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UNIFORMLY CLASSICAL QUASI-PRIMARY SUBMODULES

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ABSTRACT. In this paper we introduce the notions of uniformly quasi-primary ideals and uniformly classical quasi-primary submodules that generalize the concepts of uniformly primary ideals and uniformly classical primary submodules; respectively. Several characterizations of classical quasi-primary and uniformly classical quasi-primary submodules are given. Then we investigate for a ring R , when any finite intersection of (uniformly) primary submodules of any R -module is a (uniformly) classical quasi-primary submodule. Furthermore, the behavior of classical quasi-primary and uniformly classical quasi-primary submodules under localizations are studied. Also, we investigate the existence of (minimal) primary submodules containing classical quasi-primary submodules.

1. INTRODUCTION

Throughout this paper all rings are commutative with identity elements, and all modules are unital. Let M be an R -module. If N is a submodule (resp., proper submodule) of M , we

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write $N \leq M$ (resp., $N \not\leq M$). For every nonempty subset X of M and every submodule N of M , the ideal $\{r \in R \mid rX \subseteq N\}$ will be denoted by $(N : X)$. When $X = \{m\}$ where $m \in M$, we use $(N : m)$ instead of $(N : X)$, and when $N = 0$, we use $\text{Ann}_R X$ instead of $(0 : X)$. Note that $(N : M)$ is the annihilator of the module M/N . Also the ideal $\{r \in R \mid rm = 0 \text{ for some nonzero } m \in M\}$ will be denoted by $\text{Zd}_R(M)$, the classical Krull dimension of R will be denoted by $\dim(R)$, and for an ideal I of R , $\sqrt{I} := \{r \in R \mid r^k \in I \text{ for some } k \in \mathbb{N}\}$.

We recall that a proper ideal \mathcal{Q} of R is called a *primary* ideal if $ab \in \mathcal{Q}$ where $a, b \in R$, implies that either $a \in \mathcal{Q}$ or $b^k \in \mathcal{Q}$ for some $k \in \mathbb{N}$ (see for example [3]). Also a proper ideal \mathcal{Q} of a ring R is called *uniformly primary* if there exists a positive integer k such that whenever $r, s \in R$ satisfy $rs \in \mathcal{Q}$ and $r \notin \mathcal{Q}$, then $s^k \in \mathcal{Q}$ (see [9]). The notion of primary ideal was generalized by Fuchs [10] through defining an ideal \mathcal{Q} of a ring R to be *quasi-primary* if its radical is a prime ideal, i.e., if $ab \in \mathcal{Q}$ where $a, b \in R$, then either $a^k \in \mathcal{Q}$ or $b^k \in \mathcal{Q}$ for some $k \in \mathbb{N}$ (see also [11]). There are some extensions of these notions to modules. We recall that a proper submodule Q of M is called a *primary submodule* if $am \in Q$, where $a \in R$ and $m \in M$, then $m \in Q$ or $a^k M \subseteq Q$ for some $k \in \mathbb{N}$ (see for example [12]). Also a proper submodule Q of M is called *uniformly primary submodule* if there exists a positive integer k such that whenever $r \in R$ and $m \in M$ satisfy $rm \in Q$ and $m \notin Q$, then $r^k M \subseteq Q$ (see [1]). Moreover, Q is called *quasi-primary* if $\sqrt{(Q : M)}$ is a prime ideal of R (see [2]). Some other extensions of primary, uniformly primary and quasi-primary ideals to modules have been introduced and studied by M. Bazyar and M. Behboodi and also by M. Behboodi, R. Jahani-Nezhad and M. H. Naderi. They call a proper submodule Q of M *classical primary* (resp. *classical quasi-primary*) if $abN \subseteq Q$, where $a, b \in R$ and $N \leq M$, then either $aN \subseteq Q$ or $b^k N \subseteq Q$ (resp. $a^k N \subseteq Q$ or $b^k N \subseteq Q$) for some $k \in \mathbb{N}$. In this situation, $\mathcal{P} = \sqrt{(Q : M)}$ is a prime ideal of R and one says that Q is a classical \mathcal{P} -primary (resp. classical \mathcal{P} -quasi-primary) submodule (see [7], [8], and also [5] with a slightly different definition). Also a proper submodule Q of M is called *uniformly classical primary* if there exists a positive integer k such that whenever $a, b \in R$ and $m \in M$ satisfy $abm \in Q$ and $am \notin Q$, then $b^k m \in Q$. One says that a uniformly classical primary Q has order k if k is the smallest positive integer for which the aforementioned property holds (see [6]).

