A SHORT NOTE ON ATOMS AND COATOMS IN SUBGROUP LATTICES OF GROUPS

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ABSTRACT. In this paper we give an elementary argument about the atoms and coatoms of the lattice of all subgroups of a group. It is proved that an abelian group of finite exponent is strongly coatomic.

1. INTRODUCTION AND PRELIMINARIES

We first describe some notations and definitions that will be kept throughout. The symbols $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$ denote the set of all positive integers, integers, real numbers and complex numbers respectively. Set $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$. Then $\mathbb{T}$ is a group under multiplication. This group is called the circle group. The set of all subgroups of a group $G$ will be denoted by $\text{Sub}(G)$ which is a complete lattice with respect to inclusion. An *atom* in the lattice $\text{Sub}(G)$ is a minimal element of $\text{Sub}(G) \setminus \{\{1\}\}$. A *coatom* in the lattice $\text{Sub}(G)$ is a maximal element of $\text{Sub}(G) \setminus \{G\}$. $\text{Sub}(G)$ is said to be *atomic* if each element of $\text{Sub}(G) \setminus \{\{1\}\}$ contains an atom. $\text{Sub}(G)$ is said to be *coatomic* if each element of


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Sub\( (G) \setminus \{G\} \) is contained in a coatom. The exponent of a group \( G \) is the least element \( n = \exp G \) such that for each element \( x \in G \), we have \( x^n = 1 \). A group \( G \) is said to be \textit{locally cyclic} if for any \( a, b \in G \), the group generated by \( \{a, b\} \) is cyclic.

**Definition 1.1.** Let \( G \) be a group. An element \( a \in G \) is said to be a \textit{persistence element} if \( a \neq 1 \) and \( a \) is contained in all nontrivial subgroups of \( G \). A group \( G \) is \textit{persistent} if it has a persistence element.

Let \( a \) be a persistence element of a group \( G \). There is a prime \( p \) with \( |a| = p \). In fact, because \( a \in \langle a^2 \rangle \) there is a positive integer \( m \) with \( a = a^{2m} \). Thus \( a^{2m-1} = 1 \). So \( a \) has finite order. Let \( n = |a| \) and suppose \( n \) is not prime. There are distinct primes \( p \) and \( q \) with \( p | n, q | n \). Hence, there are \( x, y \in \langle a \rangle \) with \( |x| = p, |y| = q \) which implies that \( a \in \langle x \rangle \cap \langle y \rangle = \{1\} \), a contradiction. Therefore, \( a \) has prime order.

If \( G \) is a persistent group, it is torsion. To see this, let \( a \) be a persistence element of \( G \). If there is any \( x \in G \) with infinite order, then \( a \in \langle x \rangle \setminus \{1\} \). So, the order of \( a \) must be infinite, a contradiction. More generally, if \( p = |a| \) then \( G \) is a \( p \)-group. Otherwise, \( G \) has an element \( x \) such that \( |x| \) is divisible by a prime \( q \) different from \( p \). Choose \( b \in \langle x \rangle \) with \( |b| = q \). This implies that \( a \in \langle b \rangle \setminus \{1\} \) and so \( |a| = q \), which is impossible.

Persistence is a \textit{group property}, that is, for any two isomorphic groups, if one is persistent then the other is also persistent. More generally, let \( G \) and \( H \) be groups, \( b \) be a persistence element of \( H \), \( f : G \to H \) be a one-to-one homomorphism and \( a \in f^{-1}[\{b\}] \). Then \( a \) is a persistence element of \( G \). To prove this, it is evident that \( a \neq 1 \) and if \( K \leq G \) is nontrivial then, since \( f \) is one-to-one, \( f[K] \) is a nontrivial subgroup of \( H \). Thus \( b \in f[K] \) and so \( a \in f^{-1}[\{b\}] \subseteq f^{-1}[f[K]] = K \).

Clearly a group \( G \) is persistent if and only if \( \text{Sub}(G) \) is atomic and has a single atom. For example, \( G = \mathbb{Z}_2 \times \mathbb{Z} \) has a single atom \( H = \{(0,0), (1,0)\} \), but \( G \) is not atomic and so it is not persistent.

A group \( G \) is said to be \textit{decomposable} if there are nontrivial subgroups \( H \) and \( K \) of \( G \) such that \( G \cong H \times K \). A persisted group is \textit{indecomposable}. In fact, if \( G \) is a persistent decomposable group, there are nontrivial groups \( H \) and \( K \) with \( G \cong H \times K \). Since \( H \times K \) is persistent, it has a persistence element \( (a, b) \). We have \( (a, b) \in (H \times \{1\}) \cap (\{1\} \times K) = \{(1,1)\} \), which is impossible. As a result, if \( G \) is a finitely generated persistent abelian group, then it is a finite cyclic \( p \)-group for some prime number \( p \).

