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UNIFORMLY CLASSICAL QUASI-PRIMARY SUBMODULES

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Abstract. In this paper we introduce the notions of uniformly quasi-primary ideals and uniformly classical quasi-primary submodules that generalize the concepts of uniformly primary ideals and uniformly classical primary submodules; respectively. Several characterizations of classical quasi-primary and uniformly classical quasi-primary submodules are given. Then we investigate for a ring R, when any finite intersection of (uniformly) primary submodules of any R-module is a (uniformly) classical quasi-primary submodule. Furthermore, the behavior of classical quasi-primary and uniformly classical quasi-primary submodules under localizations are studied. Also, we investigate the existence of (minimal) primary submodules containing classical quasi-primary submodules.

1. Introduction

Throughout this paper all rings are commutative with identity elements, and all modules are unital. Let M be an R-module. If N is a submodule (resp., proper submodule) of M, we

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write $N \leq M$ (resp., $N \lneqq M$). For every nonempty subset X of M and every submodule N of M, the ideal $\{r \in R \mid rX \subseteq N\}$ will be denoted by $(N : X)$. When $X = \{m\}$ where $m \in M$, we use $(N : m)$ instead of $(N : X)$, and when $N = 0$, we use $\mathrm{Ann}_R X$ instead of $(0 : X)$. Note that $(N : M)$ is the annihilator of the module M/N. Also the ideal $\{r \in R \mid rm = 0$ for some nonzero $m \in M\}$ will be denoted by $\mathrm{Zd}_R(M)$, the classical Krull dimension of R will be denoted by $\dim(R)$, and for an ideal I of R, $\sqrt{I} := \{r \in R \mid r^k \in I$ for some $k \in \mathbb{N}\}$.

We recall that a proper ideal $\mathcal{Q}$ of R is called a *primary* ideal if $ab \in \mathcal{Q}$ where $a, b \in R$, implies that either $a \in \mathcal{Q}$ or $b^k \in \mathcal{Q}$ for some $k \in \mathbb{N}$ (see for example [3]). Also a proper ideal $\mathcal{Q}$ of a ring R is called *uniformly primary* if there exists a positive integer k such that whenever $r, s \in R$ satisfy $rs \in \mathcal{Q}$ and $r \notin \mathcal{Q}$, then $s^k \in \mathcal{Q}$ (see [9]) . The notion of primary ideal was generalized by Fuchs [10] through defining an ideal $\mathcal{Q}$ of a ring R to be *quasi-primary* if its radical is a prime ideal, i.e., if $ab \in \mathcal{Q}$ where a, $b \in R$, then either $a^k \in \mathcal{Q}$ or $b^k \in \mathcal{Q}$ for some $k \in \mathbb{N}$ (see also [11]). There are some extensions of these notions to modules. We recall that a proper submodule Q of M is called a *primary submodule* if $am \in Q$, where $a \in R$ and $m \in M$, then $m \in Q$ or $a^k M \subseteq Q$ for some $k \in \mathbb{N}$ (see for example [12]). Also a proper submodule Q of M is called *uniformly primary submodule* if there exists a positive integer k such that whenever $r \in R$ and $m \in M$ satisfy $rm \in Q$ and $m \notin Q$, then $r^k M \subseteq Q$ (see [1]). Moreover, Q is called *quasi-primary* if $\sqrt{(Q : M)}$ is a prime ideal of R (see [2]). Some other extensions of primary, uniformly primary and quasi-primary ideals to modules have been introduced and studied by M. Bazyar and M. Behboodi and also by M. Behboodi, R. Jahani-Nezhad and M. H. Naderi. Thay call a proper submodule Q of M *classical primary* (resp. *classical quasi-primary*) if $abN \subseteq Q$, where $a, b \in R$ and $N \leq M$, then either $aN \subseteq Q$ or $b^k N \subseteq Q$ (resp. $a^k N \subseteq Q$ or $b^k N \subseteq Q$) for some $k \in \mathbb{N}$. In this situation, $\mathcal{P} = \sqrt{(Q : M)}$ is a prime ideal of R and one says that Q is a classical $\mathcal{P}$-primary (resp. classical $\mathcal{P}$-quasi-primary) submodule (see [7], [8], and also [5] with a slightly different definition). Also a proper submodule Q of M is called *uniformly classical primary* if there exists a positive integer k such that whenever $a, b \in R$ and $m \in M$ satisfy $abm \in Q$ and $am \notin Q$, then $b^k m \in Q$. One says that a uniformly classical primary Q has order k if k is the smallest positive integer for which the aforementioned property holds (see [6]).

In this article, we continue the study of this construction via uniformly classical quasi-primary submodules. In Section 2, we introduce the notions of uniformly quasi-primary ideals and uniformly classical quasi-primary submodules and also we study some properties of classical quasi-primary and uniformly classical quasi-primary submodules. we conclude the section with a characterization of any Noetherian ring R with property that, any intersection of primary submodules of any R-module is a classical quasi-primary submodule. In section 3, first we get some properties of localizations of classical quasi-primary and uniformly classical quasi-primary submodules. Then we investigate the existence of (minimal) primary submodules containing a classical quasi-primary submodule.

