ON NEW CLASSES OF MULTICONE GRAPHS DETERMINED BY THEIR SPECTRUMS

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ABSTRACT. A multicone graph is defined to be join of a clique and a regular graph. A graph G is cospectral with graph H if their adjacency matrices have the same eigenvalues. A graph G is said to be determined by its spectrum or DS for short, if for any graph H with $Spec(G) = Spec(H)$, we conclude that G is isomorphic to H. In this paper, we present new classes of multicone graphs that are DS with respect to their spectrums. Also, we show that complement of these graphs are DS with respect to their adjacency spectrums. In addition, we show that graphs cospectral with these graphs are perfect. Finally, we find automorphism group of these graphs and one conjecture for further researches is proposed.

1. INTRODUCTION

In this paper, we are concerned only with finite undirected simple graph (loops and multiple edge are not allowed). All terminology and notation on graphs not defined here can be found

in [2, 5, 9]. Let $\Gamma = (V, E)$ be a simple graph, where V is the set of vertices and E is the set of edges of Γ. An edge joining the vertices u and v is denoted by $\{u, v\}$. Permutation of a set Ω is a bijection from Ω to itself. $\text{Sym}(\Omega)$ denotes the set of all permutations of Ω. S_n denotes the symmetric group $sym(\{1, 2, ..., n\})$. Automorphism of a simple graph $\Gamma = (V, E)$ is a permutation of the vertices of Γ which preserves the relation of adjacency; that is, a bijection $\pi : V \longrightarrow V$ such that $\{u, v\} \in E$ if and only if $\{\pi(u), \pi(v)\} \in E$. Under composition, the automorphisms form a group, called the automorphism group (or symmetry group) of Γ, and this is denoted by $Aut(\Gamma)$. In most situations it is very difficult to determine the automorphism group of a graph and this has been the subject of many research papers. Some of recent works appear in references [7, 8, 9]. The complement of a graph G, denoted by $\overline{G}$, is the graph on the vertex set of G such that two vertex of $\overline{G}$, are adjacent if an only if they are not adjacent in G. A graph and its complement have the same automorphisms. The automorphism group of the complete graph K_n and the empty graph $\overline{K_n}$ is the symmetric group S_n.The union of two vertex disjoint graphs G_1 and G_2 denoted by $G_1 \cup G_2$, is the graph whose vertex (respectively, edge) set is the union of vertex (respectively, edge) sets of G_1 and G_2. The join of two vertex disjoint graphs G_1 and G_2 is the graph obtained from $G_1 \cup G_2$ by joining each vertex in G_1 with every vertex in G_2. It is denoted by $G_1 \bigtriangledown G_2$. The group G is called a semidirect product of N by Q, denoted by $G = N \rtimes Q$, if G contains subgroups N and Q such that (i) $N \trianglelefteq Q(N$ is a subgroup normal of G); (ii)$NQ = G$; (iii) $N \bigcap Q = 1$. If Γ and Δ are nonempty sets then we write $Fun(\Gamma, \Delta)$ to denote the set of all functions Γ into Δ. In the case when K is a group, we can turn $Fun(\Gamma, \Delta)$ into a group by defining a product "pointwise": $fg(\gamma) = f(\gamma)g(\gamma)$, for all f, $g \in Fun(\Gamma, \Delta)$ and $\gamma \in \Gamma$. and the product on the right is in K. In the case that Γ is finite of size m, say $\Gamma = \{\gamma_1, \gamma_2, ..., \gamma_m\}$ then the group $Fun(\Gamma, \Delta)$ is isomorphic to K^m (a direct product of m copies of K) via the isomorphism. Let K and H be groups and suppose H acts on the nonempty set Γ. Then the wreath product of K by H with respct to this action is defined to be the semidirect $Fun(\Gamma, K) \rtimes H$ where H acts on the group $Fun(\Gamma, \Delta)$ via $f^x(\gamma) = f(\gamma^{x^{-1}})$ for all $f \in \text{Fun}(\Gamma, \Delta)$, $\gamma \in K$ and $x \in H$. We denote this group by $K \underset{\Gamma}{wr} H$.