The generalized quaternion group \( Q_{4n} \) is a group presented by
\[
Q_{4n} = \langle a, b \mid a^n = b^2, a^{2n} = 1, b^{-1}ab = a^{-1} \rangle.
\]
This is a group of order \( |Q_{4n}| = 4n \). By [1, p. 262], if \( G \) is a finite \( p \)-group and any of its (nontrivial) abelian subgroup is cyclic, then either \( G \) is cyclic or it is isomorphic to a generalized quaternion
2–group. Now if $G$ is a finite persistent group, then it is a $p$-group and every abelian subgroup of $G$ is a persistent abelian group and so it must be cyclic. Therefore, either $G$ is cyclic or it is isomorphic to a generalized quaternion group. As a result, a finite persistent group of odd order is cyclic.

Let $p$ be a prime number. The Prüfer $p$-group is defined to be the subgroup

$$Z_{p^\infty} = \left\{ e^{\frac{2\pi i k}{p^n}} \mid k \in \mathbb{Z}, \ n \in \mathbb{N} \right\}$$

of the circle group $\mathbb{T}$. Let $G$ be an infinite $p$-group. These statements are equivalent:

- $G$ is isomorphic to $Z_{p^\infty}$.
- $\text{Sub}(G)$ is totally ordered.
- $\text{Sub}(G)$ is distributive.
- $G$ is locally cyclic [4].

Let $G$ be an infinite abelian group with persistence element $a$ and let $p = |a|$. Then $G$ is isomorphic to $Z_{p^\infty}$. To see this, it suffices to show that $G$ is locally cyclic. Let $x, y \in G$ be non-identity elements and $H = \langle x, y \rangle$. Since $H$ is finite, abelian and persistent, it is cyclic. It follows that any persistent abelian group is locally cyclic.

**Example 1.2.** By [3, p. 339], there is an infinite group $G$ with exponent $p^2$ such that $|Z(G)| = p$ and for each $x \in G \setminus Z(G)$, $|x| = p^2$. Clearly $G$ is not abelian and is persistent and so it is not locally cyclic.

Let $G$ be an abelian group and $p$ be a prime number. The set

$$G_{(p)} = \left\{ x \in G \mid (\exists n \in \mathbb{N})(x^{p^n} = 1) \right\}$$

is said to be the $p$-primary component of $G$. $G_{(p)}$ is said to be a primary component of $G$. Any primary component of $G$ is a subgroup of $G$. According to [2] (page 88), a torsion abelian group is isomorphic to a direct sum of cyclic groups. Let $G$ be an abelian group. By [6], there is a group $H$ with $\text{Sub}(G)$ order-isomorphic to $\text{Sub}(H)$ if and only if $G$ is torsion and any of its primary components is finite.

**2. Main Results**

Let $G$ be a group and $A \subseteq G$. Then it is clear that $A$ is a normal subgroup of $G$ if and only if it is nonempty and for each $x \in G$, $xAX^{-1}x^{-1} \subseteq A$. We use this fact in the proof of the following theorem:

**Theorem 2.1.** Suppose $G$ is a group, $A \subseteq G$ and there is a $L \unlhd G$ with $A \cap L = \emptyset$. Then there exists a maximal $M \unlhd G$ with $A \cap M = \emptyset$. 
Proof. Define $\mathcal{M} = \{S \leq G \mid S \cap M = \emptyset\}$ and let $\mathcal{C} \subseteq \mathcal{M}$ be a nonempty chain. Set $N = \bigcup_{C \in \mathcal{C}} C$ and choose $x \in G$. It suffices to show that $xNN^{-1}x^{-1} \subseteq N$. Let $a, b \in N$. Since $\mathcal{C}$ is a chain, there is a $C \in \mathcal{C}$ with $a, b \in C$ and so $xab^{-1}x^{-1} \in xCC^{-1}x^{-1} \subseteq C \subseteq N$. Thus $xNN^{-1}x^{-1} \subseteq N$ which implies that $N \leq G$. Thus $N$ is an upper bound for the arbitrary nonempty chain $\mathcal{C}$ in $\mathcal{M}$. By Zorn’s lemma, $\mathcal{M}$ must have a maximal element $M$. □

Let $G$ be a group and $N$ be a normal subgroup of $G$. Then by lattice theorem, $\text{Sub}(G_N)$ is order-isomorphic to the interval $[N, G]$ in $\text{Sub}(G)$.

**Theorem 2.2.** Let $G$ be an abelian group, $a \in G$ and $M$ be a maximal subgroup with $\{a\} \cap M = \emptyset$. Then $G_M$ is persisted with persistence element $Ma$.