2. Uniformly classical primary and uniformly classical quasi-primary submodules

In this section, first we introduce uniformly quasi-primary and uniformly classical quasi-primary submodules and we get some properties of them. Then we investigate for a ring R, when any intersection of (uniformly) primary submodules of any R-module is a (uniformly) classical quasi-primary submodule.

Definition 2.1. *Let R be a ring. A proper ideal $\mathcal{Q}$ of R is called uniformly quasi-primary if there exists a positive integer k such that whenever $a, b \in R$ satisfy $ab \in \mathcal{Q}$, then $a^k \in \mathcal{Q}$ or $b^k \in \mathcal{Q}$.*

Definition 2.2. *Let M be an R-module. A proper submodule Q of M is called uniformly classical quasi-primary if there exists a positive integer k such that whenever $a, b \in R$ and $N \leq M$ satisfy $abN \subseteq Q$, then $a^kN \subseteq Q$ or $b^kN \subseteq Q$. We say that a uniformly classical quasi-primary submodule Q has order k and write $q.ord_R(Q) = k$, if k is the smallest positive integer for which the aforementioned property holds.*

An ideal I of R is called uniformly classical quasi-primary if I is a uniformly classical quasi-primary R-submodules of R.

For an R-module M, we have the following:

(1) Every uniformly classical quasi-primary submodule of M is classical quasi-primary.
(2) Every uniformly classical primary submodule of M of order k is a uniformly classical quasi-primary submodule of order k'. In this situation $k' \leq k$.

However, the converse of (1) and (2) are not true, as the following example shows.

Example 2.3.

(1): *Let K be a field and $R = K[x_1, x_2, ...]$. Let*

$$\mathcal{Q} =< \{x_i^i\}_{i=2}^{\infty} \cup \{x_1 x_i\}_{i=2}^{\infty} > .$$

One can easily see that $\mathcal{Q}$ is a (classical) primary R-submodule of R. But for every $k \in \mathbb{N}, x_1 x_{k+1} \in \mathcal{Q}$, $x_1^k \notin \mathcal{Q}$ and $x_{k+1}^k \notin \mathcal{Q}$. Therefor $\mathcal{Q}$ is not a uniformly classical *quasi-primary submodule of* R.

(2): *Let* p *be a prime integer and* $M = \{\frac{a}{p^k} + \mathbb{Z} : a \in \mathbb{Z}, k \in \mathbb{N}\}$. *It is easy to see that* $Q = (0)$ *is not a uniformly classical primary* $\mathbb{Z}$*-submodule, but* Q *is a uniformly classical quasi-primary submodule of order* 2.

(3): *Let* $R = \mathbb{Z}$ *and* $M = \mathbb{Z}_{16}$. *If* $Q_1 = (0)$ *and* $Q_2 =< 4 >$, *then* Q_1 *and* Q_2 *are uniformly classical primary* R*-submodules of* M *of orders* 2 *and* 4*; respectively. But* $q.ord_R(Q_1) = q.ord_R(Q_2) = 2$.

It is easy to see that a proper submodule Q of an R-module M is a classical quasi-primary submodule if and only if for each $L \leq M$ such that $L \nsubseteq Q$, $(Q : L)$ is a quasi-primary ideal of R (see also [7, Proposition 2.3 (2)]). The following proposition offers several other characterizations of classical quasi-primary submodules.

Proposition 2.4. *Let* M *be an* R*-module. For a proper submodule* Q *of* M *the following statements are equivalent.*

(1) Q *is a classical quasi-primary submodule.*

(2) *For each* $L \leq M$ *such that* $L \nsubseteq Q$, $(Q : L)$ *is a quasi-primary ideal of* R.

(3) *The set* $\{\sqrt{(Q : L)} \mid L \leq M,\ L \nsubseteq Q\}$ *is a chain of prime ideals.*

(4) *For each* $L \leq M$ *such that* $L \nsubseteq Q$, $\sqrt{(Q : L)}$ *is a prime ideal of* R.

Proof. (1) $\Leftrightarrow$ (2) is evident (see also [7, Proposition 2.3 (2)]).
(2) $\Rightarrow$ (3). Let K and L be two submodules of M not contained in Q. By (2), $\sqrt{(Q : L)}$, $\sqrt{(Q : K)}$ and $\sqrt{(Q : K + L)}$ are prime ideals. But it is well known that $\sqrt{(Q : K + L)} = \sqrt{(Q : K)} \cap \sqrt{(Q : L)}$, so $\sqrt{(Q : K)} \subseteq \sqrt{(Q : L)}$ or $\sqrt{(Q : L)} \subseteq \sqrt{(Q : K)}$.
(3) $\Rightarrow$ (4) is clear.
(4) $\Rightarrow$ (2) is evident by the definition of quasi-primary ideals. $\square$

In [7, Corollary 2.5], it has been shown that for a classical primary submodule of an R-module M, the set $\{\sqrt{(Q : L)} \mid\ L$ is a finitely generated submodule of M such that $L \nsubseteq Q\}$ is a chain of prime ideals of R. The following corollary generalizes the result.

