MSC(2010): Primary: 06D99. Secondary: 03B99.

Keywords: symmetric Heyting algebras, similarity, S -filter.

Received: 29 Dec 2015, Accepted: 18 March 2016.

*Corresponding author

logic programming and in all the contexts in which there is the necessity of reasoning by analogy (see [2, 5, 9, 12, 6, 13]).

In [7], G. Brunella and I. Leuştean introduced the notion of similarity on MV-algebras. A similarity MV-algebra is a pair (A, S) where $A = (A, \oplus, *, 0, 1)$ is a MV-algebra and S is a binary operation which verifies the following properties:

- (1) $S(x, x) = 1$,
- (2) $S(x, y) = S(y, x)$,
- (3) $S(x, y) \odot S(y, z) \leq S(x, z)$,
- (4) $x \odot S(x, y) \leq y$,
- (5) $S(x \leftrightarrow y, 1) \leq S(x, z) \leftrightarrow S(y, z)$.

Also introduced the similarity Łukasiewicz logic and prove a completeness theorem. Taking into account the definition of similarity MV-algebra, Chirteş in [4], defined similarity on Łukasiewicz-Moisil algebras. In particular, she study strong similarity Łukasiewicz-Moisil algebras. Taking as a guide-line the case of similarity Łukasiewicz-Moisil algebras, in the present paper we define and study similarity symmetric Heyting algebras (or similarity DH-algebras), namely symmetric Heyting algebras endowed with a *similarity*. These algebras constitute a generalization of strong similarity Łukasiewicz-Moisil algebras (see [4, pag. 64]). In addition, we introduce a propositional calculus and prove this calculus has similarity DH-algebras as algebraic counterpart.

2. PRELIMINARIES

In this paper we take for granted the concepts and results on De Morgan algebras, Heyting algebras and symmetric Heyting algebras. To obtain more information on this topics, we direct the reader to the bibliography indicated in [1, 8]. However, in order to simplify reading, in this section we summarize the fundamental concepts we use.

Recall that an algebra $\mathcal{A} = \langle A, \vee, \wedge, \rightarrow, \sim, 0, 1 \rangle$ is a symmetric Heyting algebra (or DH-algebra) if $\langle A, \vee, \wedge, \sim, 0, 1 \rangle$ is a De Morgan algebra and $\langle A, \vee, \wedge, \rightarrow, 0, 1 \rangle$ is a Heyting algebra. For simplicity, we will denote these algebras by A . In the sequel we shall denote by **DH** the variety of all DH-algebras.

A filter F of a DH-algebra A that satisfies: $\neg \sim x \in F$, for every element $x \in F$, where $\neg x = x \rightarrow 0$, is called a kernel of A [8, 11]. Equivalently, a kernel is a filter F that satisfies the contraposition law: if $x \rightarrow y \in F$, then $\sim y \rightarrow \sim x \in F$.

A. Monteiro [8] proved that if, for a given kernel F in a DH-algebra A we define $x \sim_F y$ if and only if $x \rightarrow y \in F$ and $y \rightarrow x \in F$, then \sim_F is a congruence. Moreover, every congruence on A is determined by a kernel, and the mapping $F \mapsto \sim_F$ is an isomorphism from the lattice of kernels of A onto the lattice of congruences on A .

In what follows, we will denote $(x \rightarrow y) \wedge (y \rightarrow x)$ by $x \leftrightarrow y$.

If A is a DH-álgebra then the following properties hold for any $x, y, z \in A$:

- (D1) $x \leq y$ if and only if $x \rightarrow y = 1$,
- (D2) $x \rightarrow x = y \rightarrow y$,
- (D3) $(x \rightarrow y) \wedge y = y$,
- (D4) $x \wedge (x \rightarrow y) = x \wedge y$,
- (D5) $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$,
- (D6) $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$,
- (D7) if $x \vee y = 1$ implies $y \rightarrow x = x$ and $x \rightarrow y = y$,
- (D8) $1 \rightarrow x = x \leftrightarrow 1 = x$,
- (D9) $y \leq x \rightarrow y$,
- (D10) $(x \leftrightarrow y) \wedge (y \leftrightarrow z) \leq x \leftrightarrow z$,
- (D11) $x \leftrightarrow y \leq (x \wedge z) \leftrightarrow (y \wedge z)$,
- (D12) $x \leftrightarrow y \leq (x \leftrightarrow z) \leftrightarrow (y \leftrightarrow z)$.

A. Monteiro introduced in [8, pag.123] the subvariety **L** of **DH** of totally linear DH-algebra, that is, symmetric Heyting algebras satisfying the conditions

- (L) $(x \rightarrow y) \vee (y \rightarrow x) = 1$,
- (K) $x \wedge \sim x \leq y \vee \sim y$.

and prove that

- (D13) Every totally linear DH-algebra is a subdirect product of MV-chains (see [8, pag. 127]).