Let $A(G)$ denotes the $(0, 1)$-adjacency matrix of graph G. The characteristic polynomial of G is $\det(\lambda I - A(G))$, and is denoted by $P_G(\lambda)$. The roots of $P_G(\lambda)$ are called the adjaceny eigenvalues of G and since $A(G)$ is real and symmetric, the eigenvalues are real numbers. If G has n vertices, then it has n eigenvalues in descending order as $\lambda_1 \geqslant \lambda_2 \geqslant ... \geqslant \lambda_n$. Let $\lambda_1, \lambda_2, ..., \lambda_s$ be the distinct eigenvalues of G with multiplicity $m_1, m_1, ..., m_s$, respectively. The multi-set $Spec(G) = \{\lambda_1^{m_1}, \lambda_2^{m_2}, ..., \lambda_s^{m_s}\}$ of eigenvalues of $A(G)$ is called the adjacency spectrum of G. For two graphs G and H, if $Spec(G) = Spec(H)$, we say G and H are cospectral with respect to adjacency matrix. A graph H is said to be determined by its spectrum or DS for short, if for a graph H with $Spec(G) = Spec(H)$, one has G isomorphic to H. Let G

be a graph with adjacency matrix A and D be the diagonal matrix of vertex degrees for G. The matrix $SL(G) = D + A$ and $L(G) = D - A$ are known as signless Laplacian matrix and Laplacian matrix for G, respectively.

2. Preliminaries

Lemma 2.1. [1, 15] *Let G be a graph. For the adjacency matrix and Laplacian matrix, the following can be obtained from the spectrum:*

(i) The number of vertices,

(ii) The number of edges.

For the adjacency matrix, the following follows from the spectrum

(iii) The number of closed walks of any length.

(iv) Being regular or not and the degree of regularity.

(v) Being bipartite or not.

For the Laplacian matrix, the following follows from the spectrum

(vi) The number of spanning tree.

(vii) The number of component.

(viii) The sum of squares of degrees of vertices.

Theorem 2.2. [4] *If G_1 is r_1-regular with n_1 vertices, and G_2 is r_2-regular with n_2 vertices, then the characteristic polynomial of the join $G_1 \triangledown G_2$ is given by:*

$$P_{G_1 \triangledown G_2}(x) = \frac{P_{G_1} P_{G_2}(x)}{(x-r_1)(x-r_2)}((x - r_1)(x - r_2) - n_1 n_2).$$

Proposition 2.3. [17] *Let G be a disconnected graph that is determined by the Laplacian spectrum. Then the cone over G, the graph H;that is, obtained from G by adding one vertex that is adjacent to all vertices of G, is also determined by its Laplacian spectrum.*

Theorem 2.4. [19] *A graph has exactly one positive eigenvalue if and only if its non-isolated vertices form a complete multipartite graph.*

Theorem 2.5. [1] *Let G be a simple graph with n vertices and m edges. Let $\delta = \delta(G)$ be the minimum degree of vertices of G and $\varrho(G)$ be the spectral radius of the adjacency matrix of G. Then*

$$\varrho(G) \leq \frac{\delta - 1}{2} + \sqrt{2m - n\delta + \frac{(\delta+1)^2}{4}}$$

Equality holds if and only if G is either a regular graph or a bidegreed graph in which each vertex is of degree either δ or $n - 1$.

Lemma 2.6. [3] *Let Γ be a connected non-regular graph with three distinct eigenvalues $\theta_0 > \theta_1 > \theta_2$. Then the following hold:*

(i) Γ has diameter two.

(ii) If θ_0 is not an integer, then Γ is complete bipartite.

(iii) $\theta_1 \geq 0$ with equality if and only if Γ is complete bipartite.

(iv) $\theta_2 \leq -\sqrt{2}$ with equality if and only if Γ is the path of length 2.

