Journal of Algebraic Structures and Their Applications
ISSN: 2382-9761
www.as.yazd.ac.ir

Algebraic Structures and Their Applications Vol. 1 No. 2 (2014), pp 117-122.

A SHORT NOTE ON ATOMS AND COATOMS IN SUBGROUP LATTICES OF GROUPS

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Communicated by M.A. Iranmanesh

Abstract. In this paper we give an elementary argument about the atoms and coatoms of the lattice of all subgroups of a group. It is proved that an abelian group of finite exponent is strongly coatomic.

1. Introduction and Preliminaries

We first describe some notations and definitions that will be kept throughout. The symbols $\mathbb{N}$, R, $\mathbb{C}$ denote the set of all positive integers, integers, real numbers and complex numbers respectively. Set $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$. Then $\mathbb{T}$ is a group under multiplication. This group is called the circle group. The set of all subgroups of a group G will be denoted by $\mathrm{Sub}(G)$ which is a complete lattice with respect to inclusion. An *atom* in the lattice $\mathrm{Sub}(G)$ is a minimal element of $\mathrm{Sub}(G) \setminus \{\{1\}\}$. A *coatom* in the lattice $\mathrm{Sub}(G)$ is a maximal element of $\mathrm{Sub}(G) \setminus \{G\}$. $\mathrm{Sub}(G)$ is said to be *atomic* if each element of $\mathrm{Sub}(G) \setminus \{\{1\}\}$ contains an atom. $\mathrm{Sub}(G)$ is said to be *coatomic* if each element of

MSC(2010): Primary:06C99, Secondary:20K10.

Keywords: Atom, Coatom, Group, Lattice.

Received:25 Sept 2014, Accepted: 29 December 2014.

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$\mathrm{Sub}(G) \setminus \{G\}$ is contained in a coatom. The exponent of a group G is the least element $n = \exp G$ such that for each element $x \in G$, we have $x^n = 1$. A group G is said to be *locally cyclic* if for any $a, b \in G$, the group generated by $\{a, b\}$ is cyclic.

Definition 1.1. Let G be a group. An element $a \in G$ is said to be a *persistence element* if $a \neq 1$ and a is contained in all nontrivial subgroups of G. A group G is *persistent* if it has a persistence element.

Let a be a persistence element of a group G. There is a prime p with $|a| = p$. In fact, because $a \in \left\langle a^2 \right\rangle$ there is a positive integer m with $a = a^{2m}$. Thus $a^{2m-1} = 1$. So a has finite order. Let $n = |a|$ and suppose n is not prime. There are distinct primes p and q with $p \mid n,\ q \mid n$. Hence, there are $x, y \in \langle a \rangle$ with $|x| = p,\ |y| = q$ which implies that $a \in \langle x \rangle \cap \langle y \rangle = \{1\}$, a contradiction. Therefore, a has prime order.

If G is a persistent group, it is torsion. To see this, let a be a persistence element of G. If there is any $x \in G$ with infinite order, then $a \in \langle x \rangle \setminus \{1\}$. So, the order of a must be infinite, a contradiction. More generally, if $p = |a|$ then G is a p-group. Otherwise, G has an element x such that $|x|$ is divisible by a prime q different from p. Choose $b \in \langle x \rangle$ with $|b| = q$. This implies that $a \in \langle b \rangle \setminus \{1\}$ and so $|a| = q$, which is impossible.

Persistence is a *group property*, that is, for any two isomorphic groups, if one is persistent then the other is also persistent. More generally, let G and H be groups, b be a persistence element of H, $f : G \to H$ be a one-to-one homomorphism and $a \in f^{-1}[\{b\}]$. Then a is a persistence element of G. To prove this, it is evident that $a \neq 1$ and if $K \leq G$ is nontrivial then, since f is one-to-one, $f[K]$ is a nontrivial subgroup of H. Thus $b \in f[K]$ and so $a \in f^{-1}[\{b\}] \subseteq f^{-1}[f[K]] = K$.

Clearly a group G is persistent if and only if $\mathrm{Sub}(G)$ is atomic and has a single atom. For example, $G = \mathbb{Z}_2 \times \mathbb{Z}$ has a single atom $H = \{(0,0), (1,0)\}$, but G is not atomic and so it is not persistent.

A group G is said to be *decomposable* if there are nontrivial subgroups H and K of G such that $G \cong H \times K$. A persisted group is *indecomposable.* In fact, if G is a persistent decomposable group, there are nontrivial groups H and K with $G \cong H \times K$. Since $H \times K$ is persistent, it has a persistence element (a, b). We have $(a, b) \in (H \times \{1\}) \cap (\{1\} \times K) = \{(1, 1)\}$, which is impossible. As a result, if G is a finitely generated persistent abelian group, then it is a finite cyclic p-group for some prime number p.

