Journal of Algebraic Structures and Their Applications
ISSN: 2382-9761
www.as.yazd.ac.ir

Algebraic Structures and Their Applications Vol. 2 No. 1 (2015), pp 1-9.

DOMINATION NUMBER OF TOTAL GRAPH OF MODULE

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Communicated by S. Alikhani

Abstract. Let R be a commutative ring and M be an R-module with $T(M)$ as subset, the set of torsion elements. The total graph of the module denoted by $T(\Gamma(M))$, is the (undirected) graph with all elements of M as vertices, and for distinct elements $n, m \in M$, the vertices n and m are adjacent if and only if $n + m \in T(M)$. In this paper we study the domination number of $T(\Gamma(M))$ and investigate the necessary conditions for being $\mathbb{Z}_n$ as module over $\mathbb{Z}_m$ and we find the domination number of $T(\Gamma(\mathbb{Z}_n))$.

1. Introduction

The idea of associating a graph to a ring first appears in [4]. For the vertices of the graph, Beck takes all elements of a commutative ring R. Two distinct vertices $x, y \in R$ are adjacent if $xy = 0$. This paper primarily deals with the questions of coloring and the computation of the chromatic number for some rings. Other authors have been motivated by the results of this article to research the interrelations between properties of graphs and rings, the question of the connectivity of the graph, its diameter, radius and other interesting invariants of graphs. Their interpretations in the theory of commutative rings, make Becks paper the founding paper of a new and interesting field of algebra.

MSC(2000): Primary:05C75 , Secondary: 13M05.
Keywords: total graph, domination number, module
Received:7 Mar 2015, Accepted: 4 Jul 2015.
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Of course, there are many ways to associate a graph to a given ring R. The most well-known is certainly the zero-divisor graph $\Gamma(R)$ introduced in [3]. In this paper, the set of vertices consists only of non-zero zero-divisors. The authors show that $\Gamma(R)$ is always connected, of diameter at most 3. Some other investigations into properties of this graph may be found in $[1, 4, 7-10]$.

In [1], the notion of the total graph of a commutative ring $T(\Gamma(R))$ is introduced. The vertices of this graph are all elements of the ring R. Two vertices are adjacent if their sum is a zero-divisor. This graph, unlike the zero-divisor graph, need not be connected. Even in the case when the total graph is connected, its diameter may have arbitrary value$n$, for $n \geq 1$. The structure and the properties of the total graph are thoroughly examined in [1]. We define the total graph of a module in an analogous way. Let R be a commutative ring with identity, $R^* = R - \{0\}$, $Z(R)$ the set of its zero-divisors, and $Z(R)^* = Z(R) - \{0\}$. Let M be an R-module, $M^* = M \setminus \{0\}$, and $T(M) = \{m \in M : rm = 0 \text{ for some } r \in R^*\}$ the set of its torsion elements. We will use $Tof(M) = M - T(M)$ to denote the set of non-torsion elements of M. We define the total graph of a module $T(\Gamma(M)$ as follows:

$V(T(\Gamma(M))) = M; E(T(\Gamma(M))) = \{(m_1, m_2) : m_1 + m_2 \in T(M)\}$

Let $Tof(\Gamma(M))$ be the (induced) subgraph of $T(\Gamma(M))$ with vertices $Tof(M)$, and let $Tor(\Gamma(M))$ be the (induced) subgraph of $T(\Gamma(M))$ with vertices $T(M)$.