In this article, we continue the study of this construction via uniformly classical quasi-primary submodules. In Section 2, we introduce the notions of uniformly quasi-primary ideals and uniformly classical quasi-primary submodules and also we study some properties of classical quasi-primary and uniformly classical quasi-primary submodules. We conclude the section with a characterization of any Noetherian ring R with property that, any intersection of primary submodules of any R -module is a classical quasi-primary submodule. In section 3, first we

get some properties of localizations of classical quasi-primary and uniformly classical quasi-primary submodules. Then we investigate the existence of (minimal) primary submodules containing a classical quasi-primary submodule.

2. Uniformly classical primary and uniformly classical quasi-primary submodules

In this section, first we introduce uniformly quasi-primary and uniformly classical quasi-primary submodules and we get some properties of them. Then we investigate for a ring R , when any intersection of (uniformly) primary submodules of any R -module is a (uniformly) classical quasi-primary submodule.

Definition 2.1. *Let R be a ring. A proper ideal Q of R is called uniformly quasi-primary if there exists a positive integer k such that whenever $a, b \in R$ satisfy $ab \in Q$, then $a^k \in Q$ or $b^k \in Q$.*

Definition 2.2. *Let M be an R -module. A proper submodule Q of M is called uniformly classical quasi-primary if there exists a positive integer k such that whenever $a, b \in R$ and $N \leq M$ satisfy $abN \subseteq Q$, then $a^kN \subseteq Q$ or $b^kN \subseteq Q$. We say that a uniformly classical quasi-primary submodule Q has order k and write $q.ord_R(Q) = k$, if k is the smallest positive integer for which the aforementioned property holds.*

An ideal I of R is called uniformly classical quasi-primary if I is a uniformly classical quasi-primary R -submodules of R .

For an R -module M , we have the following:

- (1) Every uniformly classical quasi-primary submodule of M is classical quasi-primary.
- (2) Every uniformly classical primary submodule of M of order k is a uniformly classical quasi-primary submodule of order k' . In this situation $k' \leq k$.

However, the converse of (1) and (2) are not true, as the following example shows.

Example 2.3.

(1): *Let K be a field and $R = K[x_1, x_2, \dots]$. Let*

$$Q = \langle \{x_i^i\}_{i=2}^\infty \cup \{x_1x_i\}_{i=2}^\infty \rangle .$$

One can easily see that Q is a (classical) primary R -submodule of R . But for every $k \in \mathbb{N}$, $x_1x_{k+1} \in Q$, $x_1^k \notin Q$ and $x_{k+1}^k \notin Q$. Therefor Q is not a uniformly classical

quasi-primary submodule of R .

(2): *Let p be a prime integer and $M = \{\frac{a}{p^k} + \mathbb{Z} : a \in \mathbb{Z}, k \in \mathbb{N}\}$. It is easy to see that $Q = (0)$ is not a uniformly classical primary \mathbb{Z} -submodule, but Q is a uniformly classical quasi-primary submodule of order 2.*

(3): *Let $R = \mathbb{Z}$ and $M = \mathbb{Z}_{16}$. If $Q_1 = (0)$ and $Q_2 = \langle 4 \rangle$, then Q_1 and Q_2 are uniformly classical primary R -submodules of M of orders 2 and 4; respectively. But $q.ord_R(Q_1) = q.ord_R(Q_2) = 2$.*

It is easy to see that a proper submodule Q of an R -module M is a classical quasi-primary submodule if and only if for each $L \leq M$ such that $L \not\subseteq Q$, $(Q : L)$ is a quasi-primary ideal of R (see also [7, Proposition 2.3 (2)]). The following proposition offers several other characterizations of classical quasi-primary submodules.

Proposition 2.4. *Let M be an R -module. For a proper submodule Q of M the following statements are equivalent.*

- (1) *Q is a classical quasi-primary submodule.*
- (2) *For each $L \leq M$ such that $L \not\subseteq Q$, $(Q : L)$ is a quasi-primary ideal of R .*
- (3) *The set $\{\sqrt{(Q : L)} \mid L \leq M, L \not\subseteq Q\}$ is a chain of prime ideals.*
- (4) *For each $L \leq M$ such that $L \not\subseteq Q$, $\sqrt{(Q : L)}$ is a prime ideal of R .*

Proof. (1) \Leftrightarrow (2) is evident (see also [7, Proposition 2.3 (2)]).

(2) \Rightarrow (3). Let K and L be two submodules of M not contained in Q . By (2), $\sqrt{(Q : L)}$, $\sqrt{(Q : K)}$ and $\sqrt{(Q : K + L)}$ are prime ideals. But it is well known that $\sqrt{(Q : K + L)} = \sqrt{(Q : K)} \cap \sqrt{(Q : L)}$, so $\sqrt{(Q : K)} \subseteq \sqrt{(Q : L)}$ or $\sqrt{(Q : L)} \subseteq \sqrt{(Q : K)}$.

