NILPOTENT GRAPHS OF MATRIX ALGEBRAS

A. MAHMOODI

Communicated by H.R. Maimani

ABSTRACT. Let R be a ring with unity. The undirected nilpotent graph of R, denoted by $\Gamma_N(R)$, is a graph with vertex set $Z_N(R)^* = \{0 \neq x \in R|\ xy \in N(R)\ for\ some\ y \in R^*\}$, and two distinct vertices x and y in $Z_N(R)^*$ are adjacent if and only if $xy \in N(R)$, or equivalently, $yx \in N(R)$, where $N(R)$ denotes the nilpotent elements of R. Here, using the concept of rank over commutative rings, we investigate basic properties of undirected nilpotent graph of matrix algebra. In particular, we give lower bound for the independence number of $\Gamma_N(M_n(F))$, when F is a finite field and $n \geqslant 2$. Also, we prove that $\Gamma_N(M_n(R))$ is not planar for all $n \geqslant 2$. Among other results, it is shown that $\mathrm{diam}(\Gamma_N(R)) \leqslant \mathrm{diam}(\Gamma_N(M_n(R)))$ for an Artinian commutative ring R. Also, we prove that for two finite fields F_1 and F_2 and integers $n, t \geqslant 2$, if $\Gamma_N(M_n(F_1)) \cong \Gamma_N(M_t(F_2))$, then $t = n$ and $F_1 \cong F_2$.

1. INTRODUCTION AND PRELIMINARIES

Let R be a commutative ring with unity. The *zero-divisor* graph of R, denoted by $\Gamma(R)$, is a graph with vertex set $Z(R)^*$, the set of nonzero zero-divisors of R, and two distinct vertices x and y are

adjacent if and only if $xy = 0$. In [12], Chen defined a graph structure on a ring R whose vertices are all the elements of R, and two distinct vertices x and y are adjacent if and only if $xy \in N(R)$, where $N(R)$ denotes the set of all nilpotent elements of R. The vertex set of $\Gamma_N(R)$ (*undirected nilpotent graph*) is $Z_N(R)^*$, where $Z_N(R) = \{x \in R | xy \in N(R) \ for \ some \ y \in R^*\}$, and two distinct vertices x and y in $Z_N(R)^*$ are adjacent if and only if $xy \in N(R)$, or equivalently, $yx \in N(R)$. This kind of graphs was introduced in [14] and [15]. It is easy to see that the usual zero-divisor graph $\Gamma(R)$ is a subgraph of $\Gamma_N(R)$.

Throughout this paper, all graphs are simple with no loops and multiple edges. Let G be a graph with vertex set $V(G)$. A *path* from x to y is a series of adjacent vertices $x - x_1 - x_2 - \cdots - x_n - y$. For $x, y \in V(G)$ with $x \neq y$, $d(x, y)$ denotes the length of a shortest path from x to y. If there is no such path, then we will make the convention $d(x, y) = \infty$. A graph G is called *connected* if there is a path between each pair x and y in $V(G)$. The *diameter* of G is defined as $\text{diam}(G) = \sup\{d(x, y) | \ x \ and \ y \ are \ vertices \ of \ G\}$. A *cycle* is a path that begins and ends at the same vertex in which no edge is repeated and all vertices other than the starting and ending vertex are distinct. A graph in which each pair of distinct vertices is joined by an edge is called a *complete* graph. We denote the complete graph with n vertices by K_n. The *girth* of G, denoted by $\text{gr}(G)$, is the length of a shortest cycle in G ($\text{gr}(G) = \infty$ if G contains no cycles). Also a subset Y of the vertices of a graph is called an *independent* set if the induced subgraph on Y has no edges. The maximum size of an independent set in a graph G is called the *independence number* of G and denoted by $\alpha(G)$. A graph is said to be *planar* if it can be drawn in the plane so that its edges intersect only at their ends. A *subdivision* of a graph is a graph obtained from it by replacing edges with pairwise internally-disjoint paths. A subset $S \subseteq V(G)$ is called a *clique* if the subgraph induced on S is complete. The number of vertices in the largest clique of the graph G is called the *clique number* of G, and is denoted by $\omega(G)$.