2. Definitions and preliminaries

For a graph G, let $\mathrm{V}(G)$ denote the set of vertices, and let $\mathrm{E}(G)$ denote the set of edges. For a graph G and vertex $x \in \mathrm{V}(G)$, the degree of x, denoted by $\deg(x)$, is the number of edges of G incident with x. A complete graph is a graph in which each pair of distinct vertices is joined by an edge. We denote the complete graph with n vertices by K_n. For a nonnegative integer r, an r-partite graph is one whose vertex-set is partitioned into r disjoint parts in such a way that the two end vertices for each edge lie in distinct partitions. A complete r-partite graph is one in which each vertex is joined to every vertex that is not in the same partition. A complete 2-partite graph (also called the complete bipartite graph) with exactly two partitions of size m and n, is denoted by $K_{m,n}$. A clique of a graph G is a complete subgraph of G. A co-clique in a graph G is a set of pairwise nonadjacent vertices. A subgraph H of a graph G is called a spanning subgraph if $\mathrm{V}(H) = \mathrm{V}(G)$. For every nonnegative integer r, a graph G is called r-regular if the degree of each vertex of G is equal to r. A 1-regular spanning subgraph H of G is called a perfect matching of G. Recall that the complement graph of a graph G is denoted by $\bar{G}$ with vertices $\mathrm{V}(G)$, and for distinct $x, y \in \mathrm{V}(\bar{G})$, the vertices x and y are adjacent if and only if $xy \notin \mathrm{E}(G)$.

For a graph $G = (V, E)$, a set S is a dominating set if every vertex in $V \setminus S$ is adjacent to a vertex in S. Also, S is called a total dominating set if every vertex in V is adjacent to a vertex in S. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G. We call a dominating set of cardinality $\gamma(G)$ a $\gamma(G)$-set. Also, the minimum cardinality among the total dominating sets of G is denoted by $\gamma_t(G)$. The study of the domination number has long been a topic of interest both in graph theory and complexity theory. It was first considered by Ore who introduced the concept of minimum dominating sets of vertices in a graph. The dominating set problem concerns testing whether $\gamma(G) \leq k$ for a given graph G and integer k. The problem is a classical NP-complete decision problem in computational complexity theory (see, for example, [7]). Therefore, it is believed that there is no efficient algorithm that finds a smallest dominating set of a given graph. The first volume of the two-volume book by Haynes, Hedetniemi and Slater [8, 9] provides a comprehensive introduction to domination in graphs.

3. Domination number of the total graph of module

Theorem 3.1. [6] *Let M be a module over a commutative ring R such that $T(M)$ is the submodule of M.Then the following hold:*

(1) $Tor(\Gamma(M))$ is a complete (induced) subgraph of $T(\Gamma(M))$ and $Tor(\Gamma(M))$is disjoint from $Tof(\Gamma(M))$.

(2) If$(0 :_R) \neq 0$,then $T(\Gamma(M))$ is a complete graph.

The next theorem gives a complete description of $T(\Gamma(M))$ .We allow α, β to be infinite, then of course $\beta - 1 = (\beta - 1)/2 = \beta$.

Theorem 3.2. [6] *Let M be a module over a commutative ring R such that $T(M)$ is the submodule of M and $|T(M)| = \alpha$ and $|\frac{M}{T(M)}| = \beta$.*

(1) If $2 = 1_R + 1_R \in Z(R)$, then $Tof(\Gamma(M))$ is the union of $\beta - 1$ disjoint K_α's.

(2) If $2 = 1_R + 1_R \notin Z(R)$, then $Tof(\Gamma(M))$ is the union of $(\beta - 1)/2$ disjoint $K_{\alpha,\alpha}$'s

Corollary 3.3. *Let M be a module over a commutative ring R such that $T(M)$ is the submodule of M, then $\gamma(Tor(\Gamma(M))) = 1$.*

Proof. This following directly from theorem 3.1. □

Theorem 3.4. *Let M be a module over a commutative ring R such that $T(M) = M$, then $\gamma(T(\Gamma(M))) = 1$.*

Proof. This is obvious. □

Theorem 3.5. *Let M be a module over a commutative ring R such that $T(M)$ is the submodule and $|T(M)| = \alpha \neq 0$ and $|\frac{M}{T(M)}| = \beta$, then $\gamma(T(\Gamma(M))) = \beta$.*

Proof. We consider two cases for $Z(R)$.