(3) \Rightarrow (4) is clear.

(4) \Rightarrow (2) is evident by the definition of quasi-primary ideals. \square

In [7, Corollary 2.5], it has been shown that for a classical primary submodule of an R -module M , the set $\{\sqrt{(Q : L)} \mid L \text{ is a finitely generated submodule of } M \text{ such that } L \not\subseteq Q\}$ is a chain of prime ideals of R . The following corollary generalizes the result.

Corollary 2.5. *Let M be an R -module and Q be a classical primary submodule of M . Then the set $\{\sqrt{(Q : L)} \mid L \leq M, L \not\subseteq Q\}$ is a chain of prime ideals.*

Proposition 2.6. (See [7, Proposition 2.6]) *Let Q be a proper submodule of a Noetherian R -module M . Then Q is classical quasi-primary if and only if for every $a, b \in R$ and $m \in M$, $abm \in Q$ implies that for some $k \in \mathbb{N}$, $a^k m \in Q$ or $b^k m \in Q$.*

The Noetherian condition in the previous proposition is essential (see [7, Example 2.2(e)]). Compare this fact with the next proposition, that it gives some characterizations of uniformly classical quasi-primary submodules.

Proposition 2.7. *Let M be an R -module and Q be a proper submodule of M . Then the following statements are equivalent.*

- (1) Q is a uniformly classical quasi-primary submodule.
- (2) For every submodule L of M such that $L \not\subseteq Q$, $(Q : L)$ is a uniformly classical quasi-primary ideal of R and $\max\{q.\text{ord}_R(Q : L) \mid L \leq M \text{ and } L \not\subseteq Q\} < \infty$.
- (3) For every $m \in M \setminus Q$, $(Q : m)$ is a uniformly classical quasi-primary ideal and $\max\{q.\text{ord}_R(Q : m) \mid m \in M \setminus Q\} < \infty$.
- (4) There is a positive integer k such that $abm \in Q$, for $a, b \in R$ and $m \in M$, implies that $a^k m \in Q$ or $b^k m \in Q$.

Proof. The implications (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) are straightforward.

(4) \Rightarrow (1). Suppose (4) and let $abN \subseteq Q$, for some submodule N of M such that $N \not\subseteq Q$ and $a, b \in R$. If $a^k N \not\subseteq Q$, then there is $n \in N$ such that $a^k n \notin Q$. Therefore $b^k n \in Q$. Now put $A := \{n \in N \mid a^k n \in Q\}$ and $B := \{n \in N \mid b^k n \in Q\}$. One can easily see that A and B are submodules of N such that $N = A \cup B$. It follows that $N = A$ or $N = B$. If $N = A$, then $a^k N \subseteq Q$, a contradiction. So $N = B$, then $b^k N \subseteq Q$. Thus, Q is a uniformly classical quasi-primary submodule of R . □

The next example shows that the conditions $\max\{q.\text{ord}_R(Q : L) \mid L \leq M \text{ and } L \not\subseteq Q\} < \infty$ and $\max\{q.\text{ord}_R(Q : m) \mid m \in M \setminus Q\} < \infty$ in parts (2) and (3) of Proposition 2.7, are necessary.

Example 2.8. *Let $M = \bigoplus_{k=1}^{\infty} \mathbb{Z}_p^k$, where p is a prime number. It is easy to see that for every nonzero submodule L of M , $\text{Ann}_{\mathbb{Z}}(L) = 0$, or for some positive integer i , $\text{Ann}_{\mathbb{Z}}(L) = p^i \mathbb{Z}$. Especially, for every positive integer i , if $L_i = \bigoplus_{k=1}^i \mathbb{Z}_p^k$, then $\text{Ann}_{\mathbb{Z}}(L_i) = p^i \mathbb{Z}$. Then for every nonzero submodule L of M , $\text{Ann}_{\mathbb{Z}}(L)$ is a uniformly classical quasi-primary ideal of \mathbb{Z} . However, the \mathbb{Z} -submodule 0 of M is not uniformly classical quasi-primary.*

Recall that M is a multiplication R -module if each submodule of M is of the form IM , where I is an ideal of R . In the next proposition, uniformly classical quasi-primary submodules in multiplication modules are characterized.

Proposition 2.9. *Let M be a multiplication R -module and Q be a proper submodule of M . Then the following statements are equivalent.*

- (1) Q is a uniformly classical quasi-primary submodule.
- (2) $Q = (Q : M)$ is a uniformly classical quasi-primary ideal of R .
- (3) $Q = \mathcal{Q}M$, where \mathcal{Q} is a uniformly classical quasi-primary ideal which is maximal with respect to this property (i.e., $IM = Q$ implies that $I \subseteq \mathcal{Q}$).

Proof. (1) \Rightarrow (2) follows from Proposition 2.7.