We denote by $M_n(R)$, R^n and I, the ring of all $n \times n$ matrices, the set of $n \times 1$ matrices over R and the identity matrix, respectively. Also, for any i and j, $1 \leq i, j \leq n$, we use E_{ij} to denote the element of $M_n(R)$ whose (i, j)-entry is 1 and other entries are 0.

In [14], the authors have proved some basic properties of $\Gamma_N(R)$, where R is an Artinian ring. Some results in our paper parallel of theirs but our interest is to use the concept of rank over commutative rings to characterize the vertices, connectedness, diameter and girth of $\Gamma_N(M_n(R))$ for all $n \geqslant 2$. In [10], the authors have investigated zero-divisor graph over commutative rings and relationship between $\text{diam}(\Gamma(R))$ and $\text{diam}(\Gamma(M_n(R)))$. Motivated by [10], we are interested to investigate the same concept for nilpotent graph over Artinian commutative rings.

This article is organized as follows. In section 2, using the notion of rank over commutative rings, the basic properties of $\Gamma_N(M_n(R))$ are obtained. Also, we prove that for two finite fields F_1 and F_2 and integers $n, t \geqslant 2$, if $\Gamma_N(M_n(F_1)) \cong \Gamma_N(M_t(F_2))$, then $t = n$ and $F_1 \cong F_2$. Here we give the lower bound showing that if F is a finite field and $n \geqslant 2$, then $\alpha(\Gamma_N(M_n(F))) \geqslant (q^n - 1)(q^n - q) \cdots (q^n - q^{n-1})$, where

$|F| = q$. In section 3, we investigate the planarity of $\Gamma_N(M_n(R))$ for all $n \geqslant 2$. More precisely, we prove that $\Gamma_N(M_n(R))$ is not planar for $n \geqslant 2$. In Section 4, among other results, we demonstrate the inequality $\operatorname{diam}(\Gamma_N(R)) \leqslant \operatorname{diam}(\Gamma_N(M_n(R)))$ and the isomorphism $\Gamma_N(M_n(R)) \cong \Gamma_N(M_n(T(R)))$, where $T(R)$ is the total quotient ring of commutative ring R.

2. Basic properties of $\Gamma_N(M_n(R))$

The main purpose of this section is to determine vertex set of $\Gamma_N(M_n(R))$ and characterize the connectedness, diameter and girth of it. Here, we appear the following useful lemma from [2].

Lemma 2.1. *Let R be a left Artinian ring. Then every nonzero element is a zero-divisor or a unit.*

Also, we will use the following well-known lemma, frequently, for the proof one can see [13].

Lemma 2.2. *Let R be an Artinian commutative ring. Then $M_n(R)$ is an Artinian ring.*

In continue, we need the *rank* of a matrix over commutative ring. Suppose that R is commutative and $A \in M_{m \times n}(R)$ (the ring of $m \times n$ matrices over R). For each $t = 1, \ldots, r = \min\{m, n\}$, $I_t(A)$ will denote the ideal in R generated by all $t \times t$ minors of A. The rank of A, denoted by $\operatorname{rank}(A)$, is the following integer:

$$\operatorname{rank}(A) = \max\{t | Ann_R(I_t(A)) = (0)\},$$

where $Ann_R(I_t(A)) = \{x \in R | xI_t(A) = 0\}$.

Before starting the main theorem of this section, we need the following well-known technical theorem in the theory of matrix algebra. For the proof one can see [11, Theorem 5.3, Theorem 9.1, Page 31].

Theorem 2.3. *Under above assumption, we have the following statements:*

(1) *The homogeneous system of equations $AX = 0$ has a nontrivial solution if and only if $\operatorname{rank}(A) < n$.*

(2) *A is a left zero-divisor in $M_n(R)$ if and only if $\det(A) \in Z(R)$.*

(3) *$\operatorname{rank}(A) < n$ if and only if $det(A) \in Z(R)$.*

In the remaining part of this section, we are interested to determine vertex set, connectivity, diameter and grith of $\Gamma_N(M_n(R))$.