case1: Suppose that $2 = 1_R + 1_R \in Z(R)$. Hence we use from theorem 3.2 ,the graph $Tof(\Gamma(M))$ is the union of $\beta - 1$ disjoint K_{α}'s and we know that $\gamma(K_\alpha) = 1$. Thus $\gamma(Tof(\Gamma(M))) = \beta - 1$ and $\gamma(Tor(\Gamma(M))) = 1$, consequently
$\gamma(T(\Gamma(M))) = \gamma(Tor(\Gamma(M)) \bigcup Tof(\Gamma(M))) = \gamma(Tor(\Gamma(M))) + \gamma(Tof(\Gamma(M)))$
$= 1 + \beta - 1 = \beta$

case2: In the second case we suppose that $2 = 1_R + 1_R \notin Z(R)$, then again we have from theorem 3.2, the graph $Tof(\Gamma(M))$ is the union of $\frac{\beta-1}{2}$ disjoint $K_{\alpha,\alpha}$'s and we know that $\gamma(K_{\alpha,\alpha}) = 2$. So
$\gamma(T(\Gamma(M))) = \gamma(Tor(\Gamma(M))) + \gamma(Tof(\Gamma(M))) = \frac{\beta-1}{2} \times 2 + 1 = \beta - 1 + 1 = \beta.$

□

Theorem 3.6. *Let M be a nonzero module over commutative ring R such that $T(M) = 0$, then $\gamma(T(\Gamma(M))) = \frac{\beta+1}{2}$.*

Proof. According hypothesis $T(M) = 0$, so $|\frac{M}{T(M)}| = |M| = \beta$. Now we show that $Z(R) = 0$. Let $0 \neq x \in Z(R)$, then there exist $0 \neq y \in R$ such that $xy = 0$. Now we consider a element $0 \neq m \in M$, and we have $(xy)m = 0$ and $x(ym) = 0$, then $ym = 0$, since $T(M) = 0$, but from $ym = 0$ we have $y = 0$ or $m = 0$, that is contradiction. Therefore $Z(R) = 0$. So $2 \notin Z(R)$ and the graph $Tof(\Gamma(M))$is the union of $\frac{\beta-1}{2}$ disjoint $K_{1,1}$'s. Therefoe we will have

$$\gamma(T(\Gamma(M))) = \gamma(Tof(\Gamma(M))) + \gamma(Tor(\Gamma(M))) = \frac{\beta-1}{2} \times 1 + 1 = \frac{\beta+1}{2}$$

□

Theorem 3.7. *Let M be a nonzero module over commutative ring R such that $T(M) = M$, then $\gamma(T(\Gamma(M))) = 1$.*

Proof. This following directly from definition. □

4. Domination number of total graph of module $\mathbb{Z}_n$ over ring $\mathbb{Z}_m$

In this section we investigate the domination number of total graph of module $\mathbb{Z}_n$ over ring $\mathbb{Z}_m$. But generally for all $n, m \in \mathbb{N}, \mathbb{Z}_n$ is not module over $\mathbb{Z}_m$. We find the necessary conditio for being module $\mathbb{Z}_n$.

Lemma 4.1. *Let $f : R \to S$ be a ring homomorphism and A be a S-module, then A is a R-module such that for all $a \in A$ and $r \in R$ we define $r * a = f(r)a$.*

Proof. We investigate the property of multiplication. For all $r.s \in R$ and $a.b \in A$,

(1) $(r + s) * a = f(r + s)a = (f(r) + f(s))a = f(r)a + f(s)a = r * a + s * a$
(2) $r * (a + b) = f(r)(a + b) = f(r)a + f(r)b = r * a + r * b$

(3) $(rs) * a = f(rs)a = f(r)f(s)a = f(r)(f(s)a) = f(r)(s * a) = r * (s * a)$

(4) $1_R * a = f(1_R)a = 1_S a = a$

□

Lemma 4.2. *Let $m, n \in \mathbb{N}$ with $gcd(m, n) = d$ and $n/d = n_1$. Then there exists a non-zero ring homomprphism $f : \mathbb{Z}_m \to \mathbb{Z}_n$ if and only if $d \mid k(kn_1 - 1)$ for some $1 \leq k \leq d - 1$.*