Corollary 2.5. *Let* M *be an* R*-module and* Q *be a classical primary submodule of* M. *Then the set* $\{\sqrt{(Q : L)} \mid L \leq M,\ L \nsubseteq Q\}$ *is a chain of prime ideals.*

Proposition 2.6. (See [7, Proposition 2.6]) *Let Q be a proper submodule of a Noetherian R-module M. Then Q is classical quasi-primary if and only if for every $a, b \in R$ and $m \in M$, $abm \in Q$ implies that for some $k \in \mathbb{N}$, $a^k m \in Q$ or $b^k m \in Q$.*

The Noetherian condition in the previous proposition is essential (see [7, Example 2.2(e)]). Compare this fact with the next proposition, that it gives some characterizations of uniformly classical quasi-primary submodules.

Proposition 2.7. *Let M be an R-module and Q be a proper submodule of M. Then the following statements are equivalent.*

(1) *Q is a uniformly classical quasi-primary submodule.*
(2) *For every submodule L of M such that $L \nsubseteq Q$, $(Q : L)$ is a uniformly classical quasi-primary ideal of R and $max\{q.ord_R(Q : L) \mid L \leq M \text{ and } L \nsubseteq Q\} < \infty$.*
(3) *For every $m \in M \setminus Q$, $(Q : m)$ is a uniformly classical quasi-primary ideal and $max\{q.ord_R(Q : m) \mid m \in M \setminus Q\} < \infty$.*
(4) *There is a positive integer k such that $abm \in Q$, for $a, b \in R$ and $m \in M$, implies that $a^k m \in Q$ or $b^k m \in Q$.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are straightforward.
(4) $\Rightarrow$ (1). Suppose (4) and let $abN \subseteq Q$, for some submodule N of M such that $N \nsubseteq Q$ and $a, b \in R$. If $a^k N \nsubseteq Q$, then there is $n \in N$ such that $a^k n \notin Q$. Therefore $b^k n \in Q$. Now put $A := \{n \in N \mid a^k n \in Q\}$ and $B := \{n \in N \mid b^k n \in Q\}$. One can easily see that A and B are submodules of N such that $N = A \cup B$. It follows that $N = A$ or $N = B$. If $N = A$, then $a^k N \subseteq Q$, a contradiction. So $N = B$, then $b^k N \subseteq Q$. Thus, Q is a uniformly classical quasi-primary submodule of R. □

The next example shows that the conditions $max\{q.ord_R(Q : L) \mid L \leq M \text{ and } L \nsubseteq Q\} < \infty$ and $max\{q.ord_R(Q : m) \mid m \in M \setminus Q\} < \infty$ in parts (2) and (3) of Proposition 2.7, are necessary.

Example 2.8. *Let $M = \bigoplus_{k=1}^{\infty} \mathbb{Z}_{p^k}$, where p is a prime number. It is easy to see that for every nonzero submodule L of M, $\mathrm{Ann}_{\mathbb{Z}}(L) = 0$, or for some positive integer i, $\mathrm{Ann}_{\mathbb{Z}}(L) = p^i\mathbb{Z}$. Especially, for every positive integer i, if $L_i = \bigoplus_{k=1}^{i} \mathbb{Z}_{p^k}$, then $\mathrm{Ann}_{\mathbb{Z}}(L_i) = p^i\mathbb{Z}$. Then for every nonzero submodule L of M, $\mathrm{Ann}_{\mathbb{Z}}(L)$ is a uniformly classical quasi-primary ideal of $\mathbb{Z}$. However, the $\mathbb{Z}-$submodule 0 of M is not uniformly classical quasi-primary.*

Recall that M is a multiplication R-module if each submodule of M is of the form IM, where I is an ideal of R. In the next proposition, uniformly classical quasi-primary submodules in multiplication modules are characterized.

Proposition 2.9. *Let M be a multiplication R-module and Q be a proper submodule of M. Then the following statements are equivalent.*

(1) *Q is a uniformly classical quasi-primary submodule.*
(2) *$\mathcal{Q} = (Q : M)$ is a uniformly classical quasi-primary ideal of R.*
(3) *$Q = \mathcal{Q}M$, where $\mathcal{Q}$ is a uniformly classical quasi-primary ideal which is maximal with respect to this property (i.e., $IM = Q$ implies that $I \subseteq \mathcal{Q}$).*

Proof. (1) $\Rightarrow$ (2) follows from Proposition 2.7.
(2) $\Rightarrow$ (3). Evidently $Q = \mathcal{Q}M$ and if $IM = Q$, for some ideal I of R, then $I \subseteq \mathcal{Q}$.
(3) $\Rightarrow$ (1). Let $Q = \mathcal{Q}M$, where $\mathcal{Q}$ is a uniformly classical quasi-primary ideal of R of order k which is maximal with respect to this property. Also let $abN \subseteq Q$ where $a, b \in R$ and $N \leq M$. Since M is a multiplication module, $N = IM$ for some ideal I of R. Thus $abIM \subseteq Q$, i.e., $abI \subseteq (Q : M) \subseteq \mathcal{Q}$. So $a^k I \subseteq \mathcal{Q}$ or $b^k I \subseteq \mathcal{Q}$. Therefore $a^k N \subseteq \mathcal{Q}M = Q$ or $b^k N \subseteq \mathcal{Q}M = Q$. Thus Q is a uniformly classical primary submodule of M. $\square$

Now, we investigate for a ring R, when any intersection of (uniformly) primary submodules of any R-module is a (uniformly) classical quasi-primary submodule.