Theorem 2.7. [13, 14] *Let G and G be two graphs with Laplacian spectrum $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$ and $\mu_1 \geq \mu_2 \geqslant ... \geq \mu_m$, respectively. Then Laplacian spectra of $\overline{G}$ and $G \bigtriangledown H$ are $n - \lambda_1, n - \lambda_2, ..., n - \lambda_{n-1}, 0$ and $n + m, m + \lambda_1, ..., m + \lambda_{n-1}, n + \mu_1, ..., m + \mu_{m-1}, 0$, respectively.*

Lemma 2.8. [14] *Let G be a graph on n vertices. Then n is Laplacian eigenvalue of G if and only if G is the join of two graphs.*

Proposition 2.9. [11] *For a graph G, the following statements are equivalent:*

(i) G is d-regular.

(ii) $\varrho(G) = d_G$, the average vertex degree.

(iii) G has $v = (1, 1, ..., 1)^t$ as an eigenvector for $\varrho(G)$.

Proposition 2.10. [4] *Let $G - j$ be the graph obtained from G by deleting the vertex j and all edges containing j. Then $P_{G-j} = P_G(x) \sum_{i=1}^{m} \frac{\alpha_{ij}^2}{x - \mu_i}$, where m is the number of distinct eigenvalues of graph G.*

Theorem 2.11. [6] *Suppose G_1 and G_2 are two graphs. If $H_1 \leq Aut(G_1)$ and $H_2 \leq Aut(G_2)$, then $H_1 \times H_2 \leq Aut(G_1 \times G_2)$. Also, if $\mid d(x_i) - d(x_j) \mid \neq \mid n_1 - n_2 \mid$, $i = 1, 2, ..., n_1$ and $j = 1, 2, ..., n_2$ then $Aut(G_1 \bigtriangledown G_2) = Aut(G_1) \times Aut(G_2)$.*

Lemma 2.12. [11] *Let $G \neq K_1$ be connected with $P(G) = \sum_{i=0}^{n} a_i t^{n-i}$ and $\lambda = \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n = \Lambda$, where P_G is characteristic polynomial of graph G and λ_i $(1 \leq i \leq n)$ is eigenvalue of G. The following are equivalent:*

(i) G is bipartite.

(ii) $a_{2i-1} = 0$ for all $1 \leq i \leq \lceil \frac{n}{2} \rceil$.

(iii) $\lambda_i = -\lambda_{n+1-i}$ for $1 \leq i \leq n$.

(iv) $\Lambda = -\lambda$.

Moreover, $m(\lambda_i) = m(-\lambda_i)$, where $m(\lambda_i)$ denotes the multiplicity of λ_i.

Theorem 2.13. [2] *The automorphism group of n disjoint copies of a graph Γ is $Aut(n\Gamma) = S_n[Aut(\Gamma)] = Aut(\Gamma) \underset{I}{wr} S_n$, where $I = \{1, 2, ..., n\}$.*

In the following, we always suppose that w and n are natural numbers. Also, $CP(n)$ denotes coktail party graph; that is, $CP(n) = K_{\underbrace{2, 2, ..., 2}_{n\,times}}$.

3. Main Results

The purpose of this section is to show multicone graphs $K_w \triangledown CP(n)$ are DS with respect to their adjacency spectrums.

3.0.1. *Connected graphs cospectral with multicone graphs $K_w \triangledown CP(n)$.*

Proposition 3.1. *Let G be a graph. If $Spec(G) = spec(K_w \triangledown CP(n))$, then $Spec(G) = \left\{ \left[\frac{\Omega+\sqrt{\Omega^2-4\Gamma}}{2}\right]^1, [-1]^{w-1}, [0]^n, [-2]^{n-1}, \left[\frac{\Omega-\sqrt{\Omega^2-4\Gamma}}{2}\right]^1 \right\}$, where $\Omega = 2n + w - 3$ and $\Gamma = (w-1)(2n-2) - 2nw$.*

Proof. Theorem 2.2 together with $Spec(CP(n)) = \left\{ [2n-2]^1, [0]^n, [-2]^{n-1} \right\}$ complete the proof. $\square$

In the next lemma we show that if $n \neq 1$, then any graph cospectral with multicone graph $K_w \triangledown CP(n)$ must be bidegreed.