Proof. Let $m/d = m_1$. ($\Longrightarrow$) Let $f(1_{\mathbb{Z}_m}) = \overline{a} \in \mathbb{Z}_n$, where $0 \leq a \leq n-1$. Since $\underbrace{1+1+1+\cdots+1}_{m\ items} = 0$, we have $0 = f(\underbrace{1+1+\cdots+1}_{m\ items}) = \underbrace{f(1)+f(1)+\cdots+f(1)}_{m\ items} = \overline{ma}$. Hence $n \mid ma$, and so $dn_1 \mid dm_1 a$, which implies that $n_1 \mid m_1 a$, and since $gcd(n_1, m_1) = 1$, we have $n_1 \mid a$. Thus $a = n_1 k$ for some $k \in F$. As $0 \neq f$, we have $0 \neq a$, and so $1 \leq a = n_1 k < n = n_1 d$. Consequently $1 \leq k \leq d - 1$.

$\overline{a} = f(1) = f(1.1) = f(1).f(1) = \overline{a^2}$, that is $n \mid a^2 - a = a(a-1)$. Hence $dn_1 \mid kn_1(kn_1 - 1)$. Therefore $d \mid k(kn_1 - 1)$, where $1 \leq k \leq d - 1$.

($\Longleftarrow$) Consider $a = kn_1$. As $1 \leq k \leq d - 1$, we have $1 \leq a < n$. Define $f : \mathbb{Z}_m \to \mathbb{Z}_n$, with $f(\overline{s}) = \overline{sa}$, for each $0 \leq s \leq m - 1$. Let $\overline{s_1}, \overline{s_2} \in \mathbb{Z}_m$, where $0 \leq s_1, s_2 \leq m - 1$.

If $\overline{s_1} = \overline{s_2}$, where $s_1 \leq s_2$, then $m \mid s_1 - s_2$. Suppose that $mt = s_1 - s_2$, where $t \in \mathbb{N}$. Then $s_1 a - s_2 a = (s_1 - s_2)a = tma = tm_1 dn_1 k = tm_1 nk$, that is $n \mid (s_1 a - s_2 a)$, and so $f(\overline{s_1}) = \overline{s_1 a} = \overline{s_2 a} = f(\overline{s_2})$. This shows that f is well defined.

Evidently $f(\overline{s_1} + \overline{s_2}) = f(\overline{s_1 + s_2}) = \overline{(s_1 + s_2)a} = \overline{s_1 a} + \overline{s_2 a} = f(\overline{s_1}) + f(\overline{s_2})$.

Note that $d \mid k(kn_1 - 1)$, and so $n = dn_1 \mid kn_1(kn_1 - 1) = a(a-1) = a^2 - a$. Thus $\overline{a} = \overline{a^2}$, in $\mathbb{Z}_n$.

Hence:

$f(\overline{s_1 s_2}) = f(\overline{s_1 s_2}) = \overline{s_1 s_2 a} = \overline{s_1 s_2 a^2} = \overline{s_1 a}.\overline{s_2 a} = f(\overline{s_1})f(\overline{s_2})$. Therefore f is a non-zero ring homomorphism. □

Corollary 4.3. *Let $m, n \in \mathbb{N}$ with $gcd(m, n) = 1$. Then there does not exist any non-zero ring homomprphism $f : \mathbb{Z}_m \to \mathbb{Z}_n$.*

Proof. Since $d = 1$, there does not exist any integer $1 \leq k \leq d - 1$, so the proof is obvious by 4.2. □

Theorem 4.4. *Let $m, n \in \mathbb{N}$. If $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_\ell^{\alpha_\ell}$ is the prime factorization of n, then there exists a non-zero ring homomorphism $f : \mathbb{Z}_m \to \mathbb{Z}_n$ if and only if $p_i^{\alpha_i} | m$ for some $1 \leq i \leq \ell$.*

Proof. Let $gcd(m, n) = d$ and $n/d = n_1$.