On the other hand, recall that an DH-algebra A satisfies the Stone condition if $\neg x \vee \neg \neg x = 1$. Moreover, a boolean element z is called strong if $\neg z = \sim z$. A filter $F \subseteq A$ is called strong filter iff for every $x \in F$ exists a strong boolean element $z \in F$ such that $z \leq x$. Taking into account these, in [8, pag.102, pag.120], the author prove that

- (D14) If A is an DH-algebra that satisfies the Stone condition, then every kernel is a strong filter.
- (D15) If A satisfies (L), then A verifying the Stone condition.
- (D16) If A satisfies (L) and (K), then the following conditions are equivalent:
 - (i) F is a maximal kernel,
 - (ii) F is a minimal prime filter.

3. SIMILARITY DH-ALGEBRAS

Definition 3.1. A similarity DH-algebra is a pair (A, S) where A is a DH-algebra and $S : A \times A \longrightarrow A$ is a binary operation on A such that the following properties hold for any $x, y, z \in A$:

- (S1) $S(x, x) = 1$,
- (S2) $S(x, y) = S(y, x)$,
- (S3) $S(x, y) \wedge S(y, z) \leq S(x, z)$,
- (S4) $S(x, y) \leq x \rightarrow y$,
- (S5) $S(x \leftrightarrow y, 1) \leq S(x, z) \leftrightarrow S(y, z)$.

The operation S which satisfies (S1)–(S5) will be called similarity operation on A (or, simply, similarity on A). If S and T are two similarities on A then we define

$$S \leq T \iff S(x, y) \leq T(x, y) \text{ for any } x, y \in A.$$

It is straightforward that the class of similarity DH-algebras is equational. The notions of subalgebra and homomorphism are defined as usual.

Remark 3.1. From (S5), (S1) and (D8) it follows that

$$S(x \leftrightarrow y, 1) \leq S(x, y) \leftrightarrow S(y, y) = S(x, y) \leftrightarrow 1 = S(x, y)$$

for every $x, y \in A$.

Example 3.1.

- (i) Let (A, S) be a strong similarity Lukasiewicz-Moisil algebra (see [4]). Then, A is a DH-algebra (see [1]). Thus, (A, S) is a similarity DH-algebra.
- (ii) On any DH-algebra A , the operation $E(x, y) := x \leftrightarrow y$ is a similarity.
- (iii) Let A be an DH-algebra and let $\Delta : A \times A \rightarrow A$ be defined by $\Delta(x, y) = \begin{cases} 1 & \text{si } x = y \\ 0 & \text{si } x \neq y \end{cases}$ for any $x, y \in A$. Then Δ is also a similarity operation on A .

Proposition 3.1. For any similarity S on A we have that:

- (a) $\Delta \leq S$
- (b) $S \leq E$

Proof. (a): Let $x, y \in A$. Then, $x = y$ or $x \neq y$. If $x = y$, by Example 3.1 we have that $\Delta(x, y) = 1$. On the other hand, applying (S1) we have that $S(x, y) = 1$, therefore $\Delta(x, y) \leq S(x, y)$. If $x \neq y$ then by Example 3.1 we obtain that $\Delta(x, y) = 0 \leq S(x, y)$.

(b): Let $x, y \in A$. From (S4) we have that $S(x, y) \leq x \rightarrow y$. On the other hand, by (S4) and (S2) we deduce that $S(x, y) \leq y \rightarrow x$ so we can ensure that $S(x, y) \leq (x \rightarrow y) \wedge (y \rightarrow x)$. So, by Example 3.1 $S(x, y) \leq E(x, y)$. \square

Proposition 3.2. For a similarity DH-algebra (A, S) , the following conditions are equivalent:

- (a) $S(x, y) = x \leftrightarrow y$ for any $x, y \in A$,
- (b) $S(x, 1) = x$ for any $x \in A$.

Proof. (a) \Rightarrow (b) Is a direct consequence of (D8). (b) \Rightarrow (a) By Proposition 3.1 (b), we only have to prove that $x \leftrightarrow y \leq S(x, y)$ for any $x, y \in A$. This easily follows by (S5), (S1) and (D8):

$$x \leftrightarrow y = S(x \leftrightarrow y, 1) \leq S(x, y) \leftrightarrow S(y, y) = S(x, y) \leftrightarrow 1 = S(x, y).$$

□

Definition 3.2. If (A, S) is a similarity DH-algebra, then $F \subseteq A$ is a S -filter if F is a kernel of A and $S(x, y) \in F$ whenever $x, y \in F$.