Proposition 2.10. *Let R be a ring. If any finite intersection of (uniformly) primary submodules of the free R-module $R \oplus R$ is a (uniformly) classical quasi-primary submodule, then every two prime ideals of R are comparable.*

Proof. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two prime ideals of R. It is clear that $N_1 = \mathcal{P}_1 \oplus R$ and $N_2 = R \oplus \mathcal{P}_2$ are (uniformly) primary submodules of the free module $M = R \oplus R$. Therefore, $N_1 \cap N_2 = \mathcal{P}_1 \oplus \mathcal{P}_2$ is a (uniformly) classical quasi-primary submodule of M and so by Proposition 2.4, either $\sqrt{(\mathcal{P}_1 \oplus \mathcal{P}_2 : \mathcal{P}_1 \oplus R)} \subseteq \sqrt{(\mathcal{P}_1 \oplus \mathcal{P}_2 : R \oplus \mathcal{P}_2)}$ or $\sqrt{(\mathcal{P}_1 \oplus \mathcal{P}_2 : R \oplus \mathcal{P}_2)} \subseteq \sqrt{(\mathcal{P}_1 \oplus \mathcal{P}_2 : \mathcal{P}_1 \oplus R)}$. It is easy to see that $\sqrt{(\mathcal{P}_1 \oplus \mathcal{P}_2 : \mathcal{P}_1 \oplus R)} = \mathcal{P}_2$ and $\sqrt{(\mathcal{P}_1 \oplus \mathcal{P}_2 : R \oplus \mathcal{P}_2)} = \mathcal{P}_1$. Thus either $\mathcal{P}_1 \subseteq \mathcal{P}_2$ or $\mathcal{P}_2 \subseteq \mathcal{P}_1$. $\square$

Let R be a ring such that every two prime ideals of R are comparable and M be an R-module. Then by [6, Theorem 2.6], any finite intersection of uniformly primary submodules of M is a uniformly classical primary submodule of M. Therefore by Proposition 2.10, we can get the following theorem that it generalizes [6, Theorem 2.6].

Theorem 2.11. *Let R be a ring. Then the following statements are equivalent.*

(1) *For any R-module M, any finite intersection of uniformly primary submodules of M is a uniformly classical quasi-primary submodule of M.*

(2) *Any finite intersection of uniformly primary submodules of the free R-module $R \oplus R$ is a uniformly classical quasi-primary submodule.*

(3) *Every two prime ideals of R are comparable (i.e.,* $\mathrm{Spec}(R)$ *is a chain).*

In the next theorem, we give a characterization of a Noetherian ring R with property that, any intersection of primary submodules of any R-module is a classical quasi-primary submodule.

Theorem 2.12. *Let R be a Noetherian ring. Then the following statements are equivalent.*

(1) *Any intersection of primary submodules of any R-module M is a classical quasi-primary submodule of M.*

(2) *Any intersection of primary submodules of the free R-module $R \oplus R$ is a classical quasi-primary submodule.*

(3) *Every proper submodule of any R-module M is a classical quasi-primary submodule of M.*

(4) *Every proper ideal of R is a quasi-primary ideal.*

(5) *Every proper ideal of R is a classical quasi-primary ideal.*

(6) *Every two prime ideals of R are comparable.*

(7) *R is local with* $\dim(R) \leq 1$.

Proof. (1) $\Rightarrow$ (2) is evident.
(2) $\Rightarrow$ (6) is true by Proposition 2.10.
(6) $\Rightarrow$ (4) is evident.
(4) $\Rightarrow$ (3) follows directly from Proposition 2.4.
(3) $\Rightarrow$ (5) $\Rightarrow$ (4) is trivial.
(4) $\Rightarrow$ (1) follows from Proposition 2.4.
(7) $\Rightarrow$ (6) is evident.
(6) $\Rightarrow$ (7). Assume that $\mathcal{P}$ is a nonzero prime ideal of R. By [12, Exercise 15.3], if there exists one prime ideal of R strictly between (0) and $\mathcal{P}$, then there are infinitely many. Since every two prime ideals of R are comparable and R is Noetherian, there is no prime ideal of R strictly between (0) and $\mathcal{P}$. Therefore R is local with $\dim(R) \leq 1$. $\square$

3. **The existence of (minimal) primary submodules contain- ing a classical quasi-primary submodule**

In this section, first we get some properties of localizations of classical quasi-primary and uniformly classical quasi-primary submodules. Then we investigate the existence of (minimal) primary submodules containing a classical quasi-primary submodule.