($\Longrightarrow$) By 4.2, $d \mid k(kn_1 - 1)$ for some $1 \leq k \leq d - 1$. This shows that $d > 1$, so we can suppose that $d = p_1^{r_1} p_2^{r_2} \cdots p_s^{r_s}$ is the prime factorization of d, for some $1 \leq s \leq \ell$.

If $p_i \mid n_1$ for all $1 \leq i \leq s$, then $gcd(kn_1-1,d)=1$, and from $d \mid k(kn_1-1)$ we get $d \mid k$, and so $d \leq k \leq d-1$, which is a contradiction. Therefore $p_i \not| n_1$, for some $1 \leq i \leq s$. This shows that $p_i^{\alpha_i}|d$, which implies that $p_i^{\alpha_i}|m$.

($\Longleftarrow$) By our assumption $p_i^{\alpha_i}|m$ for some $1 \leq i \leq \ell$, so $p_i^{\alpha_i}|d$, that is $d>1$. Let $d = p_1^{r_1}p_2^{r_2}\cdots p_s^{r_s}$ be the prime factorization of d, for some $1 \leq s \leq \ell$.

We consider the following two cases.

Case 1. $gcd(d,n_1)=1$.

Since $gcd(d,n_1)=1$, the Diophantine equation $n_1x+dy=1$ has integer solutions. Suppose (x_0,y_0) is a solution of this equation. By the division algorithm $x_0=qd+k$, for some integers q and k with $0 \leq k \leq d-1$. If $k=0$, then $d \mid x_0$, and so $d \mid (n_1x_0+dy_0)=1$, that is $d=1$, which is a contradiction, hence $1 \leq k \leq d-1$. We have $k \equiv x_0(mod\ d)$ and so $kn_1 \equiv n_1x_0(mod\ d)$ and evidently $0 \equiv dy_0(mod\ d)$; therefore $kn_1 \equiv (n_1x_0+dy_0)(mod\ d)$, that is $kn_1 \equiv 1(mod\ d)$ and so $d \mid k(kn_1-1)$, where $1 \leq k \leq d-1$. Therefore the proof is given by 4.2.

Case 2. $gcd(d,n_1)>1$. By our assumption $p_i^{\alpha_i}|m$ for some $1 \leq i \leq s$. Without loss of generality we can assume that $p_1^{\alpha_1}|m$. This shows that $r_1=\alpha_1$.

Now set $n'=p_1^{r_1}$ and $gcd(m,n')=d'$ and $n'/d'=n_1'$. Then $d'=p_1^{r_1}$ and $n_1'=1$ and evidently $gcd(d',n_1')=1$, thus by Case 1, there exists a non-zero ring homomorphism $g:\mathbb{Z}_m \to \mathbb{Z}_{n'}$. Note that $n=n'p_2^{\alpha_2}p_3^{\alpha_3}\cdots p_\ell^{\alpha_\ell}$, thus we have the ring isomorphism $\mathbb{Z}_n \cong \mathbb{Z}_{n'} \oplus \mathbb{Z}_{p_2^{\alpha_2}} \oplus \mathbb{Z}_{p_3^{\alpha_3}} \oplus \cdots \mathbb{Z}_{p_\ell^{\alpha_\ell}}$. Consequently there exists a natural ring monomorphism $\ell:\mathbb{Z}_{n'} \to \mathbb{Z}_n$, which implies the non-zero ring homomorphism $f=\ell \circ g:\mathbb{Z}_m \to \mathbb{Z}_n$. □

Let G be an abelian group and R be a ring. Then evidently G is an R-module, by the multiplication $rx=0, \forall r \in R, \forall x \in G$. In this case, we say G is an R-module by the trivial multiplication.