Proposition 3.3. Let (A, S) be a similarity DH-algebra and F a kernel of A . Then F is an S -filter iff $S(x, 1) \in F$ whenever $x \in F$.

Proof. (\Rightarrow). Because $1 \in F$ we have that $S(x, 1) \in F$ for every $x \in F$. (\Leftarrow). Let $x, y \in F$, then from (D9) we have that $x \leftrightarrow y \in F$. So, $S(x \leftrightarrow y, 1) \in F$. But, $S(x \leftrightarrow y, 1) \leq S(x, y)$ (by Remark 3.1), hence $S(x, y) \in F$. □

Proposition 3.4. If (A, S) is a similarity DH-algebra and $F \subseteq A$ is a S -filter, then \sim_F is a congruence with respect to the similarity DH-algebra (A, S) .

Proof. We only have to prove that \sim_F is compatible with S . Suppose that $x_1, x_2, y_1, y_2 \in A$ such that $x_1 \sim_F x_2$ and $y_1 \sim_F y_2$. It follows that $x_1 \leftrightarrow x_2 \in F$ and $y_1 \leftrightarrow y_2 \in F$. Hence $S(x_1 \leftrightarrow x_2, 1) \in F$ and $S(y_1 \leftrightarrow y_2, 1) \in F$. But

$$S(x_1 \leftrightarrow x_2, 1) \leq S(x_1, y_1) \leftrightarrow S(y_1, x_2)$$

and

$$S(y_1 \leftrightarrow y_2, 1) \leq S(y_1, x_2) \leftrightarrow S(x_2, y_2),$$

hence $S(x_1, y_1) \leftrightarrow S(y_1, x_2) \in F$ and $S(y_1, x_2) \leftrightarrow S(x_2, y_2) \in F$. We have that, $S(x_1, y_1) \sim_F S(y_1, x_2)$ and $S(y_1, x_2) \sim_F S(x_2, y_2)$, hence $S(x_1, y_1) \sim_F S(x_2, y_2)$. □

Remark 3.2. If (A, S) is a similarity DH-algebra and $F \subseteq A$ is a S -filter of A , then the quotient DH-algebra A / \sim_F has a canonical structure of similarity DH-algebra if we define $S_F : A / \sim_F \times A / \sim_F \rightarrow A / \sim_F$, by $S_F([x]_F, [y]_F) := [S(x, y)]_F$, for any $x, y \in A$, where $[x]_F$ is the congruence class of x with respect to \sim_F . Moreover, the canonical surjection $x \mapsto [x]_F$ is a similarity DH-algebra homomorphism.

Remark 3.3. In the rest of the paper we assume that (L) and (K) holds.

Definition 3.3. A similarity DH-algebra is called representable if it is a subdirect product of similarity DH-chains.

Lemma 3.2. Let A be a DH-algebra and $F \subseteq A$ a proper kernel. Then, for any $x \in F$ there exists $y \notin F$ such that $x \vee y = 1$.

Proof. Let F a proper kernel of A and let $x \in F$. Applying (D15) and (D14) we have that F is a strong filter. Then there exists a strong boolean element $z \in F$ such that $z \leq x$. On the other hand, since $\neg z = \sim z$ we obtain that $1 = z \vee \sim z \leq x \vee \sim z$. So, $x \vee \sim z = 1$ and $\sim z \notin F$. \square

Theorem 3.3. For a similarity DH-algebra (A, S) , the following conditions are equivalent:

- (a) (A, S) is representable,
- (b) $S(x \rightarrow y, 1) \vee (y \rightarrow x) = 1$ for any $x, y \in A$,
- (c) $x \vee y = 1$ implies $x \vee S(y, 1) = 1$ for any $x, y \in A$,
- (d) any minimal prime filter is an S -filter.

Proof. (a) \Rightarrow (b) : Because (A, S) is representable, we can consider $x \leq y$ or $y \leq x$. If $x \leq y$ then $x \rightarrow y = 1$, hence $S(x \rightarrow y, 1) \vee (y \rightarrow x) = 1$. If $y \leq x$ then $y \rightarrow x = 1$, hence $S(x \rightarrow y, 1) \vee (y \rightarrow x) = 1$.

(b) \Rightarrow (c) : From (D7) we have that $x \vee S(y, 1) = 1$.

(c) \Rightarrow (d) : Let $F \subseteq A$ be a minimal prime filter of A and suppose that $x \in F$. Then from Lemma 3.2, there exists $z \in A$ such that $z \vee x = 1$ and $z \notin F$. Using (c) it follows that $S(x, 1) \vee z = 1 \in F$, so $S(x, 1) \in F$ since $z \notin F$ and F is prime. Hence, $S(x, 1) \in F$ whenever $x \in F$ and by Proposition 3.3 we have that F is a S -filter.