(2) \Rightarrow (3). Evidently $Q = \mathcal{Q}M$ and if $IM = Q$, for some ideal I of R , then $I \subseteq \mathcal{Q}$.

(3) \Rightarrow (1). Let $Q = \mathcal{Q}M$, where \mathcal{Q} is a uniformly classical quasi-primary ideal of R of order k which is maximal with respect to this property. Also let $abN \subseteq Q$ where $a, b \in R$ and $N \leq M$. Since M is a multiplication module, $N = IM$ for some ideal I of R . Thus $abIM \subseteq Q$, i.e., $abI \subseteq (Q : M) \subseteq \mathcal{Q}$. So $a^k I \subseteq \mathcal{Q}$ or $b^k I \subseteq \mathcal{Q}$. Therefore $a^k N \subseteq \mathcal{Q}M = Q$ or $b^k N \subseteq \mathcal{Q}M = Q$. Thus Q is a uniformly classical primary submodule of M . \square

Now, we investigate for a ring R , when any intersection of (uniformly) primary submodules of any R -module is a (uniformly) classical quasi-primary submodule.

Proposition 2.10. *Let R be a ring. If any finite intersection of (uniformly) primary submodules of the free R -module $R \oplus R$ is a (uniformly) classical quasi-primary submodule, then every two prime ideals of R are comparable.*

Proof. Let \mathcal{P}_1 and \mathcal{P}_2 be two prime ideals of R . It is clear that $N_1 = \mathcal{P}_1 \oplus R$ and $N_2 = R \oplus \mathcal{P}_2$ are (uniformly) primary submodules of the free module $M = R \oplus R$. Therefore, $N_1 \cap N_2 = \mathcal{P}_1 \oplus \mathcal{P}_2$ is a (uniformly) classical quasi-primary submodule of M and so by Proposition 2.4, either $\sqrt{(\mathcal{P}_1 \oplus \mathcal{P}_2 : \mathcal{P}_1 \oplus R)} \subseteq \sqrt{(\mathcal{P}_1 \oplus \mathcal{P}_2 : R \oplus \mathcal{P}_2)}$ or $\sqrt{(\mathcal{P}_1 \oplus \mathcal{P}_2 : R \oplus \mathcal{P}_2)} \subseteq \sqrt{(\mathcal{P}_1 \oplus \mathcal{P}_2 : \mathcal{P}_1 \oplus R)}$. It is easy to see that $\sqrt{(\mathcal{P}_1 \oplus \mathcal{P}_2 : \mathcal{P}_1 \oplus R)} = \mathcal{P}_2$ and $\sqrt{(\mathcal{P}_1 \oplus \mathcal{P}_2 : R \oplus \mathcal{P}_2)} = \mathcal{P}_1$. Thus either $\mathcal{P}_1 \subseteq \mathcal{P}_2$ or $\mathcal{P}_2 \subseteq \mathcal{P}_1$. \square

Let R be a ring such that every two prime ideals of R are comparable and M be an R -module. Then by [6, Theorem 2.6], any finite intersection of uniformly primary submodules of M is a uniformly classical primary submodule of M . Therefore by Proposition 2.10, we can get the following theorem that it generalizes [6, Theorem 2.6].

Theorem 2.11. *Let R be a ring. Then the following statements are equivalent.*

- (1) *For any R -module M , any finite intersection of uniformly primary submodules of M is a uniformly classical quasi-primary submodule of M .*
- (2) *Any finite intersection of uniformly primary submodules of the free R -module $R \oplus R$ is a uniformly classical quasi-primary submodule.*
- (3) *Every two prime ideals of R are comparable (i.e., $\text{Spec}(R)$ is a chain).*

In the next theorem, we give a characterization of a Noetherian ring R with property that, any intersection of primary submodules of any R -module is a classical quasi-primary submodule.

Theorem 2.12. *Let R be a Noetherian ring. Then the following statements are equivalent.*

- (1) *Any intersection of primary submodules of any R -module M is a classical quasi-primary submodule of M .*
- (2) *Any intersection of primary submodules of the free R -module $R \oplus R$ is a classical quasi-primary submodule.*
- (3) *Every proper submodule of any R -module M is a classical quasi-primary submodule of M .*
- (4) *Every proper ideal of R is a quasi-primary ideal.*
- (5) *Every proper ideal of R is a classical quasi-primary ideal.*
- (6) *Every two prime ideals of R are comparable.*
- (7) *R is local with $\dim(R) \leq 1$.*

Proof. (1) \Rightarrow (2) is evident.

(2) \Rightarrow (6) is true by Proposition 2.10.

(6) \Rightarrow (4) is evident.

(4) \Rightarrow (3) follows directly from Proposition 2.4.