The generalized quaternion group Q_{4n} is a group presented by

$$Q_{4n} = \langle a, b \mid a^n = b^2, a^{2n} = 1, b^{-1}ab = a^{-1} \rangle.$$

This is a group of order $|Q_{4n}| = 4n$. By [1, p. 262], if G is a finite p-group and any of its (nontrivial) abelian subgroup is cyclic, then either G is cyclic or it is isomorphic to a generalized quaternion

2−group. Now if G is a finite persistent group, then it is a p-group and every abelian subgroup of G is a persistent abelian group and so it must be cyclic. Therefore, either G is cyclic or it is isomorphic to a generalized quaternion group. As a result, a finite persistent group of odd order is cyclic.

Let p be a prime number. The *Prüfer p-group* is defined to be the subgroup

$$\mathbb{Z}_{p^\infty} = \left\{ e^{\frac{2k\pi i}{p^n}} \mid k \in \mathbb{Z},\ n \in \mathbb{N} \right\}$$

of the circle group $\mathbb{T}$. Let G be an infinite p-group. These statements are equivalent:

- G is isomorphic to $\mathbb{Z}_{p^\infty}$.
- $\mathrm{Sub}(G)$ is totally ordered.
- $\mathrm{Sub}(G)$ is distributive.
- G is locally cyclic [4].

Let G be an infinite abelian group with persistence element a and let $p = |a|$. Then G is isomorphic to $\mathbb{Z}_{p^\infty}$. To see this, it suffices to show that G is locally cyclic. Let $x, y \in G$ be non-identity elements and $H = \langle x, y \rangle$. Since H is finite, abelian and persistent, it is cyclic. It follows that any persistent abelian group is locally cyclic.

Example 1.2. By [3, p. 339], there is an infinite group G with exponent p^2 such that $|Z(G)| = p$ and for each $x \in G \setminus Z(G)$, $|x| = p^2$. Clearly G is not abelian and is persistent and so it is not locally cyclic.

Let G be an abelian group and p be a prime number. The set

$$G_{(p)} = \left\{ x \in G \mid (\exists n \in \mathbb{N})(x^{p^n} = 1) \right\}$$

is said to be the *p-primary component* of G. $G_{(p)}$ is said to be a *primary component* of G. Any primary component of G is a subgroup of G. According to [2] (page 88), a torsion abelian group is isomorphic to a direct sum of cyclic groups. Let G be an abelian group. By [6], there is a group H with $\mathrm{Sub}(G)$ order-isomorphic to $\mathrm{Sub}(H)$ if and only if G is torsion and any of its primary components is finite.

2. Main Results

Let G be a group and $A \subseteq G$. Then it is clear that A is a normal subgroup of G if and only if it is nonempty and for each $x \in G$, $xAA^{-1}x^{-1} \subseteq A$. We use this fact in the proof of the following theorem:

Theorem 2.1. *Suppose G is a group, $A \subseteq G$ and there is a $L \trianglelefteq G$ with $A \cap L = \emptyset$. Then there exists a maximal $M \trianglelefteq G$ with $A \cap M = \emptyset$.*

Proof. Define $\mathcal{M} = \{S \trianglelefteq G \mid S \cap M = \emptyset\}$ and let $\mathcal{C} \subseteq \mathcal{M}$ be a nonempty chain. Set $N = \bigcup_{C \in \mathcal{C}} C$ and choose $x \in G$. It suffices to show that $xNN^{-1}x^{-1} \subseteq N$. Let $a, b \in N$. Since $\mathcal{C}$ is a chain, there is a $C \in \mathcal{C}$ with $a, b \in C$ and so $xab^{-1}x^{-1} \in xCC^{-1}x^{-1} \subseteq C \subseteq N$. Thus $xNN^{-1}x^{-1} \subseteq N$ which implies that $N \trianglelefteq G$. Thus N is an upper bound for the arbitrary nonempty chain $\mathcal{C}$ in $\mathcal{M}$. By Zorn's lemma, $\mathcal{M}$ must have a maximal element M. ■

Let G be a group and N be a normal subgroup of G. Then by lattice theorem, $\mathrm{Sub}(\frac{G}{N})$ is order-isomorphic to the interval $[N, G]$ in $\mathrm{Sub}(G)$.