(d) \Rightarrow (a) : If (A, S) is a similarity DH-algebra and let \mathcal{F} be the set of all the maximal kernel of A . It is well-known that A is a subdirectly product (as DH-algebras) of the family $\{A/\sim_F : F \in \mathcal{F}\}$ (see [8, pag.127]) and let $i : A \rightarrow \prod_{F \in \mathcal{F}} A/\sim_F$ be the corresponding representation. By hypothesis, any $F \in \mathcal{F}$ is an S -filter of (A, S) , so by Remark 3.2, $(A/\sim_F, S_F)$ is a similarity DH-algebra. It is straightforward that i is a representation of (A, S) as a subdirectly product of the family $\{(A/\sim_F, S_F) : F \in \mathcal{F}\}$. \square

Lemma 3.4. The similarity DH-algebra (A, E) in Example 3.1 is representable.

Proof. $E(x \rightarrow y, 1) \vee (y \rightarrow x) = ((x \rightarrow y) \leftrightarrow 1) \vee (y \rightarrow x) = (x \rightarrow y) \vee (y \rightarrow x) = 1$. \square

Lemma 3.5. For any DH-algebra A , the similarity DH-algebra (A, Δ) is representable iff A is a DH-chain.

Proof. (\Rightarrow) Let $x, y \in A$. If $x \not\leq y$ then $x \rightarrow y \neq 1$, so $\Delta(x \rightarrow y, 1) = 0$. But (A, Δ) is representable, hence, from Theorem 3.3, it follows that $\Delta(x \rightarrow y, 1) \vee (y \rightarrow x) = 1$, so $y \rightarrow x = 1$, hence $y \leq x$. Therefore, A is a chain. (\Leftarrow) Now, let A be a chain and $x, y \in A$. If $x \leq y$ then $x \rightarrow y = 1$ and if $y \leq x$ then $y \rightarrow x = 1$, hence in both cases, $\Delta(x \rightarrow y, 1) \vee (y \rightarrow x) = 1$. Therefore (A, Δ) is a representable similarity DH-algebra. \square

Remark 3.4. As a consequence of the previous lemma, there exists similarity DH-algebras which are not representable: (A, Δ) , where A is not a DH-chain.

Proposition 3.5. *If (A, S) is a representable similarity DH-algebra, then the following conditions are equivalent:*

- (i) $x \leq y$ implies $S(x, 1) \leq S(y, 1)$,
- (ii) $S(x \vee y, 1) = S(x, 1) \vee S(y, 1)$.

Proof. (i) \Rightarrow (ii). Without loss of generality we can suppose $x \leq y$. Then $S(x, 1) \leq S(y, 1)$, hence $S(x \vee y, 1) = S(y, 1) = S(x, 1) \vee S(y, 1)$. (ii) \Rightarrow (i) Obvious. \square

Definition 3.4. If A is an DH-algebra and S is a similarity on A , we will say that S is isotone if: $x \leq y$ implies $S(x, 1) \leq S(y, 1)$ for any $x, y \in A$

Open Problem 1. *Find an example of a similarity operation which is not isotone.*

4. THE LOGIC OF SIMILARITY DH-ALGEBRAS

In the sequel we shall denote by \mathcal{M} the modal symmetric propositional calculus (see [8]). We shall develop a propositional calculus, denoted \mathcal{SM} , which has similarity DH-algebras as model.

The language of \mathcal{SM} consists of:

- denumerable many propositional variables: v_1, \dots, v_n, \dots
- the logical connectives of \mathcal{M} : $\vee, \wedge, \rightarrow, \sim$,
- a binary logical connective: \Leftrightarrow ,
- parenthesis: (and).

The formulas of \mathcal{SM} are defined inductively as follows:

- (f1) every propositional variable is a formula,
- (f2) if φ is a formula then $\sim \varphi$ is a formula,
- (f3) if φ and ψ are formulas then $\varphi \Delta \psi$ is a formula, where $\Delta \in \{\vee, \wedge, \rightarrow, \Leftrightarrow\}$,

(f4) a string of symbols is a formula of \mathcal{SM} iff it can be shown to be a formula by a finite number of applications of (f1), (f2), (f3) and (f4).

We will denote by $Form_{\mathcal{SM}}$ the set of all formulas of \mathcal{SM} .

The axioms of \mathcal{SM} are defined as follows:

(I) any axiom of \mathcal{M} is an axiom of \mathcal{SM} , i.e a formula which has one of the following forms is an axiom (where φ, ψ and χ are arbitrary formulas):

- (M1) $\varphi \rightarrow (\psi \rightarrow \varphi)$,
- (M2) $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))$,
- (M3) $(\varphi \wedge \psi) \rightarrow \varphi$,
- (M4) $(\varphi \wedge \psi) \rightarrow \psi$,
- (M5) $((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi)) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$,
- (M6) $\varphi \rightarrow (\varphi \vee \psi)$,
- (M7) $\psi \rightarrow (\varphi \vee \psi)$,
- (M8) $((\varphi \rightarrow \chi) \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \vee \psi) \rightarrow \chi)$,
- (M9) $\varphi \rightarrow \sim\sim\varphi$,
- (M10) $\sim\sim\varphi \rightarrow \varphi$.