Lemma 3.2. *If G be a graph and $Spec(G) = Spec(K_w \triangledown CP(n))$, then G is either regular or bidegreed, in which case any vertex of G is of degree $2n + w - 2$ or $2n + w - 1$.*

Proof. If $n = 1$, then G is regular. Hence we suppose that $n \neq 1$. It is obvious that G cannot be regular, since regularity of a graph can be determined by its spectrum. By contrary, we suppose that the sequence of degrees of vertices of graph G consists of at least three number. Hence the equality in Theorem 2.5 cannot happen for any δ. But, if we put $\delta = w+2n-2$, then the equality in Theorem 2.5 holds. So, G must be bidegreed. Now, we show that $\Delta = \Delta(G) = w + 2n - 1$. By contrary, we suppose that $\Delta < w + 2n - 1$. Therefore, the equality in theorem 2.5 cannot hold for any δ. But, if we put $\delta = w + 2n - 2$, then this equality holds. This is a contradiction and so $\Delta = w-1+2n$. Now, $\delta = w+2n-2$, since G is bidegreed and G has $w+2n$ vertices, $\Delta = w-1+2n$ and $w(w-1+2n)+2n(w+2n-2) = w\Delta+2n(w+2n-2) = \sum_{i=1}^{2n+w} \deg v_i$. This completes the proof. $\square$

In the next lemma we show that multicone graphs $K_1 \triangledown CP(n)$ are DS.

Lemma 3.3. *Any graph cospectral with multicone graph $K_1 \triangledown CP(n)$ is DS.*

Proof. By Lemma 3.2, G has one vertex of degree $2n$, say j. Now, $P_G = (x - \mu_4)^n(x + 2)^{n-1}(x^2 - (2n-2)x - 2n)$. On the other hand, from Theorem 2.10, it follows that $P_{G-j} = (x - \mu_4)^{n-1}(x - \mu_3)^{n-2}[\alpha_{1j}^2 A + \alpha_{2j}^2 B + \alpha_{3j}^2 C + \alpha_{4j}^2 D]$, where
$A = (x - \mu_2)(x - \mu_3)(x - \mu_4)$,
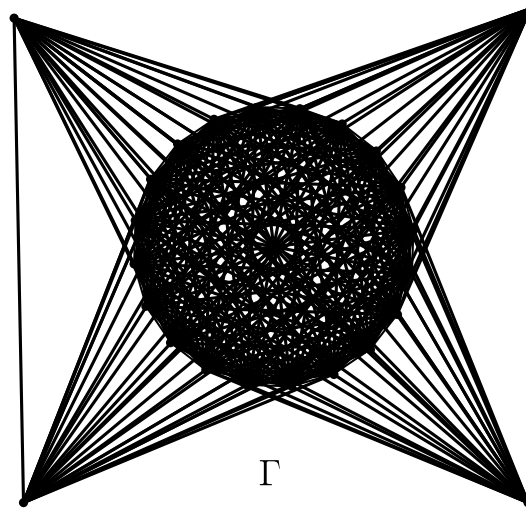$B = (x - \mu_1)(x - \mu_3)(x - \mu_4)$,

FIGURE 1. **Multicone graph $\Gamma = K_{20} \bigtriangledown CP(2)$.**

$C = (x - \mu_1)(x - \mu_2)(x - \mu_3)$,

$D = (x - \mu_1)(x - \mu_2)(x - \mu_3)$,

where $\mu_1 = \dfrac{n - 1 + \sqrt{n^2 + 1}}{2}$, $\mu_2 = \dfrac{n - 1 - \sqrt{n^2 + 1}}{2}$, $\mu_3 = -2$ and $\mu_4 = 0$.