Theorem 2.4. *Let R be an Artinian commutative ring. The vertices of $\Gamma_N(M_n(R))$ are exactly $M_n(R)^*$ for all $n \geqslant 2$.*

Proof. At first we suppose that $A \in M_n(R)^*$ is a non-singular matrix. It is clear that $A \cdot A^{-1}E_{1n}$ is nilpotent element of $M_n(R)$. Therefore, A is a vertex of $\Gamma_N(M_n(R))$. Now, suppose that $A \in M_n(R)^*$ is a singular matrix. By Lemma 2.1 and Lemma 2.2, $A \in Z(M_n(R))^*$. Using Theorem 2.3, there exist nonzero $X \in M_n(R)$ s.t $AX = 0$. Hence, A is a vertex of $\Gamma_N(M_n(R))$. ∎

It is easy to see that if R is integral domain, then $\Gamma_N(R)$ is empty graph. Hence, when F_1 and F_2 are two infinite fields, we have $\Gamma_N(F_1) \cong \Gamma_N(F_2)$, while F_1 and F_2 are not necessarily isomorphic.

Corollary 2.5. *Let F_1 be a finite field and F_2 be a field. If $\Gamma_N(M_n(F_1)) \cong \Gamma_N(M_t(F_2))$, for integers $n, t \geqslant 2$, then F_2 is a finite field, $t = n$ and $F_1 \cong F_2$.*

Proof. By Theorem 2.4, the number of vertices $\Gamma_N(M_n(F_1))$ and $\Gamma_N(M_t(F_2))$ are $|F_1|^{n^2} - 1$ and $|F_2|^{t^2} - 1$, respectively. Therefore, using hypothesis we have $|F_1| = |F_2|$. So we are done. ∎

It is clear that non-singular matrices are not vertices of $\Gamma(M_n(R))$. So, $\Gamma(M_n(R))$ is a proper subgraph of $\Gamma_N(M_n(R))$. In [10, Theorem 2.3], it was proved that $\mathrm{gr}(\Gamma(M_n(R))) \leqslant 4$. Therefore, $\mathrm{gr}(\Gamma_N(M_n(R))) \leqslant 4$. In the following theorem, we improve this inequality.

Theorem 2.6. *Let R be an Artinian commutative ring. Then, for $n \geqslant 2$, the following holds:*

(1) $\Gamma_N(M_n(R))$ *is connected.*

(2) $\mathrm{diam}(\Gamma_N(M_n(R)) \leqslant 3$.

(3) $\mathrm{gr}(\Gamma_N(M_n(R))) = 3$.

Proof. To prove assertions (1) and (2) consider the following cases:

- Let A_1 and A_2 are non-singular matrices in $M_n(R)$. Note that $A_2^{-1}E_{1n}.E_{1n}A_1^{-1} = 0$, Since $E_{1n}^2 = 0$. Hence, there is an edge between $A_2^{-1}E_{1n}$ and $E_{1n}A_1^{-1}$. Therefore, $A_1 - E_{1n}A_1^{-1} - A_2^{-1}E_{1n} - A_2$ is a path between A_1 and A_2.

- Let A is non-singular and B is singular matrices in $M_n(R)$. By Lemma 2.1, $B \in Z(M_n(R))^*$. Then, using Theorem 2.3 one can find a nonzero $C \in M_n(R)$ such that $BC = 0$. Now, if $CE_{1n} = 0$, then $A - E_{1n}A^{-1} - C - B$ is a path between A and B. Otherwise, $A - E_{1n}A^{-1} - CE_{1n} - B$ is a path between A and B.

- Let A_1 and A_2 are singular matrices in $M_n(R)$. By Lemma 2.1, A_1 and $A_2 \in Z(M_n(R))^*$. Since induced graph on singular matrices in $\Gamma(M_n(R))$ is a subgraph of induced graph on singular matrices in $\Gamma_N(M_n(R))$ so, $\mathrm{d}(A_1, A_2) \leqslant 3$ by [6, theorem 3.1].

Therefore, $\Gamma_N(M_n(R))$ is connected and $\mathrm{diam}(\Gamma_N(M_n(R))) \leqslant 3$. To prove (3) we consider two cases. First, suppose that A is a non-singular matrix. Then $E_{1n} - AE_{1n} - E_{1n}A^{-1} - E_{1n}$ is a cycle in $\Gamma_N(M_n(R))$ and we are down. Now, let A is a singular matrix. Then by Lemma 2.1, $A \in Z(M_n(R))^*$. So there is a nonzero $B \in M_n(R)$ such that $AB = 0$, by Theorem 2.3. If $BE_{1n} \neq 0$ and $E_{1n}A \neq 0$, then $E_{1n} - E_{1n}A - BE_{1n} - E_{1n}$ is a cycle in $\Gamma_N(M_n(R))$. If $BE_{1n} \neq 0$ and $E_{1n}A = 0$, then