(II) a formula which has one of the following forms is an axiom (where φ, ψ and χ are arbitrary formulas):

- (SM1) $\varphi \Leftrightarrow \varphi$,
- (SM2) $(\varphi \Leftrightarrow \psi) \rightarrow (\psi \Leftrightarrow \varphi)$,
- (SM3) $(\varphi \Leftrightarrow \psi) \rightarrow ((\psi \Leftrightarrow \chi) \rightarrow (\varphi \Leftrightarrow \chi))$,
- (SM4) $(\varphi \Leftrightarrow \psi) \rightarrow (\varphi \Leftrightarrow \psi)$,
- (SM5) $((\varphi \Leftrightarrow \psi) \Leftrightarrow (\varphi \rightarrow \psi)) \rightarrow ((\varphi \Leftrightarrow \chi) \Leftrightarrow (\psi \Leftrightarrow \chi))$,

where the derivate logical connective \Leftrightarrow is defined as usual:

$$\varphi \Leftrightarrow \psi := (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi).$$

The deduction rules of \mathcal{SM} are *modus ponens*, *contrapositions* and *similarity*:

$$\text{(MP)} \quad \frac{\varphi, \varphi \rightarrow \psi}{\psi} \qquad \text{(C)} \quad \frac{\varphi \rightarrow \psi}{\sim\psi \rightarrow \sim\varphi} \qquad \text{(Sim)} \quad \frac{\varphi, \psi}{\varphi \Leftrightarrow \psi}.$$

If $\Theta \subseteq Form_{\mathcal{SM}}$, then a formula φ is a Θ -theorem if there exist a natural number $n \geq 1$ and a sequence of formulas $\varphi_1, \varphi_2, \dots, \varphi_n = \varphi$ such that, for any $i \in \{1, 2, \dots, n\}$ one of the following condition holds:

- (c1) φ_i is an axiom,
- (c2) $\varphi_i \in \Theta$,

- (c3) there are $j, k < i$, such that φ_k is $\varphi_j \rightarrow \varphi_i$,
- (c4) there are $j, k < i$, such that φ_i is $\varphi_j \Leftrightarrow \varphi_k$.

The sequence $\varphi_1, \varphi_2, \dots, \varphi_n = \varphi$ is a Θ – proof for φ . The set of Θ – theorems will be denoted by $Theor_{\mathcal{SM}}(\Theta)$. The fact that a formula φ is a Θ – theorem will be simply denoted by $\Theta \vdash_{\mathcal{SM}} \varphi$. A formula φ will be called a *theorem* if it is probable from the emptyset. This will be denoted by $\vdash_{\mathcal{SM}} \varphi$. In this case, a proof for φ will be a sequence of formulas $\varphi_1, \varphi_2, \dots, \varphi_n = \varphi$ such that for any $i \in \{1, 2, \dots, n\}$, one of the above conditions (c1), (c3) or (c4) is satisfied. The set of all theorems of will be denoted by $Theor_{\mathcal{SM}}$.

Remark 4.1. The propositional calculus \mathcal{SM} is an extension of \mathcal{M} . It is straightforward that $Form_{\mathcal{M}} \subseteq Form_{\mathcal{SM}}$. Moreover, if $\Theta \subseteq Form_{\mathcal{M}}$ and $\varphi \in Form_{\mathcal{M}}$ such that $\Theta \vdash_{\mathcal{M}} \varphi$, then $\Theta \vdash_{\mathcal{SM}} \varphi$, since any Θ -proof in \mathcal{M} is a Θ -proof en \mathcal{SM} . Consequently , any theorem of \mathcal{M} is also a theorem of \mathcal{SM} .

Proposition 4.1. *The following is a derivate deduction rule of \mathcal{SM} :*

$$(R) \frac{\varphi \leftrightarrow \psi}{(\varphi \leftrightarrow \chi) \leftrightarrow (\psi \leftrightarrow \chi)}.$$

Proof. Let Θ be an arbitrary set of formulas of \mathcal{SM} .

$$\begin{array}{ll} \Theta \vdash \varphi \leftrightarrow \psi & \text{(hypothesis)} \\ \vdash \varphi \rightarrow \varphi & \text{(theorem of } \mathcal{M}) \\ \Theta \vdash (\varphi \leftrightarrow \psi) \leftrightarrow (\varphi \rightarrow \varphi) & \text{(Sim)} \\ \Theta \vdash (\varphi \leftrightarrow \chi) \leftrightarrow (\psi \leftrightarrow \chi) & ((S5),(MP)) \end{array}$$

The Lindenbaum-Tarski algebra of \mathcal{SM} . In the sequel $\Theta \subseteq Form_{\mathcal{SM}}$ is a fixed set of formulas.