Lemma 4.5. *Let $m,n \in \mathbb{N}$ with $m<n$ and $gcd(m,n)=d$ and $n/d=n_1$. Then $\mathbb{Z}_n$ is a non trivial $\mathbb{Z}_m$-module if and only if $d \mid k(kn_1-1)$ for some $1 \leq k \leq d-1$.*

Proof. ($\Longrightarrow$) Consider $\overline{1}_{\mathbb{Z}_m} \in \mathbb{Z}_m$ and $\overline{1}_{\mathbb{Z}_n} \in \mathbb{Z}_n$. Since $\underbrace{\overline{1}_{\mathbb{Z}_n}+\overline{1}_{\mathbb{Z}_n}+\overline{1}_{\mathbb{Z}_n}+\cdots\overline{1}_{\mathbb{Z}_n}}_{m\ items}=0$, we have $0=(\underbrace{\overline{1}_{\mathbb{Z}_n}+\overline{1}_{\mathbb{Z}_n}+\overline{1}_{\mathbb{Z}_n}+\cdots\overline{1}_{\mathbb{Z}_n}}_{m\ items}).\overline{1}_{\mathbb{Z}_n}=\underbrace{\overline{1}_{\mathbb{Z}_m}.\overline{1}_{\mathbb{Z}_n}+\overline{1}_{\mathbb{Z}_m}.\overline{1}_{\mathbb{Z}_n}+\cdots+\overline{1}_{\mathbb{Z}_m}.\overline{1}_{\mathbb{Z}_n}}_{m\ items}=m(\overline{1}_{\mathbb{Z}_m}.\overline{1}_{\mathbb{Z}_n})$. Thus in the group $\mathbb{Z}_n$, we have $o(\overline{1}_{\mathbb{Z}_m}.\overline{1}_{\mathbb{Z}_n})|m$. Also as $\overline{1}_{\mathbb{Z}_m}.\overline{1}_{\mathbb{Z}_n} \in \mathbb{Z}_n$, obviously $o(\overline{1}_{\mathbb{Z}_m}.\overline{1}_{\mathbb{Z}_n})|n$, therefore $o(\overline{1}_{\mathbb{Z}_m}.\overline{1}_{\mathbb{Z}_n})|gcd(m,n)=d$.

Let $\overline{1}_{\mathbb{Z}_m}.\overline{1}_{\mathbb{Z}_n}=\overline{t}$, where $0 \leq t \leq n-1$. Then we know that $o(\overline{1}_{\mathbb{Z}_m}.\overline{1}_{\mathbb{Z}_n})=n/gcd(n,t) \mid d$. Thus $n_1d=n \mid d \times gcd(n,t)$, and so $n_1 \mid gcd(n,t) \mid t$, that is $t=n_1.k$, for some $0 \leq k \leq d-1$. Hence

$\overline{1}_{\mathbb{Z}_m}.\overline{1}_{\mathbb{Z}_n}=\overline{n_1k}$, for some $0 \leq k \leq d-1$.

For every $\overline{r} \in \mathbb{Z}_m$, we have $\overline{r}.\overline{1}_{\mathbb{Z}_n} = (\underbrace{\overline{1}_{\mathbb{Z}_m} + \overline{1}_{\mathbb{Z}_m} + \cdots + \overline{1}_{\mathbb{Z}_m}}_{r \ items}).\overline{1}_{\mathbb{Z}_n} = \overline{1}_{\mathbb{Z}_m}.\overline{1}_{\mathbb{Z}_n} + \overline{1}_{\mathbb{Z}_m}.\overline{1}_{\mathbb{Z}_n} + \cdots + \overline{1}_{\mathbb{Z}_m}.\overline{1}_{\mathbb{Z}_n} = r(\overline{1}_{\mathbb{Z}_m}.\overline{1}_{\mathbb{Z}_n}) = \overline{rn_1k}$. Also for every $\overline{s} \in \mathbb{Z}_n$, we have $\overline{r}.\overline{s} = \overline{r}.(\underbrace{\overline{1}_{\mathbb{Z}_n} + \overline{1}_{\mathbb{Z}_n} + \cdots + \overline{1}_{\mathbb{Z}_n}}_{s \ items}) = \overline{r}.\overline{1}_{\mathbb{Z}_n} + \overline{r}.\overline{1}_{\mathbb{Z}_n} + \cdots + \overline{r}.\overline{1}_{\mathbb{Z}_n} = s(\overline{r}.\overline{1}_{\mathbb{Z}_n}) = s(\overline{rn_1k}) = \overline{rsn_1k}$. This shows that this multiplication of the module is defined by

$$\overline{r}.\overline{s} = \overline{rsn_1k} \quad \forall \overline{r} \in \mathbb{Z}_m \in \quad \forall \overline{s} \in \mathbb{Z}_n \qquad (*)$$