$E_{1n} - A - BE_{1n} - E_{1n}$ is a cycle in $\Gamma_N(M_n(R))$. So, suppose that $BE_{1n} = 0$ and $E_{1n}A \neq 0$, then $E_{1n} - E_{1n}A - B - E_{1n}$ is a cycle in $\Gamma_N(M_n(R))$. Finally, let $BE_{1n} = 0$ and $E_{1n}A = 0$. Hence, $E_{1n} - A - B - E_{1n}$ is a cycle in $\Gamma_N(M_n(R))$. ∎

In the graph $\Gamma_N(M_n(R))$, the vertices I_n and E_{nn} are not adjacent. Also, there is no edge between non-singular matrices. Therefore, we have the following corollary.

Corollary 2.7. *Let R be an Artinian commutative ring. Then $\Gamma_N(M_n(R))$ is not a complete graph for all $n \geqslant 2$.*

A graph is bipartite if its vertex set can be partitioned into two subsets X and Y so that every edge has one end in X and one end in Y. There is a characterization of bipartite graphs. One can see [9, Theorem 4.7].

Theorem 2.8. *A graph is bipartite if and only if it contains no odd cycle.*

By Theorem 2.6, we have following corollary.

Corollary 2.9. *Let R be an Artinian commutative ring. Then, for all $n \geqslant 2$, the $\Gamma_N(M_n(R))$ is not bipartite graph.*

Theorem 2.10. *Let F be a finite field and q be a prime integer such that $|F| = q$. Then $\alpha(\Gamma_N(M_n(F))) \geqslant (q^n - 1)(q^n - q) \cdots (q^n - q^{n-1})$.*

Proof. By Theorem 2.4, every non-singular matrix is a vertex of $\Gamma_N(M_n(F))$. But it is clear that non-singular matrices are not adjacent in $\Gamma_N(M_n(F))$. On the other hand, it is well-known that the number of non-singular matrix is $(q^n - 1)(q^n - q) \cdots (q^n - q^{n-1})$. Therefore, $\alpha(\Gamma_N(M_n(F))) \geqslant (q^n - 1)(q^n - q) \cdots (q^n - q^{n-1})$. ∎

Using Theorem 2.4 and Theorem 2.10, we conclude the following corollary.

Corollary 2.11. *Let F be a finite field and q be a prime integer such that $|F| = q$. Then $\omega(\Gamma_N(M_n(F))) \leqslant q^{n^2} - (q^n - 1)(q^n - q) \cdots (q^n - q^{n-1}) - 1$.*

3. PLANARITY OF $\Gamma_N(M_n(R))$

Let R be a commutative ring. A natural question is posed by Anderson, Frazier, Lauve, and Livingston in [3]: For which finite commutative rings R is $\Gamma(R)$ planar? S. Akbari, H.R. Maimani and

S. Yassemi gave an answer to this question. More precisely, they proved that if R is a local ring with at least 33 elements, and $\Gamma(R) \neq \emptyset$, then $\Gamma(R)$ is not planar [1]. Motivated by above question, we investigate the planarity of $\Gamma_N(M_n(R))$ for all $n \geqslant 2$, where R is a (not necessarily local) commutative ring. Since $\Gamma(R)$ is a subgraph of $\Gamma_N(R)$, by [1] we can state the following result, when $n = 1$.

Corollary 3.1. *Let R be a local commutative ring with at least 33 elements, and $\Gamma(R) \neq \emptyset$. Then $\Gamma_N(R)$ is not planar.*

Let R be a commutative ring. In the sequel, we prove that $\Gamma_N(M_n(R))$ for all $n \geqslant 2$ is not planar. A remarkably simple characterization of planar graphs was given by Kuratowski in 1930, [9, P. 153]. Indeed, a graph is planar if and only if it contains no subdivision homeomorphic to K_5 or $K_{3,3}$.

Lemma 3.2. *Let R be commutative ring and $n \geqslant 3$. Then $\Gamma_N(M_n(R))$ is not planar.*

Proof. It is not hard to see that the induced subgraph over vertices $E_{11}, E_{22}, E_{33}, E_{23}$ and E_{13} of $\Gamma_N(M_n(R))^*$ is complete graph K_5. So, using Kuratowski's theorem, $\Gamma_N(M_n(R))$ for all $n \geqslant 3$ is not planar.