For any two formulas φ and ψ we defined

$$\varphi \equiv_{\Theta} \psi \text{ iff } \Theta \vdash_{\mathcal{SM}} \varphi \rightarrow \psi \text{ and } \Theta \vdash_{\mathcal{SM}} \psi \rightarrow \varphi.$$

It is straightforward that the relation \equiv_{Θ} is an equivalence relation on $Form_{\mathcal{SM}}$. For any formula $\varphi \in Form_{\mathcal{SM}}$ we will denote by $[\varphi]_{\Theta}$ the equivalence class of φ with respect to \equiv_{Θ} . The set $Form_{\mathcal{SM}} / \equiv_{\Theta} = \{[\varphi]_{\Theta} : \varphi \in Form_{\mathcal{SM}}\}$ is the quotient of $Form_{\mathcal{SM}}$ with respect to \equiv_{Θ} .

On $Form_{\mathcal{SM}} / \equiv_{\Theta}$ we define the following operations:

$$\begin{aligned} \sim [\varphi]_{\Theta} &:= [\sim \varphi]_{\Theta} \\ [\varphi]_{\Theta} \Delta [\psi]_{\Theta} &:= [\varphi \Delta \psi]_{\Theta}, \text{ where } \Delta \in \{\vee, \wedge, \rightarrow\}, \\ 1_{\Theta} &:= Theor_{\mathcal{SM}}(\Theta), \\ 0_{\Theta} &:= \sim 1_{\Theta}, \end{aligned}$$

$$S([\varphi]_{\Theta}, [\psi]_{\Theta}) := [\varphi \Leftrightarrow \psi]_{\Theta}.$$

Proposition 4.2. *The structure $SM(\Theta) = ((Form_{\mathcal{SM}} / \equiv_{\Theta}, \vee, \wedge, \rightarrow, \sim, 0, 1), S)$ is a similarity DH-algebra.*

Proof. By Remark 4.1 and the fact that the Lindenbaum-Tarski algebra of \mathcal{M} is an DH-algebra (see [8], page 62), it follows that $(Form_{\mathcal{SM}} / \equiv_{\Theta}, \vee, \wedge, \rightarrow, \sim, 0, 1)$ is an DH-algebra. By (R), the binary operation S is well defined. By (SM1)-(SM5), S is a similarity on $Form_{\mathcal{SM}} / \equiv_{\Theta}$. \square

If $\Theta \subseteq Form_{\mathcal{SM}}$ then the similarity DH-algebra $SM(\Theta)$ is called the *Lindenbaum-Tarski algebra of Θ* . In the particular case when $\Theta = \emptyset$, the relation \equiv_{Θ} will be simply denoted by \equiv . Thus, we have

$$\varphi \equiv \psi \text{ iff } \vdash \varphi \rightarrow \psi \text{ and } \vdash \psi \rightarrow \varphi.$$

The equivalence class of a formula φ with respect to \equiv will be denoted by $[\varphi]$. Consequently, the similarity DH-algebra $SM(\emptyset)$ will be denoted by SM and this is the Lindenbaum-Tarski algebra of the propositional calculus \mathcal{SM} .

The semantics of \mathcal{SM} . Our models are similarity DH-algebra. We shall define the valuations, the tautologies and we shall prove the completeness of \mathcal{SM} .

If (A, S) is a similarity DH-algebra an (A, S) -**valuation** is a function $e : Form_{\mathcal{SM}} \rightarrow (A, S)$ which satisfies the following conditions:

$$\begin{aligned} e(\sim \varphi) &= \sim e(\varphi), \\ e(\varphi \Delta \psi) &= e(\varphi) \Delta e(\psi), \text{ where } \Delta \in \{\vee, \wedge, \rightarrow\}, \\ e(\varphi \Leftrightarrow \psi) &= S(e(\varphi), e(\psi)), \end{aligned}$$

As usual, an (A, S) -valuation is uniquely determined by the values of the propositional variables.