Let R be a commutative ring and M be an R-module. Suppose S is a multiplicative closed subset of R which contains 1 but not 0. Let M_S be the localization of M at S. Let $f : M \to M_S$ be the natural map. For any R_S-submodule Q of M_S, we define $Q^c = \{m \in M : f(m) \in Q\}$. The above notations are fixed for this section.

One can easily see that the following lemma is true.

Lemma 3.1. *Let S be a multiplicatively subset of a ring R and N be a proper submodule of an R-module M. Then the following statements are hold:*
(1) $(N : M) \subseteq \mathrm{Zd_R}(\frac{\mathrm{M}}{\mathrm{N}}) = \cup_{\mathrm{x}\in \mathrm{M}\setminus \mathrm{N}}(\mathrm{N} : \mathrm{x}) = \cup_{\mathrm{x}\in \mathrm{M}\setminus \mathrm{N}}\sqrt{(\mathrm{N} : \mathrm{x})}$.
(2) *If* $\{\sqrt{(N : M)} \mid L$ *is a submodule of* M *such that* $L \nsubseteq N\}$ *is a chain of prime ideals, then* $\mathrm{Zd_R}(\frac{\mathrm{M}}{\mathrm{N}})$ *is a prime ideal of* R.
(3) $\mathrm{Zd_R}(\frac{\mathrm{M}}{\mathrm{N}}) \cap \mathrm{S} = \emptyset$ *if and only if* $(N_S)^c = N$.
(4) *If* $(N_S)^c = N$, *then* $(N :_R M)_S = (N_S :_{R_S} M_S)$.
(5) *For every submodule* W *of* M_S, $\mathrm{Zd_R}(\frac{\mathrm{M}}{\mathrm{W^c}}) \cap \mathrm{S} = \emptyset$.

By applying Proposition 2.4 and Lemma 3.1, we have the following corollary.

Corollary 3.2. *Let Q be a classical quasi-primary submodule of an R-module M. Then* $\mathrm{Zd_R}(\frac{\mathrm{M}}{\mathrm{Q}})$ *is a prime ideal of R.*

Proposition 3.3. *Let S be a multiplicatively closed subset of a ring R and M be an R-module.*
(1) *If Q is a uniformly classical quasi-primary submodule of M such that $Q_S \neq M_S$, then Q_S is a uniformly classical quasi-primary submodule of M_S as an R_S-module. Furthermore $q.ord(Q_S) \le q.ord(Q)$.*
(2) *If W is a uniformly classical quasi-primary submodule of M_S, then W^c is a uniformly classical quasi-primary submodule of M. Furthermore, $q.ord(W^c) \le q.ord(W)$.*

Proof. We only prove (1), the proof of (2) is similar.
(1). Let Q be a uniformly classical quasi-primary submodule of M of order k. Also let $\frac{r_1}{s_1}\frac{r_2}{s_2}\frac{m}{s} \in Q_S$ and $(\frac{r_1}{s_1})^k\frac{m}{s} \notin Q_S$, for some $\frac{r_1}{s_1}, \frac{r_2}{s_2} \in R_S$ and $\frac{m}{s} \in M_S$. Then there is some $s_0 \in S$ such that $r_1r_2s_0m \in Q$ but $(r_1s_0)^k m \notin Q$. Now by Proposition 2.7, $r_2^k m \in Q$, so $(\frac{r_2}{s_2})^k\frac{m}{s} \in Q_S$.

Thus Q_S is a classical quasi-primary submodule of M_S such that $q.ord(Q_S) \leq q.ord(Q)$. $\square$

In the next proposition, we state some properties of classical quasi-primary submodules, similar to Proposition 3.3.

Proposition 3.4. *Let S be a multiplicatively closed subset of a ring R and M be an R-module.*
(1) If M is Noetherian and Q is a classical quasi-primary submodule of M such that $Q_S \neq M_S$, then Q_S is a classical quasi-primary submodule of M_S as an R_S-module.
(2) If Q is a classical quasi-primary submodule of M such that $(Q_S)^c = Q$, then Q_S is a classical quasi-primary submodule of M_S as an R_S-module.
(3) If W is a classical quasi-primary submodule of M_S, then W^c is a classical quasi-primary submodule of M.

Proof. The proof of (1) is straightforward by applying Proposition 2.6. Also (2) and (3) can be easily proved by the definition of classical quasi-primary submodules. $\square$

In the next proposition, the behavior of classical quasi-primary submodules under localizations has been summarized.

