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A SHORT NOTE ON ATOMS AND COATOMS IN SUBGROUP LATTICES OF GROUPS

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Abstract. In this paper we give an elementary argument about the atoms and coatoms of the lattice of all subgroups of a group. It is proved that an abelian group of finite exponent is strongly coatomic.

1. Introduction and Preliminaries

We first describe some notations and definitions that will be kept throughout. The symbols $\mathbb{N}$, R, $\mathbb{C}$ denote the set of all positive integers, integers, real numbers and complex numbers respectively. Set $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$. Then $\mathbb{T}$ is a group under multiplication. This group is called the circle group. The set of all subgroups of a group G will be denoted by $\mathrm{Sub}(G)$ which is a complete lattice with respect to inclusion. An *atom* in the lattice $\mathrm{Sub}(G)$ is a minimal element of $\mathrm{Sub}(G) \setminus \{\{1\}\}$. A *coatom* in the lattice $\mathrm{Sub}(G)$ is a maximal element of $\mathrm{Sub}(G) \setminus \{G\}$. $\mathrm{Sub}(G)$ is said to be *atomic* if each element of $\mathrm{Sub}(G) \setminus \{\{1\}\}$ contains an atom. $\mathrm{Sub}(G)$ is said to be *coatomic* if each element of

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$\mathrm{Sub}(G) \setminus \{G\}$ is contained in a coatom. The exponent of a group G is the least element $n = \exp G$ such that for each element $x \in G$, we have $x^n = 1$. A group G is said to be *locally cyclic* if for any $a, b \in G$, the group generated by $\{a, b\}$ is cyclic.

Definition 1.1. Let G be a group. An element $a \in G$ is said to be a *persistence element* if $a \neq 1$ and a is contained in all nontrivial subgroups of G. A group G is *persistent* if it has a persistence element.

Let a be a persistence element of a group G. There is a prime p with $|a| = p$. In fact, because $a \in \langle a^2 \rangle$ there is a positive integer m with $a = a^{2m}$. Thus $a^{2m-1} = 1$. So a has finite order. Let $n = |a|$ and suppose n is not prime. There are distinct primes p and q with $p \mid n,\ q \mid n$. Hence, there are $x, y \in \langle a \rangle$ with $|x| = p,\ |y| = q$ which implies that $a \in \langle x \rangle \cap \langle y \rangle = \{1\}$, a contradiction. Therefore, a has prime order.

If G is a persistent group, it is torsion. To see this, let a be a persistence element of G. If there is any $x \in G$ with infinite order, then $a \in \langle x \rangle \setminus \{1\}$. So, the order of a must be infinite, a contradiction. More generally, if $p = |a|$ then G is a p-group. Otherwise, G has an element x such that $|x|$ is divisible by a prime q different from p. Choose $b \in \langle x \rangle$ with $|b| = q$. This implies that $a \in \langle b \rangle \setminus \{1\}$ and so $|a| = q$, which is impossible.

Persistence is a *group property*, that is, for any two isomorphic groups, if one is persistent then the other is also persistent. More generally, let G and H be groups, b be a persistence element of H, $f : G \to H$ be a one-to-one homomorphism and $a \in f^{-1}[\{b\}]$. Then a is a persistence element of G. To prove this, it is evident that $a \neq 1$ and if $K \leq G$ is nontrivial then, since f is one-to-one, $f[K]$ is a nontrivial subgroup of H. Thus $b \in f[K]$ and so $a \in f^{-1}[\{b\}] \subseteq f^{-1}[f[K]] = K$.

Clearly a group G is persistent if and only if $\mathrm{Sub}(G)$ is atomic and has a single atom. For example, $G = \mathbb{Z}_2 \times \mathbb{Z}$ has a single atom $H = \{(0,0), (1,0)\}$, but G is not atomic and so it is not persistent.

A group G is said to be *decomposable* if there are nontrivial subgroups H and K of G such that $G \cong H \times K$. A persisted group is *indecomposable.* In fact, if G is a persistent decomposable group, there are nontrivial groups H and K with $G \cong H \times K$. Since $H \times K$ is persistent, it has a persistence element (a, b). We have $(a, b) \in (H \times \{1\}) \cap (\{1\} \times K) = \{(1,1)\}$, which is impossible. As a result, if G is a finitely generated persistent abelian group, then it is a finite cyclic p-group for some prime number p.