Now if $k = 0$, then by $(*)$ the multiplication is the trivial zero multiplication, thus $1 \leq k \leq d-1$. Consequently $\overline{1}_{\mathbb{Z}_m}.\overline{1}_{\mathbb{Z}_n} = \overline{n_1k}$, for some $1 \leq k \leq d-1$.

Evidently $(\overline{1}_{\mathbb{Z}_m}\overline{1}_{\mathbb{Z}_m}).\overline{1}_{\mathbb{Z}_n} = \overline{1}_{\mathbb{Z}_m}.\overline{1}_{\mathbb{Z}_n} = \overline{n_1k}$. Also by $(*)$, $\overline{1}_{\mathbb{Z}_m}.(\overline{1}_{\mathbb{Z}_m}.\overline{1}_{\mathbb{Z}_n}) = \overline{1}_{\mathbb{Z}_m}.(\overline{n_1k}) = \overline{1.n_1kn_1k} = \overline{n_1^2k^2}$. Hence $\overline{n_1k} = \overline{n_1^2k^2}$, which implies that $n_1d = n \mid n_1^2k^2 - n_1k = n_1k(kn_1 - 1)$, that is $d \mid k(kn_1 - 1)$.

$(\Longleftarrow)$ For each $\overline{r} \in \mathbb{Z}_m$ and for each $\overline{s} \in \mathbb{Z}_n$, define:

$$\overline{r}.\overline{s} = \overline{rsn_1k} \qquad (**)$$

First let $\overline{r_1} = \overline{r_2} \in \mathbb{Z}_m$, and $\overline{s_1} = \overline{s_2} \in \mathbb{Z}_n$. Then $d \mid m \mid r_1 - r_2$ and $n \mid s_1 - s_2$. Let $dr^* = r_1 - r_2$ and $ns^* = s_1 - s_2$.

Then in $\mathbb{Z}_n$ we have

$r_1s_1n_1k - r_2s_2n_1k = (r_1 - r_2)s_1n_1k + r_2(s_1 - s_2)n_1k = r^*dn_1s_1k - ns^*r_2n_1k = r^*ns_1k - ns^*r_2n_1k = n(r^*s_1k + s^*r_2n_1k)$, thus $\overline{r_1s_1n_1k} = \overline{r_2s_2n_1k}$ in $\mathbb{Z}_n$, that is the multiplication is well defined.

Also $1_{\mathbb{Z}_m}.1_{\mathbb{Z}_n} = \overline{1.1.n_1k} = \overline{n_1k}$, which is not zero because $1 \leq k \leq d-1$. So it is not a trivial zero multiplication.

One can easily check that $\mathbb{Z}_n$ is a $\mathbb{Z}_m$-module by the multiplication defined in $(**)$.

Corollary 4.6. *Let $m, n \in \mathbb{N}$ with $m < n$ and $gcd(m, n) = 1$. Then $\mathbb{Z}_n$ is an $\mathbb{Z}_m$-module only by the trivial zero multiplication.*

Proof. Since $d = 1$, there does not exist any integer $1 \leq k \leq d-1$, so the proof is obvious by 4.5.