Proposition 3.5. *Let S be a multiplicatively closed subset of a ring R, and let Q be a classical quasi-primary submodule of an R-module M.*

(1) If M is Noetherian and for some $x \in M \setminus Q$, $(Q : x) \cap S = \emptyset$, then Q_S is a classical quasi-primary submodule of M_S.
(2) If $\mathcal{P}$ is a prime ideal of R such that $\sqrt{(Q : M)} = \mathcal{P}$ and $\mathrm{Zd_R}(\frac{M}{Q}) \cap \mathrm{S} = \emptyset$, then Q_S is a $\mathcal{P}_S$-quasi-primary submodule of M_S such that $(Q_S)^c = Q$, $(Q : M)_S = (Q_S : M_S)$ and $\mathrm{Zd_{R_S}}(\frac{M_S}{Q_S}) = (\mathrm{Zd_R}(\frac{M}{Q}))_S$.
(3) If $\mathcal{Q}$ is a prime ideal of R_S and W is a classical $\mathcal{Q}$-quasi-primary submodule of M_S, then W^c is a classical $\mathcal{Q}^c$-quasi-primary submodule of M such that $(W^c)_S = W$, $\mathrm{Zd_R}(\frac{M}{W^c}) \cap \mathrm{S} = \emptyset$ and $\mathrm{Zd_R}(\frac{M}{W^c}) = (\mathrm{Zd_{R_S}}(\frac{M_S}{W}))^c$.

Proof. (1) By Proposition 2.4, $\sqrt{(Q : x)}$ is a prime ideal of R, so $\sqrt{(Q_S : \frac{x}{1})} = (\sqrt{(Q : x)})_S$ is a prime ideal of R_S. Therefore $Q_S \neq M_S$. Now by Proposition 3.4(1), Q_S is a classical quasi-primary submodule of M_S.
(2) It suffices to prove that $\mathrm{Zd}_{R_S}(\frac{M_S}{Q_S}) = (\mathrm{Zd}_R(\frac{M}{Q}))_S$, one can easily get other statements by Lemma 3.1 and Proposition 3.4.

First note that $\frac{y}{1} \in M_S \setminus Q_S$ if and only if $y \in M \setminus Q$, because if $y \in M \setminus Q$ and $\frac{y}{1} \in Q_S$, then $sy \in Q$, for some $s \in S$. Therefore $\emptyset \neq (Q : y) \cap S \subseteq \mathrm{Zd_R}(\frac{M}{Q}) \cap \mathrm{S}$, which is a contradiction. Now since for every $\frac{x}{s} \in M_S \setminus Q_S$, $(Q_S : \frac{x}{s}) = (Q_S : \frac{x}{1})$, then

$\mathrm{Zd}_{\mathrm{R_S}}(\frac{\mathrm{M_S}}{\mathrm{Q_S}}) = \bigcup_{\mathrm{x}\in \mathrm{M}\setminus \mathrm{Q}}(\mathrm{Q_S} : \frac{\mathrm{x}}{1}) = \bigcup_{\mathrm{x}\in \mathrm{M}\setminus \mathrm{Q}}(\mathrm{Q} : \mathrm{x})_{\mathrm{S}} = (\bigcup_{\mathrm{x}\in \mathrm{M}\setminus \mathrm{Q}}(\mathrm{Q} : \mathrm{x}))_{\mathrm{S}}$.

so $\mathrm{Zd}_{\mathrm{R_S}}(\frac{\mathrm{M_S}}{\mathrm{Q_S}}) = (\mathrm{Zd}_{\mathrm{R}}(\frac{\mathrm{M}}{\mathrm{Q}}))_{\mathrm{S}}$.

(3) By Lemma 3.1 and Proposition 3.4(3), W^c is a classical primary submodule of M such that $(W^c)_S = W$ and $\mathrm{Zd}_{\mathrm{R}}(\frac{\mathrm{M}}{\mathrm{W^c}}) \cap \mathrm{S} = \emptyset$.
Now by part (2), $(W^c : M)_S = ((W^c)_S : M_S) = (W : M_S)$. Note that $(W^c : M)$ is a primary ideal and $(W^c : M) \cap S \subseteq \mathrm{Zd}_{\mathrm{R}}(\frac{\mathrm{M}}{\mathrm{W^c}}) \cap \mathrm{S} = \emptyset$. Then $(W^c : M) = ((W^c : M)_S)^c = (W : M_S)^c$. Again by part (2), we get $(\mathrm{Zd}_{\mathrm{R}}(\frac{\mathrm{M}}{\mathrm{W^c}}))_{\mathrm{S}} = \mathrm{Zd}_{\mathrm{R_S}}(\frac{\mathrm{M_S}}{(\mathrm{W^c})_{\mathrm{S}}}) = \mathrm{Zd}_{\mathrm{R_S}}(\frac{\mathrm{M_S}}{\mathrm{W}})$, then $((\mathrm{Zd}_{\mathrm{R}}(\frac{\mathrm{M}}{\mathrm{W^c}}))_{\mathrm{S}})^{\mathrm{c}} = (\mathrm{Zd}_{\mathrm{R_S}}(\frac{\mathrm{M_S}}{\mathrm{W}}))^{\mathrm{c}}$. By Corollary 3.2, $\mathrm{Zd}_{\mathrm{R}}(\frac{\mathrm{M}}{\mathrm{W^c}})$ is a prime ideal, so $\mathrm{Zd}_{\mathrm{R}}(\frac{\mathrm{M}}{\mathrm{W^c}}) = ((\mathrm{Zd}_{\mathrm{R}}(\frac{\mathrm{M}}{\mathrm{W^c}}))_{\mathrm{S}})^{\mathrm{c}}$. Hence $\mathrm{Zd}_{\mathrm{R}}(\frac{\mathrm{M}}{\mathrm{W^c}}) = (\mathrm{Zd}_{\mathrm{R_S}}(\frac{\mathrm{M_S}}{\mathrm{W}}))^{\mathrm{c}}$. □