$\blacksquare$

Lemma 3.3. *Let R be a commutative ring with at least five elements. Then $\Gamma_N(M_2(R))$ is not planar.*

Proof. Since $|R^*| \geqslant 4$, one can choose nonzero elements a_1, a_2 and $a_3 \in R \setminus \{1\}$. Induced subgraph over vertices $I, E_{12}, a_1 E_{12}, a_2 E_{12}$ and $a_3 E_{12}$ is K_5. So, by Kuratowski's theorem, $\Gamma_N(M_2(R))$ is not planner. $\blacksquare$

It is not hard to see that the only commutative rings with two or three elements are $\mathbb{Z}_2$ or $\mathbb{Z}_3$. This leads us to the following theorem.

Lemma 3.4. *Let R be a commutative ring with $|R^*| \leqslant 2$. Then $\Gamma_N(M_2(R))$ is not planar.*

Proof. First, suppose that $R = \mathbb{Z}_3$. Let $A := E_{11} + E_{12} + E_{21}, B := E_{11} + E_{12} + E_{22}, C := E_{11} + E_{21} + E_{22}, D := E_{12} + E_{21} + E_{22}, P := \sum_{1 \leqslant i,j \leqslant 2} E_{i,j}, X := E_{11} + E_{12}, Y := E_{21} + E_{22}$ and $Z := E_{11} + E_{21}$. Graph in Figure (i) is a subgraph of $\Gamma_N(M_2(\mathbb{Z}_3))$. So, by Kuratowski's theorem, $\Gamma_N(M_2(R))$ is not planar. For $R = \mathbb{Z}_2$ graph in Figure (ii) is a subgraph of $\Gamma_N(M_2(\mathbb{Z}_2))$. Again by Kuratowski's theorem, $\Gamma_N(M_2(R))$ is not planar.

(i) (ii)

■
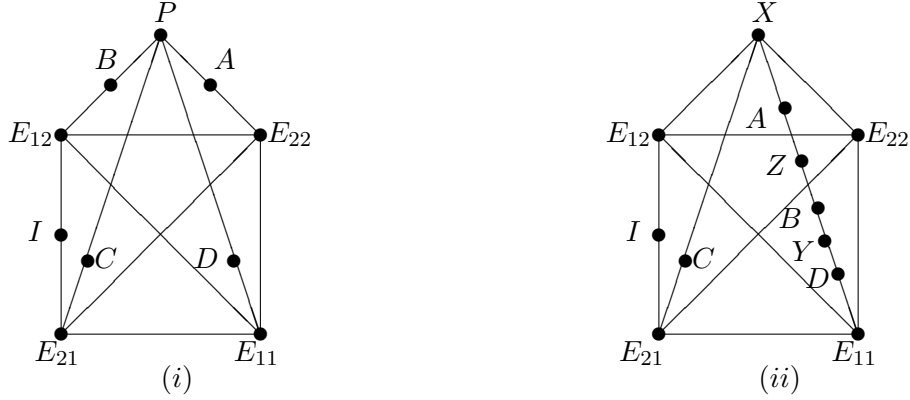
The only remaining case which is not covered by the above theorems is $|R| = 4$. For any prime number p, rings of order p^2 were classified in [8, P. 248]. It was proved that for any prime number p there are, up to isomorphism, exactly 11 rings of order p^2. Using this fact and simple verifications, we have the following theorem.

Theorem 3.5. *Let R be a commutative ring with unity of order 4. Then $R = \mathbb{Z}_4$, $R = \mathbb{Z}_2 \times \mathbb{Z}_2$, $R =< a, b; 2a = 2b = 0, a^2 = 0, b^2 = b, ab = a, ba = a >$ or $R = GF_2(\mathbb{Z}_2)$ (The Galois field of order 4).*

Lemma 3.6. *Let R be a commutative ring with $|R| = 4$. Then $\Gamma_N(M_2(R))$ is not planar.*