(3) \Rightarrow (5) \Rightarrow (4) is trivial.

(4) \Rightarrow (1) follows from Proposition 2.4.

(7) \Rightarrow (6) is evident.

(6) \Rightarrow (7). Assume that \mathcal{P} is a nonzero prime ideal of R . By [12, Exercise 15.3], if there exists one prime ideal of R strictly between (0) and \mathcal{P} , then there are infinitely many. Since every two prime ideals of R are comparable and R is Noetherian, there is no prime ideal of R strictly between (0) and \mathcal{P} . Therefore R is local with $\dim(R) \leq 1$. \square

3. The existence of (minimal) primary submodules containing a classical quasi-primary submodule

In this section, first we get some properties of localizations of classical quasi-primary and uniformly classical quasi-primary submodules. Then we investigate the existence of (minimal) primary submodules containing a classical quasi-primary submodule.

Let R be a commutative ring and M be an R -module. Suppose S is a multiplicative closed subset of R which contains 1 but not 0. Let M_S be the localization of M at S . Let $f : M \rightarrow M_S$ be the natural map. For any R_S -submodule Q of M_S , we define $Q^c = \{m \in M : f(m) \in Q\}$. The above notations are fixed for this section.

One can easily see that the following lemma is true.

Lemma 3.1. *Let S be a multiplicatively subset of a ring R and N be a proper submodule of an R -module M . Then the following statements are hold:*

- (1) $(N : M) \subseteq \text{Zd}_R(\frac{M}{N}) = \cup_{x \in M \setminus N} (N : x) = \cup_{x \in M \setminus N} \sqrt{(N : x)}$.
- (2) *If $\{\sqrt{(N : M)} \mid L \text{ is a submodule of } M \text{ such that } L \not\subseteq N\}$ is a chain of prime ideals, then $\text{Zd}_R(\frac{M}{N})$ is a prime ideal of R .*
- (3) $\text{Zd}_R(\frac{M}{N}) \cap S = \emptyset$ if and only if $(N_S)^c = N$.
- (4) *If $(N_S)^c = N$, then $(N :_R M)_S = (N_S :_{R_S} M_S)$.*
- (5) *For every submodule W of M_S , $\text{Zd}_R(\frac{M}{W^c}) \cap S = \emptyset$.*

By applying Proposition 2.4 and Lemma 3.1, we have the following corollary.

Corollary 3.2. *Let Q be a classical quasi-primary submodule of an R -module M . Then $\text{Zd}_R(\frac{M}{Q})$ is a prime ideal of R .*

Proposition 3.3. *Let S be a multiplicatively closed subset of a ring R and M be an R -module.*

- (1) *If Q is a uniformly classical quasi-primary submodule of M such that $Q_S \neq M_S$, then Q_S is a uniformly classical quasi-primary submodule of M_S as an R_S -module. Furthermore $q.\text{ord}(Q_S) \leq q.\text{ord}(Q)$.*
- (2) *If W is a uniformly classical quasi-primary submodule of M_S , then W^c is a uniformly classical quasi-primary submodule of M . Furthermore, $q.\text{ord}(W^c) \leq q.\text{ord}(W)$.*

Proof. We only prove (1), the proof of (2) is similar.

- (1). Let Q be a uniformly classical quasi-primary submodule of M of order k . Also let $\frac{r_1 r_2 m}{s_1 s_2 s} \in Q_S$ and $(\frac{r_1}{s_1})^k \frac{m}{s} \notin Q_S$, for some $\frac{r_1}{s_1}, \frac{r_2}{s_2} \in R_S$ and $\frac{m}{s} \in M_S$. Then there is some $s_0 \in S$ such that $r_1 r_2 s_0 m \in Q$ but $(r_1 s_0)^k m \notin Q$. Now by Proposition 2.7, $r_2^k m \in Q$, so $(\frac{r_2}{s_2})^k \frac{m}{s} \in Q_S$.

Thus Q_S is a classical quasi-primary submodule of M_S such that $q.ord(Q_S) \leq q.ord(Q)$. \square

In the next proposition, we state some properties of classical quasi-primary submodules, similar to Proposition 3.3.

Proposition 3.4. *Let S be a multiplicatively closed subset of a ring R and M be an R -module.*

- (1) *If M is Noetherian and Q is a classical quasi-primary submodule of M such that $Q_S \neq M_S$, then Q_S is a classical quasi-primary submodule of M_S as an R_S -module.*
- (2) *If Q is a classical quasi-primary submodule of M such that $(Q_S)^c = Q$, then Q_S is a classical quasi-primary submodule of M_S as an R_S -module.*
- (3) *If W is a classical quasi-primary submodule of M_S , then W^c is a classical quasi-primary submodule of M .*

Proof. The proof of (1) is straightforward by applying Proposition 2.6. Also (2) and (3) can be easily proved by the definition of classical quasi-primary submodules. \square

In the next proposition, the behavior of classical quasi-primary submodules under localizations has been summarized.