Corollary 3.6. *Let M be an R-module and S a multiplicatively closed subset of R. Let $\mathcal{P}$ be a prime ideal of R and Q be a proper submodule of M such that $\mathrm{Zd}_{\mathrm{R}}(\frac{\mathrm{M}}{\mathrm{Q}}) \cap \mathrm{S} = \emptyset$. Then Q is a uniformly classical $\mathcal{P}$-quasi-primary submodule of M of order k if and only if Q_S is a uniformly classical $\mathcal{P}_S$-quasi-primary submodule of M_S of order k.*

Corollary 3.7. *Let M be an R-module and S a multiplicatively closed subset of R.*

(1) If M is Noetherian and Q is a (uniformly) classical quasi-primary submodule of M such that $(Q : M) \cap S = \emptyset$, then Q_S is a (uniformly) classical quasi-primary submodule of M_S.

(2) If W is a (uniformly) classical quasi-primary submodule of M_S, then W^c is a (uniformly) classical quasi-primary submodule of M and $(W^c)_S = W$. Furthermore, $(W^c : M) \cap S = \emptyset$.

Proof. (1) Since $(Q : M) \cap S = \emptyset$, then $(Q_S : M_S) \neq R_S$, i.e., $Q_S \neq M_S$. Now the results get by Proposition3.4(1) and Proposition 3.3.
(2) By Lemma 3.1, $(W^c : M) \subseteq \mathrm{Zd}_{\mathrm{R}}(\frac{\mathrm{M}}{\mathrm{W^c}})$, so the results can get by Proposition 3.3 and Proposition 3.5. □

The next example shows that even for a free module of finite rank M, the function $Q \longrightarrow Q_S$ does not define a one-to-one correspondence between (uniformly) classical quasi-primary submodules Q of M with $(Q : M) \cap S = \emptyset$ and (uniformly) classical quasi-primary submodules of M_S.

Example 3.8. (See also [4, Example 1]) Let $M = R \oplus R, Q = P_1 \oplus P_2, Q' = P_1 \oplus R$, and $S = R \setminus P_1$, where P_1 and P_2 are prime ideals of R such that $P_1 \subsetneq P_2$. It is easy to see that Q and Q' are (uniformly) classical quasi-primary submodules of M. Also $Q_S = (P_1)_S \oplus (P_2)_S = (P_1)_S \oplus R_S = Q'_S$ and so $(Q_S)^c = P_1 \oplus R \neq Q$.

Now we try to investigate the existence of (minimal) primary submodules containing a classical quasi-primary submodule.

Proposition 3.9. *Let M be an R-module and N be a (classical) $\mathcal{P}$-quasi-primary submodule of M such that for some $x \in M$, $\sqrt{(N : x)} = \mathcal{P}$. Then there exists a $\mathcal{P}$-primary submodule Q of M containing N.*

Proof. By the definition $\sqrt{(Q : M)} = \mathcal{P}$. Now, consider the following set:

$$T = \{C \mid N \subseteq C, C \text{ is a submodule of } M \text{ and } \sqrt{(C : x)} = \mathcal{P}\}.$$

By Zorn's lemma T has a maximal element. Let Q be a maximal element of T. Evidently, $\sqrt{(Q : M)} = \mathcal{P}$. Now we show that Q is a primary submodule of M. Let $ra \in Q$, where $a \in M \setminus Q$ and $r \in R$. Therefore $\sqrt{(Q : M)} \subsetneq \sqrt{(Q + Ra : x)}$. Consider $r_1 \in \sqrt{(Q + Ra : x)} \setminus \mathcal{P}$. Thus $r(r_1)^k x \in rQ + Rra \subseteq Q$, for some $k \in \mathbb{N}$. Thus $r(r_1)^k \in (Q : x) \subseteq \mathcal{P}$, so $r \in \mathcal{P} = \sqrt{(Q : M)}$, i. e., $r^l M \subseteq Q$, for some positive integer l. □

In the following, we will introduce a certain primary submodule containing a classical quasi-primary submodule.

Theorem 3.10. *Let $\mathcal{P}$ be a prime ideal of a ring R and N be a (classical) $\mathcal{P}$-quasi-primary submodule of an R-module M such that for some $x \in M$, $\sqrt{(N : x)} = \mathcal{P}$.*

(1) If $\mathcal{Q}$ is a $\mathcal{P}$-quasi-primary ideal of R such that $\mathcal{Q} \subseteq (N : M)$, then $((\mathcal{Q}M)_{\mathcal{P}})^c$ is a $\mathcal{P}$-primary submodule of M.

(2) $(N_{\mathcal{P}})^c = \{y \in M \setminus N \mid \sqrt{(N : M)} \neq \sqrt{(N : y)}\} \cup N$, and $(N_{\mathcal{P}})^c$ is a $\mathcal{P}$-primary submodule of M and minimal primary over N.