Proof. By Theorem 3.5, there are four possibilities for R. First, suppose that $R = \mathbb{Z}_4$. Let $A = 2E_{11} + 2E_{21}$. Induced subgraph over vertices $I, A, E_{12}, 2E_{12}$ and $3E_{12}$ of $\Gamma_N(M_2(\mathbb{Z}_4))$ is complete graph K_5. So, by Kuratowski's theorem, $\Gamma_N(M_2(R))$ is not planar. Now, let $R = \mathbb{Z}_2 \times \mathbb{Z}_2$. Consider the matrices

$$X = \begin{pmatrix} (0,0) & (0,0) \\ (1,0) & (0,0) \end{pmatrix}, Y = \begin{pmatrix} (0,0) & (0,0) \\ (0,0) & (1,0) \end{pmatrix}, Z = \begin{pmatrix} (0,0) & (0,0) \\ (0,1) & (0,0) \end{pmatrix},$$

$$X' = \begin{pmatrix} (0,0) & (0,1) \\ (0,0) & (0,0) \end{pmatrix}, Y' = \begin{pmatrix} (0,0) & (1,1) \\ (0,0) & (0,0) \end{pmatrix} \text{ and } Z' = \begin{pmatrix} (0,0) & (1,0) \\ (0,0) & (0,0) \end{pmatrix}$$

of $\Gamma_N(M_n(R)^*)$. Induced subgraph over vertices I, X, X', Y' and Z' of $\Gamma_N(M_2(\mathbb{Z}_2 \times \mathbb{Z}_2))$ is a subdivision of complete graph K_5. It should be noted that the paths $X - Y - Y'$ and $X - Z - Z'$ are placed instead of edges XY' and XZ', respectively. So, by Kuratowski's theorem, $\Gamma_N(M_2(R))$ is not planar.

According to Theorem 3.5, $R =< a, b; 2a = 2b = 0, a^2 = 0, b^2 = b, ab = a, ba = a >$ is the third case that can happen. Induced subgraph over vertices $E_{12}, aE_{12}, (a+1)E_{12}, aE_{21}$ *and* aE_{22} of $\Gamma_N(M_2(R))$ is complete graph K_5. Again by Kuratowski's theorem, $\Gamma_N(M_2(R))$ is not planar. Finally, we can

assume that $R = GF_2(\mathbb{Z}_2) = \mathbb{Z}_2(\alpha) = \{0, 1, \alpha, 1+\alpha\}$; where $\alpha^2+\alpha+1 = 0$. Consider induced subgraph over vertices $E_{12}, \alpha E_{12}, (\alpha+1)E_{22}, \alpha E_{11}$ and αE_{22}. It is a subdivision of complete graph K_5. Again we note that the path $(\alpha+1)E_{22} - (\alpha+1)E_{11} - \alpha E_{22}$ is placed instead of edge $(\alpha+1)E_{22}\alpha E_{22}$. Hence, by Kuratowski's theorem, $\Gamma_N(M_2(R))$ is not planar.

$\blacksquare$

We can summarized the above discussions in the following theorem.

Theorem 3.7. *Let R be a commutative ring. Then $\Gamma_N(M_n(R))$, for all $n \geqslant 2$ is not planar.*

$$4. \quad \text{diam}(\Gamma_N(M_n(R))) \text{ AND } \text{diam}(\Gamma_N(R))$$

it is easy to see that $\Gamma_N(R)$ is isomorphic by a subgraph of $\Gamma_N(M_n(R))$ for all $n \geqslant 2$. It is worthy to ask: Is there relationship between diameter of $\Gamma_N(R)$ and $\Gamma_N(M_n(R))$, for all $n \geqslant 2$. In this section, we give an answer to this question, when R is an Artinian commutative ring. By [12, Theorem 2.1] we have $\text{diam}(\Gamma_N(R)) \leqslant 3$. On the other hand, I_n and E_{nn} are not adjacent in $\Gamma_N(M_n(R))$, by Corollary 2.7. Using this fact and Theorem 2.6, we have $2 \leqslant \text{diam}(\Gamma_N(M_n(R))) \leqslant 3$. Now, we consider two following cases:

- First, suppose that $\text{diam}(\Gamma_N(R)) = 2$. Then there exist x and y in $V(\Gamma_N(R))$ such that $d(x, y) = 2$. Since xy is not nilpotent and R is an Artinian ring; so that xy is unit. Let $A = xE_{11}$ and $B = yE_{11}$. Hence $AB = xyE_{11}$ is not nilpotent, so $d(A, B) \geqslant 2$. Therefore, we have
$$\text{diam}(\Gamma_N(M_n(R))) \geqslant 2 = \text{diam}(\Gamma_N(R)).$$
- Now, suppose that $\text{diam}(\Gamma_N(R)) = 3$. In this case, we show that
$$\text{diam}(\Gamma_N(M_n(R))) = 3.$$
 For the proof, let x and y are in $V(\Gamma_N(R))$ such that $d(x, y) = 3$. Since xy is not nilpotent and R is an Artinian ring, so xy is unit. Consider the matrices $A = xE_{11}$ and $B = yE_{11}$. We know AB is not nilpotent matrix. So, $d(A, B) \geqslant 2$. We claim that $d(A, B) = 3$. Otherwise, there exists a nonzero matrix $C \in M_n(R)$ such that AC or CA (Also BC or CB) is a nilpotent matrix. Note that since C is not zero matrix, it contains a nonzero entry say c_{ti}. One can replace A with xE_{it} and B with yE_{it}. With out lose of generality, suppose that AC and BC are nilpotent. So there is $k, k' \in \mathbb{N}$ such that $(AC)^k = 0$ and $(BC)^{k'} = 0$. Therefore, we have $(xc_{ti})^k = 0$ and $(yc_{ti})^{k'} = 0$. Hence $d(x, y) = 2$ and this is a contradiction. This shows that $\text{diam}(\Gamma_N(M_n(R))) = 3$.

The following theorem describes the relationship between the diameter of $\Gamma_N(R)$ and $\Gamma_N(M_n(R))$.

Theorem 4.1. *Let R be an Artinian commutative ring. Then*

$$\text{diam}(\Gamma_N(R)) \leqslant \text{diam}(\Gamma_N(M_n(R))).$$

Let $T(R)$ be the total quotient ring of a commutative ring R. In [4], the authors have been proved that $\Gamma(R) \cong \Gamma(T(R))$. Then, in [10], the authors proved this claim for $\Gamma(M_n(R))$ and $\Gamma(M_n(T(R)))$. In the sequel we will provide simple proof for the above claim. Also, we will prove similar result for $\Gamma_N(M_n(R))$ and $\Gamma_N(M_n(T(R)))$.

Theorem 4.2. *Let R be an Artinian commutative ring with quotient ring $T(R)$. Then $\Gamma(M_n(R)) \cong \Gamma(M_n(T(R)))$.*

Proof. One can define the function $\varphi : \Gamma(M_n(R)) \to \Gamma(M_n(T(R)))$ such that $\varphi((a_{ij})) = (\frac{a_{ij}}{1})$ for every $(a_{ij}) \in M_n(R)$. It is straightforward to see that φ is one-to-one function. Suppose that $(\frac{a_{ij}}{s_{ij}}) \in V(\Gamma(M_n(T(R)))^*)$. Hence, there is and $(\frac{b_{ij}}{t_{ij}}) \in M_n(T(R))^*$ such that $(\frac{a_{ij}}{s_{ij}})(\frac{b_{ij}}{t_{ij}}) = 0$. Since s_{ij} and $t_{ij} \in S = R - Z(R)$ for $1 \leq i, j \leq n$, by Lemma 2.1, s_{ij} and t_{ij} are invertible. So, $(s_{ij}^{-1}a_{ij})$ and $(t_{ij}^{-1}b_{ij}) \in M_n(R)^*$. Now, we have

$$\varphi((\Sigma_k(s_{ik}^{-1}a_{ik})(t_{kj}^{-1}b_{kj}))) = (\Sigma_k \frac{s_{ik}^{-1}a_{ik}}{1} \frac{t_{kj}^{-1}b_{kj}}{1}) = (\Sigma_k \frac{a_{ik}}{s_{ik}} \frac{b_{kj}}{t_{kj}}) = 0.$$

Thus, $(s_{ij}^{-1}a_{ij})(t_{ij}^{-1}b_{ij}) = 0$. This shows that $(s_{ij}^{-1}a_{ij}) \in V(\Gamma(M_n(R))^*)$ and it is not hard to see that $\varphi((s_{ij}^{-1}a_{ij})) = (\frac{a_{ij}}{s_{ij}})$. Therefore, φ is a bijection. With similar method, one can see that $\varphi^{-1}(A')$ and $\varphi^{-1}(B')$ are adjacent in $\Gamma(M_n(R))$ if A' and B' are adjacent in $\Gamma(M_n(T(R)))$. Also, it is easy to see that if A and B are adjacent in $\Gamma(M_n(R))$, then $\varphi(A)$ and $\varphi(B)$ are adjacent in $\Gamma(M_n(T(R)))$. Therefore, we are down.