Theorem 2.2. *Let G be an abelian group, $a \in G$ and M be a maximal subgroup with $\{a\} \cap M = \emptyset$. Then $\frac{G}{M}$ is persisted with persistence element Ma.*

Proof. Let $\mathcal{A}$ be a nontrivial subgroup of $\frac{G}{M}$. There is a $N \leq G$ with $M \subset N$ and $\mathcal{A} = \frac{N}{M}$. Since $M \subset N$, there exists $b \in N \setminus M$. Define $M' = \langle M \cup \{b\} \rangle$. Taking into account the maximality of M, $\{a\} \cap M' \neq \emptyset$. Thus $a \in M'$ and so there are $m \in M$ and $k \in \mathbb{Z}$ with $a = mb^k$. Thus $Ma = Mb^k$ and because $b^k \in N$, $Ma \in \frac{N}{M} = \mathcal{A}$. ■

Theorem 2.3. *Let G be an abelian group with finite exponent. Then* $\mathrm{Sub}(G)$ *is coatomic.*

Proof. Let $N < G$. There is some $a \in G \setminus N$. Since $\{a\} \cap N = \emptyset$, there is a maximal (normal) subgroup M of G with $\{a\} \cap M = \emptyset$. To prove the theorem, it suffices to prove that the interval $[M, G]$, or equivalently $\mathrm{Sub}(\frac{G}{M})$, has a coatom. By the previous theorem, $\frac{G}{M}$ is persistent. Thus either $\frac{G}{M}$ is a finite cyclic group or it is isomorphic to a Prüfer group. If $\frac{G}{M}$ is isomorphic to a Prüfer group, for every positive integer n, there is some $x \in G$ with $n \leq |Mx|$. Therefore, for all $n \in N$, $n \leq |x| \leq \exp G$. So, $\exp G = \infty$, which is a contradiction. Thus $\frac{G}{M}$ is not isomorphic to a Prüfer group and so is finite. This proves that $\mathrm{Sub}(\frac{G}{M})$ has a coatom. ■

Let G be a group. By [5] (page 324), these statements are equivalent:

- $\mathrm{Sub}(G)$ has a coatom.
- There is some proper normal subgroup N of G such that $\frac{G}{N}$ is finite.
- G is not divisible.

So, if G is abelian and none of the its quotient groups are divisible, then according to the proof of Theorem 3, $\mathrm{Sub}(G)$ is coatomic. The converse is also true:

Theorem 2.4. *Let G be an abelian group and* $\mathrm{Sub}(G)$ *be coatomic. Then none of the quotient groups of G are divisible.*

The main idea of this proof is due to Derek Holt.

Proof. Suppose there is some $N \trianglelefteq G$ such that $\frac{G}{N}$ is divisible. There is a coatom M containing N. Since the interval $[M, G]$ is order-isomorphic to $\frac{G}{M}$, $|\operatorname{Sub}(\frac{G}{M})| = 2$ and so there is a prime number p with $|\frac{G}{M}| = p$. Choose $b \in G \setminus M$. Since $\frac{G}{N}$ is divisible, there exists $a \in G$ such that $Na^p = Nb$. But, $a^p b^{-1} \in N \subseteq M$ and so $Mb = Ma^p = M$ which implies $b \in M$. This leads to a contradiction. ■

Theorem 2.5. *Let G be a group. Then* Sub(G) *is atomic if and only if G is torsion.*

Proof. →) If there is $x \in G$ with infinite order, then $\langle x \rangle$ does not contain any atoms. ←) Let $H \leq G$ be nontrivial. There is non-identity element $a \in H$ and a prime p such that $|a|$ is finite and $p \mid |a|$. Thus, $\langle a \rangle$ has an element b of order p. Clearly $\langle b \rangle$ is an atom in Sub(G) and $\langle b \rangle \subseteq \langle a \rangle \subseteq H$. ■

Let G be a torsion abelian group. Let $N \in \operatorname{Sub}(G)$ and $N \neq G$. Since $\frac{G}{N}$ is torsion and abelian, it has an atom $\mathcal{A}$. $\mathcal{A}$ is a successor element to $\{N\}$. Since $\operatorname{Sub}(\frac{G}{N})$ is order-isomorphic to the interval $[N, G]$, there is a successor element to N in Sub(G). Therefore every element of Sub(G) except the greatest one, has an immediate successor, that is, Sub(G) is strongly atomic.

Let G be an abelian group with finite exponent. Let $N \in \operatorname{Sub}(G)$ be nontrivial. Since N is abelian and has finite exponent, it has a coatom C. We have $[C, N] = \{C, N\}$, that is, C is an immediate predecessor element to N. Therefore every nontrivial element of Sub(G) has an immediate predecessor, that is, Sub(G) is strongly coatomic.

3. Concluding Remarks

In this paper, the structure of atomic and coatomic subgroup lattice of infinite abelian groups are considered into account. In the case of non-abelian groups, the problem of characterizing infinite groups with coatomic subgroup lattice still remains unanswered. On the other hand, the structure of a group G in which every quotient group $\frac{G}{N}$ is persistent is a good problem for future research work.

Acknowledgment. The third author is partially supported by the university of Kashan under grant number 364988/19.

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