We know that P_{G-j} has $2n$ roots. Hence we have:

$x + y + 2n - 2 = (2 - n)\mu_3$,

$x^2 + y^2 + (2n - 2)^2 = 4n^2 - 4n - [(n - 2).\mu_2^2]$,

$x^3 + y^3 + (2n - 2)^3 = 48\binom{n}{3} - [(n - 2).\mu_2^3]$,

where x and y are eigenvalues of P_{G-j}. If we solve the above equations, we obtain $x = 0$ and $y = -2$. So, $Spec(G - j) = \left\{ [2n - 2]^1, [0]^n, [-2]^{n-1} \right\}$. We know that graph $G - j$ is a regular graph such that degree of its regularity is $2n - 2$. Theorem 2.4 implies that G is a complete multipartite graph. Hence $G - j \cong K_{\underbrace{2, 2, ..., 2}_{n \; times}} = CP(n)$ and so $G \cong K_1 \bigtriangledown CP(n)$. This completes the proof. $\square$

Hitherto, we have shown multicone graphs $K_1 \bigtriangledown CP(n)$ are DS. The natural questions is; what happens for multicone graphs $K_w \bigtriangledown CP(n)$? We answer this question in the next theorem.
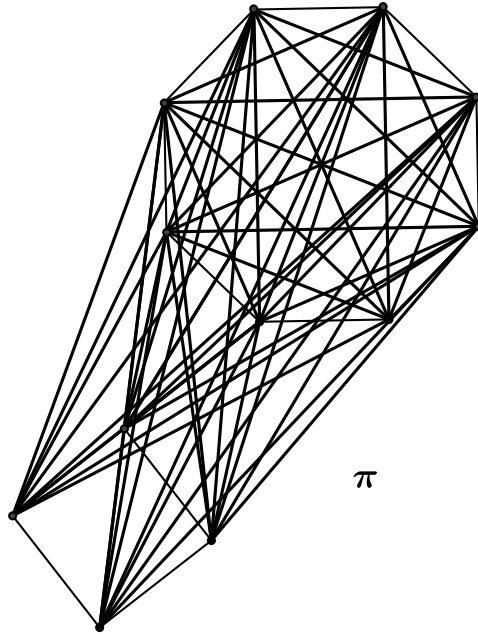
FIGURE 2. Multicone graph $\pi = K_8 \bigtriangledown C_4$.

Theorem 3.4. *Multicone graphs $K_w \bigtriangledown CP(n)$ are DS with respect to their adjacency spectrums.*

Proof. We solve the problem by induction on w. If $w = 1$, there is nothing for proof. Let the claim be true for w; that is, if $Spec(G_1) = Spec(K_w \bigtriangledown CP(n))$, then $G_1 \cong K_w \bigtriangledown CP(n)$, where G_1 is a graph. We show that $Spec(G) = Spec(K_{w+1} \bigtriangledown CP(n))$ follows that $K_w \bigtriangledown CP(n)$, where G is a graph. By lemma 3.2, $G = K_1 \bigtriangledown G_1$. This completes the proof. □

Corollary 3.5. *Multicone graphs $K_w \bigtriangledown C_4 = K_w \bigtriangledown CP(2)$ are DS with respect to their adjacency spectrums.*

In the following section we show that multicone graphs $K_w \bigtriangledown CP(n)$ are DS with respect to their Laplacian spectrums.

4. CONNECTED GRAPHS COSPECTRAL WITH MULTICONE GRAPH $K_w \bigtriangledown CP(n)$ WITH RESPECT TO LAPLACIAN SPECTRUMS

Theorem 4.1. *Multicone graphs $K_w \bigtriangledown CP(n)$ are DS with respect to their Laplacian spectra.*