Proposition 3.5. *Let S be a multiplicatively closed subset of a ring R , and let Q be a classical quasi-primary submodule of an R -module M .*

- (1) *If M is Noetherian and for some $x \in M \setminus Q$, $(Q : x) \cap S = \emptyset$, then Q_S is a classical quasi-primary submodule of M_S .*
- (2) *If \mathcal{P} is a prime ideal of R such that $\sqrt{(Q : M)} = \mathcal{P}$ and $Zd_R(\frac{M}{Q}) \cap S = \emptyset$, then Q_S is a \mathcal{P}_S -quasi-primary submodule of M_S such that $(Q_S)^c = Q$, $(Q : M)_S = (Q_S : M_S)$ and $Zd_{R_S}(\frac{M_S}{Q_S}) = (Zd_R(\frac{M}{Q}))_S$.*
- (3) *If \mathcal{Q} is a prime ideal of R_S and W is a classical \mathcal{Q} -quasi-primary submodule of M_S , then W^c is a classical \mathcal{Q}^c -quasi-primary submodule of M such that $(W^c)_S = W$, $Zd_R(\frac{M}{W^c}) \cap S = \emptyset$ and $Zd_R(\frac{M}{W^c}) = (Zd_{R_S}(\frac{M_S}{W}))^c$.*

Proof. (1) By Proposition 2.4, $\sqrt{(Q : x)}$ is a prime ideal of R , so $\sqrt{(Q_S : \frac{x}{1})} = (\sqrt{(Q : x)})_S$ is a prime ideal of R_S . Therefore $Q_S \neq M_S$. Now by Proposition 3.4(1), Q_S is a classical quasi-primary submodule of M_S .

(2) It suffices to prove that $Zd_{R_S}(\frac{M_S}{Q_S}) = (Zd_R(\frac{M}{Q}))_S$, one can easily get other statements by Lemma 3.1 and Proposition 3.4.

First note that $\frac{y}{1} \in M_S \setminus Q_S$ if and only if $y \in M \setminus Q$, because if $y \in M \setminus Q$ and $\frac{y}{1} \in Q_S$, then $sy \in Q$, for some $s \in S$. Therefore $\emptyset \neq (Q : y) \cap S \subseteq Zd_R(\frac{M}{Q}) \cap S$, which is a contradiction. Now since for every $\frac{x}{s} \in M_S \setminus Q_S$, $(Q_S : \frac{x}{s}) = (Q_S : \frac{x}{1})$, then

$$\text{Zd}_{R_S}(\frac{M_S}{Q_S}) = \bigcup_{x \in M \setminus Q} (Q_S : \frac{x}{1}) = \bigcup_{x \in M \setminus Q} (Q : x)_S = (\bigcup_{x \in M \setminus Q} (Q : x))_S.$$

$$\text{so } \text{Zd}_{R_S}(\frac{M_S}{Q_S}) = (\text{Zd}_R(\frac{M}{Q}))_S.$$

(3) By Lemma 3.1 and Proposition 3.4(3), W^c is a classical primary submodule of M such that $(W^c)_S = W$ and $\text{Zd}_R(\frac{M}{W^c}) \cap S = \emptyset$.

Now by part (2), $(W^c : M)_S = ((W^c)_S : M_S) = (W : M_S)$. Note that $(W^c : M)$ is a primary ideal and $(W^c : M) \cap S \subseteq \text{Zd}_R(\frac{M}{W^c}) \cap S = \emptyset$. Then $(W^c : M) = ((W^c : M)_S)^c = (W : M_S)^c$. Again by part (2), we get $(\text{Zd}_R(\frac{M}{W^c}))_S = \text{Zd}_{R_S}(\frac{M_S}{(W^c)_S}) = \text{Zd}_{R_S}(\frac{M_S}{W})$, then $((\text{Zd}_R(\frac{M}{W^c}))_S)^c = (\text{Zd}_{R_S}(\frac{M_S}{W}))^c$. By Corollary 3.2, $\text{Zd}_R(\frac{M}{W^c})$ is a prime ideal, so $\text{Zd}_R(\frac{M}{W^c}) = ((\text{Zd}_R(\frac{M}{W^c}))_S)^c$. Hence $\text{Zd}_R(\frac{M}{W^c}) = (\text{Zd}_{R_S}(\frac{M_S}{W}))^c$. \square