Theorem 4.7. *Let $m, n \in \mathbb{N}$ with $gcd(m, n) = 1$, then $\gamma(T(\Gamma(\mathbb{Z}_n))) = 1$.*

Proof. Since $gcd(m, n) = 1$ thus $\mathbb{Z}_n$ is an $\mathbb{Z}_m$-module only by the trivial zero multiplication 4.6, so $T(\mathbb{Z}_n) = \mathbb{Z}_n$. Therefore according to 3.7 we have $\gamma(T(\Gamma(\mathbb{Z}_n))) = 1$. □

Theorem 4.8. *Let $m, n \in \mathbb{N}$ with $gcd(m, n) = d > 1$ and $n/d = n_1$. If the prime factorization of d is $d = p_1^{r_1}p_2^{r_2}\cdots p_s^{r_s}$, then $\mathbb{Z}_n$ is an $\mathbb{Z}_m$-module if and only if $p_i \not| n_1$ for some $1 \leq i \leq s$.*

Proof. ($\Longrightarrow$) By 4.6, $d \mid k(kn_1 - 1)$ for some $1 \leq k \leq d-1$. If $p_i \mid n_1$ for all $1 \leq i \leq s$, then $gcd(kn_1 - 1, d) = 1$, and from $d \mid k(kn_1 - 1)$ we get $d \mid k$, and so $d \leq k \leq d-1$, which is a contradiction.

($\Longleftarrow$) By 4.6, there is a non-zero ring homomorphism $f : \mathbb{Z}_m \to \mathbb{Z}_n$. For each $r \in \mathbb{Z}_m$ and each $x \in \mathbb{Z}_n$, define $r.x = f(r)x$. One can easily see that $\mathbb{Z}_n$ is a non trivial $\mathbb{Z}_m$-module by this multiplication.

Proposition 4.9. *Let $n \in \mathbb{N}$ and G be an abelian group with $nG = 0$. Then G is an $\mathbb{Z}_n$-module by the natural definition $k.x = kx$, for each $k \in \mathbb{Z}_n$ and every $x \in G$. Particularly if G is a finite group of order n, then G is an $\mathbb{Z}_n$-module.*

Proof. Note that if M is an R-module and I is an ideal of R with $I \subseteq Ann_R\, M$, then M is an R/I-module. Now as G is an $\mathbb{Z}$-module, and $n\mathbb{Z} \subseteq Ann_{\mathbb{Z}}G$, obviously G is an $\mathbb{Z}/n\mathbb{Z}$-module, and since $\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}_n$, the abelian group G is an $\mathbb{Z}_n$-module.

For the particular case, note that $nG = 0$.

Corollary 4.10. *Let $n < m$ and $n \mid m$, then $\mathbb{Z}_n$ is a module over $\mathbb{Z}_m$ and $\gamma(T(\Gamma(\mathbb{Z}_n))) = 1$.*

Proof. Since $n \mid m$ so $m\mathbb{Z}_n = 0$ and according to 4.9 $\mathbb{Z}_n$ is a module over ring $\mathbb{Z}_m$ by the way since $n < m$ so $\bar{n}$ in not zero in $\mathbb{Z}_m$ thus $T(\mathbb{Z}_n) = \mathbb{Z}_n$ and according to 3.7 we have $\gamma(T(\Gamma(\mathbb{Z}_n))) = 1$. □

Example 4.11. Find the domination number of total graph of module $\mathbb{Z}_6$ over ring $\mathbb{Z}_3$.

Solution : We apply 4.2 and we define ring homomorphism $f : \mathbb{Z}_3 \to \mathbb{Z}_6$ with $f(\bar{t}) = \overline{4t}$. Now $\mathbb{Z}_6$ is an $\mathbb{Z}_3$-module and for all $\bar{x} \in \mathbb{Z}_3$ and for all $\bar{a} \in \mathbb{Z}_6$ we have $\bar{x} * \bar{a} = \overline{4xa}$. With this multiplication we obtain $T(\mathbb{Z}_6) = \{\bar{0}, \bar{3}\}$ thus $\mid T(\mathbb{Z}_6) \mid= 2$ and $\mid \frac{\mathbb{Z}_6}{T(\mathbb{Z}_6)} \mid= 3$ and from 3.5 we conclude that $\gamma(T(\Gamma(\mathbb{Z}_6))) = 3$

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