Proof. Let $A$ be a nontrivial subgroup of $G_M$. There is a $N \leq G$ with $M \subset N$ and $A = G_N$. Since $M \subset N$, there exists $b \in N \setminus M$. Define $M' = \langle M \cup \{b\} \rangle$. Taking into account the maximality of $M$, $\{a\} \cap M' \neq \emptyset$. Thus $a \in M'$ and so there are $m \in M$ and $k \in \mathbb{Z}$ with $a = mb^k$. Thus $Ma = Mb^k$ and because $b^k \in N$, $Ma \in G_N = A$. □

**Theorem 2.3.** Let $G$ be an abelian group with finite exponent. Then $\text{Sub}(G)$ is coatomic.

Proof. Let $N < G$. There is some $a \in G \setminus N$. Since $\{a\} \cap N = \emptyset$, there is a maximal (normal) subgroup $M$ of $G$ with $\{a\} \cap M = \emptyset$. To prove the theorem, it suffices to prove that the interval $[M, G]$, or equivalently $\text{Sub}(G_M)$, has a coatom. By the previous theorem, $G_M$ is persistent. Thus either $G_M$ is a finite cyclic group or it is isomorphic to a Prüfer group. If $G_M$ is isomorphic to a Prüfer group, for every positive integer $n$, there is some $x \in G$ with $n \leq |Mx|$. Therefore, for all $n \in N$, $n \leq |x| \leq \exp G$. So, $\exp G = \infty$, which is a contradiction. Thus $G_M$ is not isomorphic to a Prüfer group and so is finite. This proves that $\text{Sub}(G_M)$ has a coatom. □

Let $G$ be a group. By [5] (page 324), these statements are equivalent:

- $\text{Sub}(G)$ has a coatom.
- There is some proper normal subgroup $N$ of $G$ such that $G_N$ is finite.
- $G$ is not divisible.

So, if $G$ is abelian and none of its quotient groups are divisible, then according to the proof of Theorem 3, $\text{Sub}(G)$ is coatomic. The converse is also true:

**Theorem 2.4.** Let $G$ be an abelian group and $\text{Sub}(G)$ be coatomic. Then none of the quotient groups of $G$ are divisible.

The main idea of this proof is due to Derek Holt.
Proof. Suppose there is some $N \leq G$ such that $\frac{G}{N}$ is divisible. There is a coatom $M$ containing $N$. Since the interval $[M, G]$ is order-isomorphic to $\frac{G}{M}$, $|\text{Sub}(\frac{G}{M})| = 2$ and so there is a prime number $p$ with $|\frac{G}{M}| = p$. Choose $b \in G \setminus M$. Since $\frac{G}{N}$ is divisible, there exists $a \in G$ such that $Na^p = Nb$. But, $a^p b^{-1} \in N \subseteq M$ and so $Mb = Ma^p = M$ which implies $b \in M$. This leads to a contradiction. ■

**Theorem 2.5.** Let $G$ be a group. Then $\text{Sub}(G)$ is atomic if and only if $G$ is torsion.

Proof. $\Rightarrow$) If there is $x \in G$ with infinite order, then $\langle x \rangle$ does not contain any atoms. $\Leftarrow$) Let $H \leq G$ be nontrivial. There is non-identity element $a \in H$ and a prime $p$ such that $|a|$ is finite and $p \mid |a|$. Thus, $\langle a \rangle$ has an element $b$ of order $p$. Clearly $\langle b \rangle$ is an atom in $\text{Sub}(G)$ and $\langle b \rangle \subseteq \langle a \rangle \subseteq H$. ■

Let $G$ be a torsion abelian group. Let $N \in \text{Sub}(G)$ and $N \neq G$. Since $\frac{G}{N}$ is torsion and abelian, it has an atom $A$. $A$ is a successor element to $\{N\}$. Since $\text{Sub}(\frac{G}{N})$ is order-isomorphic to the interval $[N, G]$, there is a successor element to $N$ in $\text{Sub}(G)$. Therefore every element of $\text{Sub}(G)$ except the greatest one, has an immediate successor, that is, $\text{Sub}(G)$ is strongly atomic.

Let $G$ be an abelian group with finite exponent. Let $N \in \text{Sub}(G)$ be nontrivial. Since $N$ is abelian and has finite exponent, it has a coatom $C$. We have $[C, N] = \{C, N\}$, that is, $C$ is an immediate predecessor element to $N$. Therefore every nontrivial element of $\text{Sub}(G)$ has an immediate predecessor, that is, $\text{Sub}(G)$ is strongly coatomic.

3. Concluding Remarks

In this paper, the structure of atomic and coatomic subgroup lattice of infinite abelian groups are considered into account. In the case of non-abelian groups, the problem of characterizing infinite groups with coatomic subgroup lattice still remains unanswered. On the other hand, the structure of a group $G$ in which every quotient group $\frac{G}{N}$ is persistent is a good problem for future research work.

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**References**


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