∎

Theorem 4.3. *Let R be an Artinian commutative ring with quotient ring $T(R)$. Then $\Gamma_N(M_n(R)) \cong \Gamma_N(M_n(T(R)))$.*

Proof. One can define the function $\varphi : \Gamma_N(M_n(R)) \to \Gamma_N(M_n(T(R)))$ such that $\varphi((a_{ij})) = (\frac{a_{ij}}{1})$ for every $(a_{ij}) \in M_n(R)$. With similar method of Theorem 4.2, it is not hard to see that φ is a bijection. Note that one can replace two adjacent metrics $A' = (\frac{a_{ij}}{s_{ij}}), B' = (\frac{b_{ij}}{t_{ij}}) \in \Gamma_N(M_n(T(R)))$ with $A' = (\frac{a'_{ij}}{1})$ and $B' = (\frac{b'_{ij}}{1})$, respectively, where $a'_{ij} = s_{ij}^{-1}a_{ij}$ and $b'_{ij} = t_{ij}^{-1}b_{ij}$. So, it is clear that if $A'B'$ is nilpotent matrix in $M_n(T(R))$, then $\varphi^{-1}(A')\varphi^{-1}(B')$ is nilpotent matrix in $\Gamma_N(M_n(R))$. ∎

The pervious theorem concludes the following corollary.

Corollary 4.4. *Let R be an Artinian commutative ring with quotient ring $T(R)$. Then $\text{diam}(\Gamma_N(M_n(R))) = \text{diam}(\Gamma_N(M_n(T(R))))$.*

REFERENCES

[1] S. Akbari, H.R. Maimani and S. Yassemi, *When a zero-divisor graph is planar or a complete r-partite graph*, J. Algebra 270, 169-180 (2003).

[2] S. Akbari, A. Mohammadian, *Zero-divisor graphs of non-commutative rings*, J. Algebra 296, 462–479 (2006).

[3] D. F. Anderson, A. Frazier, A. Lauve, P. Livingston, *The zero-divisor graph of a commutative ring.* II Lect. Notes Pure and Appl. Math 220, 61–72 (2001).

[4] D. F. Anderson, R. Levy, J. Shapiro, *Zero-divisor graphs, Von-Neumann regular rings and Boolean algebras*, J. Pure Appl. Algebra 180, 221–241 (2003).

[5] D. F. Anderson and P. Livingston. *The zero-divisor graph of a commutative ring*, J. Algebra 217, 434–447 (1999).

[6] D. F. Anderson, S. B. Mulay,*On the diameter and girth of a zero-divisor graph*, J. Pure Appl. Algebra 210, 543–550 (2007).

[7] I. Beck, *Coloring of commutative rings.* J. Algebra 116, 208–226 (1988).

[8] B. Fine, *Classification of finite rings of order p^2*, Mathematics Magazine, VOL. 66, NO. 4, 246–252 (1993)

[9] J.A. Bondy, U.S.R. Murty,*Graph Theory with Applications*, American Elsevier, New York, 1976.

[10] I. Bozic, Z. Petrovic,*Zero-divisor graphs of matrices over commutative rings*, Communication in Algebra 37, 1186–1192 (2009)

[11] W. C. Brown, *Matrices Over Commutative Rings*, Marcel Dekker, Inc, New York, Basel, Hong Kong.

[12] P. W. Chen, *A kind of graph structure of rings*, Algebra Colloq 10(2), 229–238 (2003).

[13] T. Y. Lam, *A First Course in Noncommutative Rings*, Springer-Verlag, New York, Inc, 1976.

[14] A. H. Li,Q. H. Li, *A kind of graph structure on non-reduced rings*, Algebra Colloq 17(1), 173–180 (2010).

[15] A. H. Li, Q. S. Li, *A kind of graph structure on Von-Neumann regular rings*, International J. Algebra 4, 291–302 (2010).

Akram Mahmoodi

Department of Mathematics,

Payame Noor University, P.O. BOX 19395-3697 Tehran, Iran

akmahmoodi@yahoo.com