Proof. (1) Let $r \in \sqrt{((N_{\mathcal{P}})^c : x)}$. Then $sr^k x \in N$ for some $s \in R \setminus \mathcal{P}$, and $k \in \mathbb{N}$, So $sr^k \in (N : x) \subseteq \mathcal{P}$. Then $r \in \mathcal{P}$, i.e., $\sqrt{((N_{\mathcal{P}})^c : x)} = \sqrt{(N : x)} = \mathcal{P}$. Therefore $(N_{\mathcal{P}})^c \neq M$ and $((\mathcal{Q}M)_{\mathcal{P}})^c \subseteq (N_{\mathcal{P}})^c$, so $((\mathcal{Q}M)_{\mathcal{P}})^c$ is a proper submodule of M. Now we show that $((\mathcal{Q}M)_{\mathcal{P}})^c$ is a $\mathcal{P}$-primary submodule of M.

Since $\mathcal{Q} \subseteq (((\mathcal{Q}M)_{\mathcal{P}})^c : M) \subseteq (((\mathcal{Q}M)_{\mathcal{P}})^c : x) \subseteq ((N_{\mathcal{P}})^c : x)$, therefore $\mathcal{P} \subseteq \sqrt{(((\mathcal{Q}M)_{\mathcal{P}})^c : M)} \subseteq \sqrt{((N_{\mathcal{P}})^c : x)} = \mathcal{P}$. So $\sqrt{(((\mathcal{Q}M)_{\mathcal{P}})^c : M)} = \mathcal{P}$.

Now let $ra \in ((\mathcal{Q}M)_{\mathcal{P}})^c$, where $r \in R \setminus \mathcal{P}$ and $a \in M$. So $sra = y$, for some $s \in R \setminus \mathcal{P}$ and $y \in \mathcal{Q}M$. Therefore $\frac{a}{1} = \frac{y}{sr} \in (\mathcal{Q}M)_{\mathcal{P}}$. Hence $a \in ((\mathcal{Q}M)_{\mathcal{P}})^c$.

(2) Let $N' = \{y \in M \setminus N \mid \sqrt{(N : M)} \neq \sqrt{(N : y)}\}$. If $y \in (N_{\mathcal{P}})^c \setminus N$, then $sy \in N$, for some $s \in R \setminus \mathcal{P}$, so $s \in (N : y) \setminus \sqrt{(N : M)}$, i.e., $y \in N'$. Now suppose $z \in N'$, then $z \notin N$ and $\sqrt{(N : M)} \subsetneq \sqrt{(N : z)}$. Let $s_0 \in \sqrt{(N : z)} \setminus \sqrt{(N : M)}$. Then $\frac{z}{1} = \frac{s_0^k z}{s_0^k} \in N_{\mathcal{P}}$, for some $k \in \mathbb{N}$. So $z \in (N_{\mathcal{P}})^c$. Obviously, $N \subseteq (N_{\mathcal{P}})^c$, Whence $(N_{\mathcal{P}})^c = N' \cup N$.

Evidently $\mathcal{P} = \sqrt{(N : M)} \subseteq \sqrt{((N_{\mathcal{P}})^c : M)} \subseteq \sqrt{((N_{\mathcal{P}})^c : x)} = \mathcal{P}$, i.e., $\sqrt{((N_{\mathcal{P}})^c : M)} = \mathcal{P}$.

Now let $ra \in (N_{\mathcal{P}})^c$, where $r \in R$ and $a \in M \setminus (N_{\mathcal{P}})^c$. Then $sra \in N$ for some $s \in R \setminus \mathcal{P}$, so $sr \in (N : a)$. On the other hand, since $a \notin (N_{\mathcal{P}})^c = N' \cup N$, $\sqrt{(N : a)} = \mathcal{P}$. Thus $sr \in \mathcal{P}$, so $r \in \mathcal{P}$. Therefore $(N_{\mathcal{P}})^c$ is a $\mathcal{P}$-primary submodule of M.
Finally, if $N \subseteq L \subseteq (N_{\mathcal{P}})^c$, where L is a primary submodule of M, then $(N_{\mathcal{P}})^c \subseteq (L_{\mathcal{P}})^c = L$, so $L = (N_{\mathcal{P}})^c$. Consequently $(N_{\mathcal{P}})^c$ is a minimal primary submodule over N. □

Corollary 3.11. *Let M be an R-module and N a classical $\mathcal{P}$-quasi-primary submodule of M. If M is finitely generated, then $(N_{\mathcal{P}})^c$ is a $\mathcal{P}$-primary submodule of M and a minimal primary submodule over N.*

Proof. Suppose $\{x_1, x_2, x_3, ..., x_k\}$ is a generating set of M. Therefore $\sqrt{(Q : M)} = \cap_{i=1}^{k} \sqrt{(Q : x_i)}$. Therefore by Proposition 2.4, $\sqrt{(Q : M)} = \sqrt{(Q : x_i)}$, for some $i, 1 \leq i \leq k$. Now apply Theorem 3.10. □

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