Corollary 3.6. *Let M be an R -module and S a multiplicatively closed subset of R . Let \mathcal{P} be a prime ideal of R and Q be a proper submodule of M such that $\text{Zd}_R(\frac{M}{Q}) \cap S = \emptyset$. Then Q is a uniformly classical \mathcal{P} -quasi-primary submodule of M of order k if and only if Q_S is a uniformly classical \mathcal{P}_S -quasi-primary submodule of M_S of order k .*

Corollary 3.7. *Let M be an R -module and S a multiplicatively closed subset of R .*

(1) *If M is Noetherian and Q is a (uniformly) classical quasi-primary submodule of M such that $(Q : M) \cap S = \emptyset$, then Q_S is a (uniformly) classical quasi-primary submodule of M_S .*

(2) *If W is a (uniformly) classical quasi-primary submodule of M_S , then W^c is a (uniformly) classical quasi-primary submodule of M and $(W^c)_S = W$. Furthermore, $(W^c : M) \cap S = \emptyset$.*

Proof. (1) Since $(Q : M) \cap S = \emptyset$, then $(Q_S : M_S) \neq R_S$, i.e., $Q_S \neq M_S$. Now the results get by Proposition 3.4(1) and Proposition 3.3.

(2) By Lemma 3.1, $(W^c : M) \subseteq \text{Zd}_R(\frac{M}{W^c})$, so the results can get by Proposition 3.3 and Proposition 3.5. \square

The next example shows that even for a free module of finite rank M , the function $Q \rightarrow Q_S$ does not define a one-to-one correspondence between (uniformly) classical quasi-primary submodules Q of M with $(Q : M) \cap S = \emptyset$ and (uniformly) classical quasi-primary submodules of M_S .

Example 3.8. (See also [4, Example 1]) Let $M = R \oplus R$, $Q = P_1 \oplus P_2$, $Q' = P_1 \oplus R$, and $S = R \setminus P_1$, where P_1 and P_2 are prime ideals of R such that $P_1 \subsetneq P_2$. It is easy to see that Q and Q' are (uniformly) classical quasi-primary submodules of M . Also $Q_S = (P_1)_S \oplus (P_2)_S = (P_1)_S \oplus R_S = Q'_S$ and so $(Q_S)^c = P_1 \oplus R \neq Q$.

Now we try to investigate the existence of (minimal) primary submodules containing a classical quasi-primary submodule.

Proposition 3.9. *Let M be an R -module and N be a (classical) \mathcal{P} -quasi-primary submodule of M such that for some $x \in M$, $\sqrt{(N : x)} = \mathcal{P}$. Then there exists a \mathcal{P} -primary submodule Q of M containing N .*

Proof. By the definition $\sqrt{(Q : M)} = \mathcal{P}$. Now, consider the following set:

$$T = \{C \mid N \subseteq C, C \text{ is a submodule of } M \text{ and } \sqrt{(C : x)} = \mathcal{P}\}.$$

By Zorn's lemma T has a maximal element. Let Q be a maximal element of T . Evidently, $\sqrt{(Q : M)} = \mathcal{P}$. Now we show that Q is a primary submodule of M . Let $ra \in Q$, where $a \in M \setminus Q$ and $r \in R$. Therefore $\sqrt{(Q : M)} \subsetneq \sqrt{(Q + Ra : x)}$. Consider $r_1 \in \sqrt{(Q + Ra : x)} \setminus \mathcal{P}$. Thus $r(r_1)^k x \in rQ + Rra \subseteq Q$, for some $k \in \mathbb{N}$. Thus $r(r_1)^k \in (Q : x) \subseteq \mathcal{P}$, so $r \in \mathcal{P} = \sqrt{(Q : M)}$, i. e., $r^l M \subseteq Q$, for some positive integer l . \square

In the following, we will introduce a certain primary submodule containing a classical quasi-primary submodule.

Theorem 3.10. *Let \mathcal{P} be a prime ideal of a ring R and N be a (classical) \mathcal{P} -quasi-primary submodule of an R -module M such that for some $x \in M$, $\sqrt{(N : x)} = \mathcal{P}$.*

(1) *If \mathcal{Q} is a \mathcal{P} -quasi-primary ideal of R such that $\mathcal{Q} \subseteq (N : M)$, then $((\mathcal{Q}M)_{\mathcal{P}})^c$ is a \mathcal{P} -primary submodule of M .*

(2) *$(N_{\mathcal{P}})^c = \{y \in M \setminus N \mid \sqrt{(N : M)} \neq \sqrt{(N : y)}\} \cup N$, and $(N_{\mathcal{P}})^c$ is a \mathcal{P} -primary submodule of M and minimal primary over N .*

Proof. (1) Let $r \in \sqrt{((N_{\mathcal{P}})^c : x)}$. Then $sr^k x \in N$ for some $s \in R \setminus \mathcal{P}$, and $k \in \mathbb{N}$. So $sr^k \in (N : x) \subseteq \mathcal{P}$. Then $r \in \mathcal{P}$, i.e., $\sqrt{((N_{\mathcal{P}})^c : x)} = \sqrt{(N : x)} = \mathcal{P}$. Therefore $(N_{\mathcal{P}})^c \neq M$ and $((\mathcal{Q}M)_{\mathcal{P}})^c \subseteq (N_{\mathcal{P}})^c$, so $((\mathcal{Q}M)_{\mathcal{P}})^c$ is a proper submodule of M . Now we show that $((\mathcal{Q}M)_{\mathcal{P}})^c$ is a \mathcal{P} -primary submodule of M .