The generalized quaternion group Q_{4n} is a group presented by

$$Q_{4n} = \langle a, b \mid a^n = b^2, a^{2n} = 1, b^{-1}ab = a^{-1} \rangle.$$

This is a group of order $|Q_{4n}| = 4n$. By [1, p. 262], if G is a finite p-group and any of its (nontrivial) abelian subgroup is cyclic, then either G is cyclic or it is isomorphic to a generalized quaternion

2−group. Now if G is a finite persistent group, then it is a p-group and every abelian subgroup of G is a persistent abelian group and so it must be cyclic. Therefore, either G is cyclic or it is isomorphic to a generalized quaternion group. As a result, a finite persistent group of odd order is cyclic.

Let p be a prime number. The *Prüfer p-group* is defined to be the subgroup

$$\mathbb{Z}_{p^\infty} = \left\{ e^{\frac{2k\pi i}{p^n}} \mid k \in \mathbb{Z},\ n \in \mathbb{N} \right\}$$

of the circle group $\mathbb{T}$. Let G be an infinite p-group. These statements are equivalent:

- G is isomorphic to $\mathbb{Z}_{p^\infty}$.
- $\mathrm{Sub}(G)$ is totally ordered.
- $\mathrm{Sub}(G)$ is distributive.
- G is locally cyclic [4].

Let G be an infinite abelian group with persistence element a and let $p = |a|$. Then G is isomorphic to $\mathbb{Z}_{p^\infty}$. To see this, it suffices to show that G is locally cyclic. Let $x, y \in G$ be non-identity elements and $H = \langle x, y \rangle$. Since H is finite, abelian and persistent, it is cyclic. It follows that any persistent abelian group is locally cyclic.

Example 1.2. By [3, p. 339], there is an infinite group G with exponent p^2 such that $|Z(G)| = p$ and for each $x \in G \setminus Z(G)$, $|x| = p^2$. Clearly G is not abelian and is persistent and so it is not locally cyclic.

Let G be an abelian group and p be a prime number. The set

$$G_{(p)} = \left\{ x \in G \mid (\exists n \in \mathbb{N})(x^{p^n} = 1) \right\}$$

is said to be the *p-primary component* of G. $G_{(p)}$ is said to be a *primary component* of G. Any primary component of G is a subgroup of G. According to [2] (page 88), a torsion abelian group is isomorphic to a direct sum of cyclic groups. Let G be an abelian group. By [6], there is a group H with $\mathrm{Sub}(G)$ order-isomorphic to $\mathrm{Sub}(H)$ if and only if G is torsion and any of its primary components is finite.

2. Main Results

Let G be a group and $A \subseteq G$. Then it is clear that A is a normal subgroup of G if and only if it is nonempty and for each $x \in G$, $xAA^{-1}x^{-1} \subseteq A$. We use this fact in the proof of the following theorem:

Theorem 2.1. *Suppose G is a group, $A \subseteq G$ and there is a $L \trianglelefteq G$ with $A \cap L = \emptyset$. Then there exists a maximal $M \trianglelefteq G$ with $A \cap M = \emptyset$.*

Proof. Define $\mathcal{M} = \{S \trianglelefteq G \mid S \cap M = \emptyset\}$ and let $\mathcal{C} \subseteq \mathcal{M}$ be a nonempty chain. Set $N = \bigcup_{C \in \mathcal{C}} C$ and choose $x \in G$. It suffices to show that $xNN^{-1}x^{-1} \subseteq N$. Let $a, b \in N$. Since $\mathcal{C}$ is a chain, there is a $C \in \mathcal{C}$ with $a, b \in C$ and so $xab^{-1}x^{-1} \in xCC^{-1}x^{-1} \subseteq C \subseteq N$. Thus $xNN^{-1}x^{-1} \subseteq N$ which implies that $N \trianglelefteq G$. Thus N is an upper bound for the arbitrary nonempty chain $\mathcal{C}$ in $\mathcal{M}$. By Zorn's lemma, $\mathcal{M}$ must have a maximal element M. ■

Let G be a group and N be a normal subgroup of G. Then by lattice theorem, $\mathrm{Sub}(\frac{G}{N})$ is order-isomorphic to the interval $[N, G]$ in $\mathrm{Sub}(G)$.