Proof. We solve the problem by induction on w. If $w = 1$, there is nothing to prove. Let the claim be true for w; that is, if $Spec(L(G_1)) = Spec(L(K_w \bigtriangledown CP(n))) = \left\{ [w+2n]^{w-1+n}, [w+2n-2]^n, [0]^1 \right\}$, then $G_1 \cong K_w \bigtriangledown CP(n)$, where G_1 is a graph. We
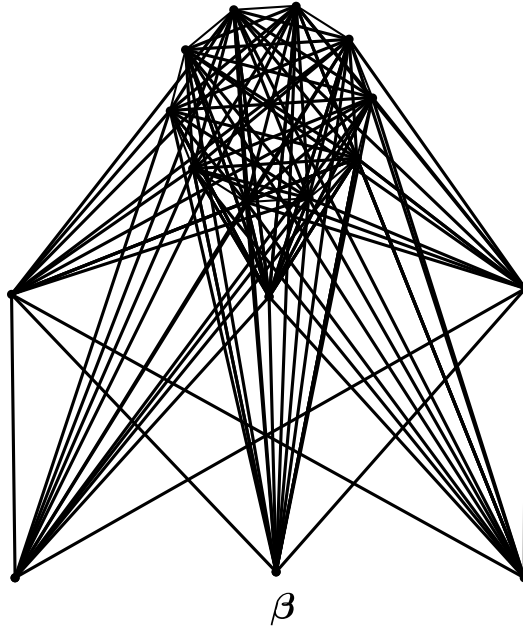
FIGURE 3. Multicone graph $\beta = K_{10} \bigtriangledown CP(3)$.

show that from $Spec(K_{w+1} \bigtriangledown CP(n)) = \left\{[w+2n+1]^{w+n}, [w+2n-1]^n, [0]^1\right\}$ follows that $G \cong K_{w+1} \bigtriangledown CP(n)$. We conclude from Lemma 2.8 that G_1 and G are join of two graphs. On the other hand G has one vertex and $2n+w$ edges more than G_1. Hence G has one vertex of degree $2n+w$, say j. So, $Spec(L(G-j)) = Spec(K_w \bigtriangledown CP(n))$. Hence $G - j \cong K_w \bigtriangledown CP(n)$ and so $G \cong K_1 \bigtriangledown (K_w \bigtriangledown CP(n))$. This completes the proof. □

Corollary 4.2. *Multicone graphs $K_w \bigtriangledown C_4$ are DS with respect to their Laplacian spectrums.*

In the next section we show that split graphs $\overline{K_w \bigtriangledown CP(n)}$ are DS with respect to their adjacency spectrums.

5. SPLIT GRAPHS WHOSE ARE COSPECTRAL WITH COMPLEMENT OF MULTICONE GRAPHS
$$K_w \bigtriangledown CP(n).$$

Proposition 5.1. *Let G be cospectral with adjacency spectrum of complement of multicone graph $K_w \bigtriangledown CP(n)$. Then $Spec(G) = \{[1]^n, [0]^w, [-1]^n\}$.*

Proof. Straightforward. □

Theorem 5.2. *If $Spec(G) = Spec(\overline{K_w \bigtriangledown CP(n)})$, then $G \cong \overline{K_w \bigtriangledown CP(n)}$.*
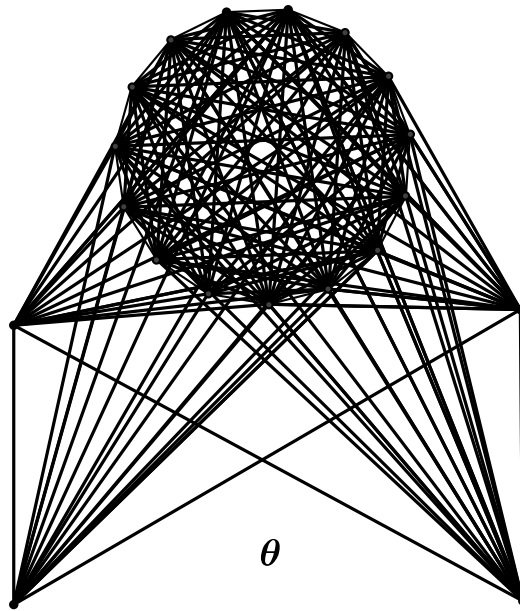
FIGURE 4. Multicone graph $\Theta = K_{15} \bigtriangledown CP(2)$