Let Θ be a set of formulas and φ a formula. For a similarity DH-algebra (A, S) , we say that φ is a Θ -*tautology with respect to (A, S)* (or, Θ -*tautology w.r.t. (A, S)*) if

$$e(\Theta) = \{e(\psi) : \psi \in \Theta\} = \{1\} \text{ implies } e(\varphi) = 1,$$

for any (A, S) -valuation $e : Form_{\mathcal{SM}} \rightarrow (A, S)$. The fact that φ is a Θ -tautology w.r.t. (A, S) will be denoted by $\Theta \models_{(A, S)} \varphi$. If Θ is empty, a \emptyset -tautology w.r.t. (A, S) will be simply called a *tautology w.r.t. (A, S)* and the fact that φ is a tautology w.r.t. (A, S) will be denoted by $\models_{(A, S)} \varphi$. One can easily see that

$$\models_{(A, S)} \varphi \text{ iff } e(\varphi) = 1 \text{ for any } (A, S)\text{-valuation } e : Form_{\mathcal{SM}} \rightarrow (A, S).$$

Theorem 4.1 shows that DH-algebras are the algebraic counterpart of \mathcal{SM} .

Theorem 4.1. (Completeness of \mathcal{SM})

If Θ is a set formulas and φ is a formula of \mathcal{SM} , then the following are equivalent:

- (a) $\Theta \vdash_{\mathcal{SM}} \varphi$,
- (b) $\Theta \models_{(A,S)} \varphi$ for any similarity DH-algebra (A, S) ,
- (c) $[\varphi]_{\Theta} = [1]_{\Theta} \in SM(\Theta)$.

Proof. (a) \Rightarrow (b) : (**Soundness**) Let $\varphi_1, \dots, \varphi_n$ be a Θ -proof for φ in \mathcal{SM} , (A, S) a similarity DH-algebra and e an (A, S) -valuation. We shall prove by induction on $i \in \{1, 2, \dots, n\}$ that $e(\varphi_i) = 1$. If φ_i is an axiom, then an easy calculation shows that $e(\varphi_i) = 1$. If $\varphi_i \in \Theta$, then $e(\varphi_i) \in \{1\}$ therefore $e(\varphi_i) = 1$. Suppose there are $j, k < i$ such that φ_k is $\varphi_j \rightarrow \varphi_i$. By induction hypothesis, $1 = e(\varphi_k) = e(\varphi_j)$, so $e(\varphi_i) = 1 \rightarrow e(\varphi_i) = e(\varphi_j \rightarrow \varphi_i) = e(\varphi_k) = 1$. If there are $j, k < i$ such that φ_i is $\varphi_j \Leftrightarrow \varphi_k$, then by induction $e(\varphi_j) = e(\varphi_k) = 1$ so $e(\varphi_i) = S(e(\varphi_j), e(\varphi_k)) = S(1, 1) = 1$. Hence, for $i = n$ we get $e(\varphi) = 1$.

(b) \Rightarrow (c) : Follows by the fact that $SM(\Theta)$ is a similarity MV-algebra.

(c) \Rightarrow (a) : Is straightforward. \square

In the sequel, we shall prove a stronger completeness result for the axiomatic extension of \mathcal{SM} which correspond to the class of representable similarity DH-algebras. Let \mathcal{SM}_{tl} be the axiomatic of \mathcal{SM} , which has the additional axioms:

- (SM6) $(\varphi \rightarrow \psi) \vee (\psi \rightarrow \varphi)$,
- (SM7) $(\varphi \wedge \sim \psi) \rightarrow (\varphi \vee \sim \psi)$,
- (SM8) $((\varphi \rightarrow \psi) \Leftrightarrow (\varphi \rightarrow \varphi)) \vee (\psi \rightarrow \varphi)$.

Theorem 4.2. (Completeness of \mathcal{SM}_{tl})

If Θ is a set of formulas and φ is a formula of \mathcal{SM}_{tl} , then the following are equivalent:

- (a) $\Theta \vdash_{\mathcal{SM}} \varphi$,
- (b) $\Theta \models_{(A,S)} \varphi$ for any similarity DH-algebra (A, S) ,
- (c) $\Theta \models_{(A,S)} \varphi$ for any similarity DH-chain (A, S) ,
- (d) $[\varphi]_{\Theta} = [1]_{\Theta} \in SM(\Theta)$.

Proof. By Theorem 4.1, Theorem 3.3 and the fact that Lindenbaum-Tarski algebra of \mathcal{SM}_{tl} is a representable similarity DH-algebra. \square

Proposition 4.3. (Conservative extension property)

\mathcal{SM} and \mathcal{SM}_{tl} are conservative extensions of \mathcal{M} , i.e.

$$\Theta \vdash_{\mathcal{M}} \varphi \text{ iff } \Theta \vdash_{\mathcal{SM}} \varphi \text{ iff } \Theta \vdash_{\mathcal{SM}_{tl}} \varphi,$$

where $\Theta \subseteq \text{Form}_{\mathcal{M}}$ and φ is a formula of \mathcal{M} .