Since $\mathcal{Q} \subseteq (((\mathcal{Q}M)_{\mathcal{P}})^c : M) \subseteq (((\mathcal{Q}M)_{\mathcal{P}})^c : x) \subseteq ((N_{\mathcal{P}})^c : x)$, therefore $\mathcal{P} \subseteq \sqrt{(((\mathcal{Q}M)_{\mathcal{P}})^c : M)} \subseteq \sqrt{((N_{\mathcal{P}})^c : x)} = \mathcal{P}$. So $\sqrt{(((\mathcal{Q}M)_{\mathcal{P}})^c : M)} = \mathcal{P}$.

Now let $ra \in ((\mathcal{Q}M)_{\mathcal{P}})^c$, where $r \in R \setminus \mathcal{P}$ and $a \in M$. So $sra = y$, for some $s \in R \setminus \mathcal{P}$ and $y \in \mathcal{Q}M$. Therefore $\frac{a}{1} = \frac{y}{sr} \in (\mathcal{Q}M)_{\mathcal{P}}$. Hence $a \in ((\mathcal{Q}M)_{\mathcal{P}})^c$.

(2) Let $N' = \{y \in M \setminus N \mid \sqrt{(N : M)} \neq \sqrt{(N : y)}\}$. If $y \in (N_{\mathcal{P}})^c \setminus N$, then $sy \in N$, for some $s \in R \setminus \mathcal{P}$, so $s \in (N : y) \setminus \sqrt{(N : M)}$, i.e., $y \in N'$. Now suppose $z \in N'$, then $z \notin N$ and $\sqrt{(N : M)} \subsetneq \sqrt{(N : z)}$. Let $s_0 \in \sqrt{(N : z)} \setminus \sqrt{(N : M)}$. Then $\frac{z}{1} = \frac{s_0^k z}{s_0^k} \in N_{\mathcal{P}}$, for some $k \in \mathbb{N}$. So $z \in (N_{\mathcal{P}})^c$. Obviously, $N \subseteq (N_{\mathcal{P}})^c$, Whence $(N_{\mathcal{P}})^c = N' \cup N$.

Evidently $\mathcal{P} = \sqrt{(N : M)} \subseteq \sqrt{((N_{\mathcal{P}})^c : M)} \subseteq \sqrt{((N_{\mathcal{P}})^c : x)} = \mathcal{P}$, i.e., $\sqrt{((N_{\mathcal{P}})^c : M)} = \mathcal{P}$.

Now let $ra \in (N_{\mathcal{P}})^c$, where $r \in R$ and $a \in M \setminus (N_{\mathcal{P}})^c$. Then $sra \in N$ for some $s \in R \setminus \mathcal{P}$, so $sr \in (N : a)$. On the other hand, since $a \notin (N_{\mathcal{P}})^c = N' \cup N$, $\sqrt{(N : a)} = \mathcal{P}$. Thus $sr \in \mathcal{P}$, so $r \in \mathcal{P}$. Therefore $(N_{\mathcal{P}})^c$ is a \mathcal{P} -primary submodule of M .

Finally, if $N \subseteq L \subseteq (N_{\mathcal{P}})^c$, where L is a primary submodule of M , then $(N_{\mathcal{P}})^c \subseteq (L_{\mathcal{P}})^c = L$, so $L = (N_{\mathcal{P}})^c$. Consequently $(N_{\mathcal{P}})^c$ is a minimal primary submodule over N . \square

Corollary 3.11. *Let M be an R -module and N a classical \mathcal{P} -quasi-primary submodule of M . If M is finitely generated, then $(N_{\mathcal{P}})^c$ is a \mathcal{P} -primary submodule of M and a minimal primary submodule over N .*

Proof. Suppose $\{x_1, x_2, x_3, \dots, x_k\}$ is a generating set of M . Therefore $\sqrt{(Q : M)} = \bigcap_{i=1}^k \sqrt{(Q : x_i)}$. Therefore by Proposition 2.4, $\sqrt{(Q : M)} = \sqrt{(Q : x_i)}$, for some $i, 1 \leq i \leq k$. Now apply Theorem 3.10. \square

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