Proof. If $n = 1$, there is nothing to prove. Hence we suppose that $n \neq 1$. It is obvious that G cannot be regular, since regularity of a graph can be determined by its spectrum. Firstly, we show that G is disconnected. By contrary, we suppose that G is connected. So, by Lemma 2.6, G must be a complete bipartite graph and so $G \cong K_2$. This is a contradiction. Hence $G = G_1 \cup G_2 \cup ... \cup G_k$, where G_i is a connected component of G and $1 \leq i \leq k$. Now, we show that G_i cannot have three distinct eigenvalues. By contrary, we suppose that G_i has three distinct eigenvalues. We show that G_i must be regular. By contrary, we assume that G_i is non-regular. Hence Theorem 2.6 implies that G_i is a complete bipartite graph. So, $G_i \cong K_2$. This is a contradiction. Therefore, G_i is a regular graph and also multiplicity of eigenvalue 1, is 1, since G_i is connected. Now, from Lemma 2.6, it follows that G_i is a complete multipartite and so $G_i = mK_2$, where $m \geqslant 1$. This implies that $G_i \cong K_2$. This is a contradiction. Therefore G_i has one or two eigenvalue(s). Hence, any of connected components of G is either K_1 or a complete graph. Hence $G \cong wK_1 \cup nK_2$. This completes the proof. $\square$

In the following, we show that any graph cospectral with multicone graph $K_w \bigtriangledown CP(n)$ must be perfect. Also, we show that any graph cospectral with multicone graph $K_w \bigtriangledown CP(n)$ with respect to Laplacian spectrum is perfect. In addition, we prove that any graph cospectral

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with split graph $\overline{K_w \triangledown CP(n)}$ must be perfect. Finally, we obtain automorphism group of multicone graphs $K_w \triangledown CP(n)$.

6. Some of algebraic properties about multicone graphs $K_w \triangledown CP(n)$.

Suppose $\chi(G)$ and $\omega(G)$ are chromatic number and clique number of graph G, respectively. A graph is **perfect** if $\chi(H) = \omega(H)$ for every induced subgraph H of G. It is proved that a graph G is perfect if and only if G is Berge; that is, it contains no odd hole or antihole as induced subgraph, where odd hole and antihole are odd cycle, C_m for $m \geq 5$, and its complement, respectively. Also, in 1972 Lovásaz proved that, A graph is perfect if and only if its complement is perfect [1].

Theorem 6.1. *Let graph G be cospectral with multicone graph $K_w \triangledown CP(n)$. Then G and $\overline{G}$ are perfect.*

Proof. Straightforward. □

Theorem 6.2. *Let $Spec(L(G)) = Spec(L(K_w \triangledown CP(n)))$. Then G and $\overline{G}$ are perfect.*

Proof. Straightforward. □

Theorem 6.3. *Let graph G be cospectral with graph $\overline{K_w \triangledown CP(n)}$. Then G and $\overline{G}$ are perfect.*

Proof. Straightforward. □

Proposition 6.4. $Aut(K_w \triangledown CP(n)) \cong S_w \times [S_2 \underset{I}{wr} S_n]$, *where* $I = \{1, 2, ..., n\}$.

Proof. It is clear that $Aut(CP(n)) \cong Aut(nK_2)$. Now, Theorem 2.11 together with Theorem 2.13 complete the proof. □

In the following, we pose one conjecture.

7. Corollaries and Conjecture

Corollary 7.1. *Let G be a graph. The following statements are equivalent:*

(i) $G \cong K_w \bigtriangledown CP(n)$.

(ii) $Spec(G) = Spec(K_w \bigtriangledown CP(n))$.

(iii) $Spec(L(G)) = Spec(L(K_w \bigtriangledown CP(n)))$.

Corollary 7.2. *Let G be a graph. The following statements are equivalent:*

(i) $G \cong \overline{K_w \bigtriangledown CP(n)}$.

(ii) $Spec(G) = Spec(\overline{K_w \bigtriangledown CP(n)})$.

(iii) $Spec(L(G)) = Spec(L(\overline{K_w \bigtriangledown CP(n)})$.

Conjecture 7.3. *Multicone graphs $K_w \bigtriangledown CP(n)$ are DS with respect to their signless Laplacian spectrums.*

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