Proof. One implication follows by Remark 4.1. For the other implication, let $\Theta \vdash_{S\mathcal{M}} \varphi$ and suppose that $\Theta \not\vdash_{\mathcal{M}} \varphi$. It follows that there exists an DH-chain A and an A -valuation $e : Form_{\mathcal{M}} \rightarrow A$ such that $e(\Theta) = \{1\}$ and $e(\varphi) \neq 1$. It is obvious that $S : A \times A \rightarrow A$ defined by $S(x, y) := x \leftrightarrow y$ is a similarity on A . Hence we consider $e_S(\Theta) : Form_{S\mathcal{M}} \rightarrow (A, S)$ to be the unique (A, S) -valuation with $e_S(x) = e(x)$ for any propositional variable x . One can easily prove by structural induction that $e_S(\psi) = e(\psi)$ for any formula ψ of \mathcal{M} . Thus, $e_S(\Theta) = e(\Theta) = \{1\}$ and $e_S(\varphi) \neq 1$. By Theorem 4.1 this a contradiction of our hypothesis, so $\Theta \vdash_{\mathcal{M}} \varphi$. \square

5. CONCLUSION AND FUTURE RESEARCH

In this paper we have studied symmetric Heyting algebras (or DH-algebras) endowed with an operator of similarity S . These algebras are a generalization of strong similarity Łukasiewicz-Moisil algebras [4]. Also in [7] similarities are studied on MV-algebras. Since Wajsberg algebras are polynomially equivalent to MV-algebras (see [10]), in a future paper we will study the notion of similarity on Heyting-Wajsberg algebras (see [3]) as a common framework for similarity DH-algebras and similarity MV-algebras.

Competing Interests

The author declare that no competing interests exist.

REFERENCES

- [1] V. Boicescu, A. Filipoiu, G. Georgescu and S. Rudeanu, *Lukasiewicz-Moisil Algebras*, *Anal. of Discrete Math.* 49, North-Holland, The Netherlands, 1991.
- [2] J.L. Castro, F. Klawonn, *Similarity in Fuzzy Reasoning, using Fuzzy Logic.*, *Mathware Soft Comput.*, **2** (1995) 197-228.
- [3] G. Cattaneo, D. Ciucci, R. Giuntini, M. König, *Algebraic structures related to many valued logical systems. I., Heyting Wajsberg algebras.* *Fund. Inform.* **63** (2004), no. 4, 331–355.
- [4] F. Chirteş, *Similarity Łukasiewicz–Moisil algebras*, *An. Univ. Craiova Ser. Mat. Inform.* **35** (2008), 54–75.
- [5] F. Formato, G. Gerla, M. Sessa, *Similarity-based unification*, *Fund. Inform.*, **41** (2000), 393-414.
- [6] G. Georgescu, A. Popescu, *Concept lattices and similarity in non-commutative fuzzy logic*, *Fund. Inform.*, **53** (2002) 23-54.
- [7] B. Gerla and I. Leuştean, *Similarity MV-algebras*, *Fund. Inform.* **69** (2006), no. 3, 287–300.
- [8] A. Monteiro, *Sur les algèbres de Heyting Simetriques*, *Special issue in honor of António Monteiro. Portugal. Math.* **39** (1980) 1–4.

- [9] L. Pasi, S. Kalle, *Fuzzy similarity based classification in normal Lukasiewicz algebra with geometric and harmonic mean*, Third eusflat, an international conference in fuzzy logic and technology, Zittau, Saks. (2003) 242–245.
- [10] A. Rodríguez, A. Torrens, V. Verdú. Lukasiewicz, Logic and Wajsberg algebras. Polish Acad. Sci. Inst. Philos. Sociol. Bull. Sect. Logic **19** (1990), no. 2, 51–55.
- [11] H.P. Sankappanavar, *Heyting algebras with a dual lattice endomorphism*, Z. Math. Logik Grundlag. Math. **33** (1987), no. 6, 565–573.
- [12] E. Turunen, *A Lukasiewicz-style many-valued similarity reasoning*. Review. in Beyond Two: Theory and Applications of Multiple Valued Logic, Melvin Fitting and Ewa Orłowska editors, 311-321, 2003.
- [13] L. Zadeh, *Similarity relations and fuzzy orderings*, Information Sciences, **3** (1971) 177–200.

Federico Gabriel Almiñana

Departamento de Matemática,

Universidad Nacional de San Juan. Av. José Ignacio de la Roza 230 (O). Capital. CP 5400 San Juan. Argentina

federicogabriel17@gmail.com

Mathias Exequiel Pelaye

Departamento de Matemática,

Universidad Nacional de San Juan. Av. José Ignacio de la Roza 230 (O). Capital. CP 5400 San Juan. Argentina

pelayesmathias@gmail.com