Theorem 2.2. *Let G be an abelian group, $a \in G$ and M be a maximal subgroup with $\{a\} \cap M = \emptyset$. Then $\frac{G}{M}$ is persisted with persistence element Ma.*

Proof. Let $\mathcal{A}$ be a nontrivial subgroup of $\frac{G}{M}$. There is a $N \leq G$ with $M \subset N$ and $\mathcal{A} = \frac{N}{M}$. Since $M \subset N$, there exists $b \in N \setminus M$. Define $M' = \langle M \cup \{b\} \rangle$. Taking into account the maximality of M, $\{a\} \cap M' \neq \emptyset$. Thus $a \in M'$ and so there are $m \in M$ and $k \in \mathbb{Z}$ with $a = mb^k$. Thus $Ma = Mb^k$ and because $b^k \in N$, $Ma \in \frac{N}{M} = \mathcal{A}$. ■

Theorem 2.3. *Let G be an abelian group with finite exponent. Then* $\mathrm{Sub}(G)$ *is coatomic.*

Proof. Let $N < G$. There is some $a \in G \setminus N$. Since $\{a\} \cap N = \emptyset$, there is a maximal (normal) subgroup M of G with $\{a\} \cap M = \emptyset$. To prove the theorem, it suffices to prove that the interval $[M, G]$, or equivalently $\mathrm{Sub}(\frac{G}{M})$, has a coatom. By the previous theorem, $\frac{G}{M}$ is persistent. Thus either $\frac{G}{M}$ is a finite cyclic group or it is isomorphic to a Prüfer group. If $\frac{G}{M}$ is isomorphic to a Prüfer group, for every positive integer n, there is some $x \in G$ with $n \leq |Mx|$. Therefore, for all $n \in N$, $n \leq |x| \leq \exp G$. So, $\exp G = \infty$, which is a contradiction. Thus $\frac{G}{M}$ is not isomorphic to a Prüfer group and so is finite. This proves that $\mathrm{Sub}(\frac{G}{M})$ has a coatom. ■

Let G be a group. By [5] (page 324), these statements are equivalent:

- $\mathrm{Sub}(G)$ has a coatom.
- There is some proper normal subgroup N of G such that $\frac{G}{N}$ is finite.
- G is not divisible.

So, if G is abelian and none of the its quotient groups are divisible, then according to the proof of Theorem 3, $\mathrm{Sub}(G)$ is coatomic. The converse is also true:

Theorem 2.4. *Let G be an abelian group and* $\mathrm{Sub}(G)$ *be coatomic. Then none of the quotient groups of G are divisible.*

The main idea of this proof is due to Derek Holt.

Proof. Suppose there is some $N \trianglelefteq G$ such that $\frac{G}{N}$ is divisible. There is a coatom M containing N. Since the interval $[M, G]$ is order-isomorphic to $\frac{G}{M}$, $|\operatorname{Sub}(\frac{G}{M})| = 2$ and so there is a prime number p with $|\frac{G}{M}| = p$. Choose $b \in G \setminus M$. Since $\frac{G}{N}$ is divisible, there exists $a \in G$ such that $Na^p = Nb$. But, $a^p b^{-1} \in N \subseteq M$ and so $Mb = Ma^p = M$ which implies $b \in M$. This leads to a contradiction. ■

Theorem 2.5. *Let G be a group. Then* $\operatorname{Sub}(G)$ *is atomic if and only if G is torsion.*

Proof. $\rightarrow$) If there is $x \in G$ with infinite order, then $\langle x \rangle$ does not contain any atoms. $\leftarrow$) Let $H \leq G$ be nontrivial. There is non-identity element $a \in H$ and a prime p such that $|a|$ is finite and $p \mid |a|$. Thus, $\langle a \rangle$ has an element b of order p. Clearly $\langle b \rangle$ is an atom in $\operatorname{Sub}(G)$ and $\langle b \rangle \subseteq \langle a \rangle \subseteq H$. ■

Let G be a torsion abelian group. Let $N \in \operatorname{Sub}(G)$ and $N \neq G$. Since $\frac{G}{N}$ is torsion and abelian, it has an atom $\mathcal{A}$. $\mathcal{A}$ is a successor element to $\{N\}$. Since $\operatorname{Sub}(\frac{G}{N})$ is order-isomorphic to the interval $[N, G]$, there is a successor element to N in $\operatorname{Sub}(G)$. Therefore every element of $\operatorname{Sub}(G)$ except the greatest one, has an immediate successor, that is, $\operatorname{Sub}(G)$ is strongly atomic.

Let G be an abelian group with finite exponent. Let $N \in \operatorname{Sub}(G)$ be nontrivial. Since N is abelian and has finite exponent, it has a coatom C. We have $[C, N] = \{C, N\}$, that is, C is an immediate predecessor element to N. Therefore every nontrivial element of $\operatorname{Sub}(G)$ has an immediate predecessor, that is, $\operatorname{Sub}(G)$ is strongly coatomic.

3. Concluding Remarks

In this paper, the structure of atomic and coatomic subgroup lattice of infinite abelian groups are considered into account. In the case of non-abelian groups, the problem of characterizing infinite groups with coatomic subgroup lattice still remains unanswered. On the other hand, the structure of a group G in which every quotient group $\frac{G}{N}$ is persistent is a good problem for future research work.

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