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INDEPENDENT SETS OF SOME GRAPHS ASSOCIATED TO COMMUTATIVE RINGS

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Abstract. Let $G = (V, E)$ be a simple graph. A set $S \subseteq V$ is independent set of G, if no two vertices of S are adjacent. The independence number $\alpha(G)$ is the size of a maximum independent set in the graph. In this paper we study and characterize the independent sets of the zero-divisor graph $\Gamma(R)$ and ideal-based zero-divisor graph $\Gamma_I(R)$ of a commutative ring R.

1. Introduction

A simple graph $G = (V, E)$ is a finite nonempty set $V(G)$ of objects called vertices together with a (possibly empty) set $E(G)$ of unordered pairs of distinct vertices of G called edges. The concept of zero-divisor graph of a commutative ring with identity was introduced by Beck in [8] and has been studied in [1, 2, 4, 5, 7]. Redmond in [14] has extended this concept to any arbitrary ring. Let R be a commutative ring with 1. The zero-divisor graph of R, denoted $\Gamma(R)$, is an undirected graph whose

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vertices are the nonzero zero-divisors of R and two distinct vertices x and y are adjacent if and only if $xy = 0$. Thus $\Gamma(R)$ is an empty graph if and only if R is an integral domain.
The concept of dominating set in zero-divisor graph has implicitly been studied in [11] and [13]. Throughout this article, all rings are commutative with identity $1 \neq 0$. For a subset A of a ring R, we let $A^* = A \setminus \{0\}$. We will denote the rings of integers modulo n, the integers, and a finite field with q elements by $\mathbb{Z}_n, \mathbb{Z}$ and F_q, respectively. For commutative ring theory, see [6, 12].
An independent set of a graph G is a set of vertices where no two vertices are adjacent. The independence number $\alpha(G)$ is the size of a maximum independent set in the graph. An independent set with cardinality $\alpha(G)$ is called a α-set ([3, 9, 10]).
A graph G is called bipartite if its vertex set can be partitioned into X and Y such that every edge of G has one endpoint in X and other endpoint in Y. A graph G is said to be star if G contains one vertex in which all other vertices are joined to this vertex and G has no other edges. A complete r-partite graph is one whose vertex set can be partitioned into r subsets so that no edge has both ends in any one subset and each vertex of a partite set is joined to every vertex of the another partite sets. We denote a complete bipartite graph by $K_{m,n}$. The graph $K_{1,n}$ is called a star graph, and a bistar graph is a graph generated by two $K_{1,n}$ graphs, where their centers are joined. For a nontrivial connected graph G and a pair vertices u and v of G, the distance $d(u, v)$ between u and v is the length of a shortest path from u to v in G. The girth of a graph G, containing a cycle, is the smallest size of the length of the cycles of G and is denoted by $gr(G)$. If G has no cycles, we define the girth of G to be infinite. A graph in which each pair of distinct vertices is joined by an edge is called a complete graph K_n on n vertices. For a graph G, a complete subgraph of G is called a clique. The clique number, $\omega(G)$, is the greatest integer $n \geq 1$ such that $K_n \subseteq G$, and $\omega(G)$ is infinite if $K_n \subseteq G$ for all $n \geq 1$, see [17].
Similar to paper [13], in this paper, we study the independent sets and independence number of zero-divisor graphs and ideal-based zero-divisor graphs. In Section 2 we review some preliminary results related to independence number of a graph. In Section 3, we study the independence number of zero-divisor graphs associated to commutative rings. In Section 4, investigate the independence number of an ideal based zero-divisor graph. Finally in Section 5, we list tables for graphs associated to small commutative ring R, and write independence, domination and clique number of $\Gamma(R)$.

2. Preliminary results

There are several classes of graphs whose independent sets and independence numbers are clear. We state some of them in the following Lemma, which their proofs are straightforward.

Lemma 2.1. ([17])

(i) $\alpha(K_n) = 1$.

(ii) *Let G be a complete r-partite graph ($r \geq 2$) with partite sets $V_1, ..., V_r$. If $|V_i| \geq 2$ for $1 \leq i \leq r$, then $\alpha(G) = max|V_i|$.*

(iii) $\alpha(K_{1,n}) = n$ *for a star graph* $K_{1,n}$.

(iv) *The independence number of a bistar graph is* $2n$.

(v) *Let C_n, P_n be a cycle and a path with n vertices, respectively. Then $\alpha(P_n) = \lfloor \frac{n+1}{2} \rfloor$ and $\alpha(C_n) = \lfloor \frac{n}{2} \rfloor$.*

Corollary 2.2. *Let F_1 and F_2 be finite fields with $|F_1^*| = m$ and $|F_2^*| = n$. Then*

(i) $\alpha(\Gamma(F_1 \times F_2)) = \max\{m, n\}$.

(ii) $\alpha(\Gamma(F_1 \times \mathbb{Z}_4)) = \max\{2m, 3\}$.

Proof.

(i) The graph $\Gamma(F_1 \times F_2)$ is bipartite ([4]) and we have the result by Lemma 2.1 (ii).

(ii) We have $Z^*(F_1 \times \mathbb{Z}_4) = \{(x, y)|x \in \; F_1^*, y = 0, 2\} \cup \{(0, y)|y = 1, 2, 3\}$.

If $F_1 = \mathbb{Z}_2$ then $\{(0, y)|y = 1, 2, 3\}$ is a maximum independent set in the graph and so $\alpha(\Gamma(F_1 \times Z_4)) = 3$. If $F_1 \neq \mathbb{Z}_2$ then $\{(x, y)|x \in \; F_1^*, y = 0, 2\}$ is a maximum independent set in the graph and so $\alpha(\Gamma(F_1 \times Z_4)) = 2m$. Therefore $\alpha(\Gamma(F_1 \times \mathbb{Z}_4)) = \max\{2m, 3\}$. □

3. Independence number of a zero-divisor graph

We begin this section with the following lemma:

Lemma 3.1. *Let R be a ring and $r \geq 3$. If $\Gamma(R)$ is a r-partite graph with parts $V_1, \ldots, V_r$, then $\alpha(\Gamma(R)) = max|V_i|$.*

Note that, for any prime number p and any positive integer n, there exists a finite ring R whose zero-divisor graph $\Gamma(R)$ is a complete p^n-partite graph. For example, if $\Gamma(R)$ is a finite field with p^n elements, then $R = F_{p^n}[x, y]/(xy, y^2 - x)$ is the desired ring.

Remark. It is easy to see that a graph G has independence number equal to 1, if for every $x, y \in Z(R)^*$, $xy = 0$, this means $\Gamma(R)$ is a complete graph.

We need the following theorem:

Theorem 3.2. ([5]) *If R is a commutative ring which is not an integral domain, then exactly one of the following holds:*

(i) $\Gamma(R)$ *has a cycle of length* 3 *or* 4 *(i.e., $gr(R) \leq 4$);*

(ii) $\Gamma(R)$ *is a star graph; or*

(iii) $\Gamma(R)$ *is the zero-divisor graph of* $R \cong \mathbb{Z}_2 \times \mathbb{Z}_4$ *or* $R \cong \mathbb{Z}_2 \times \mathbb{Z}[X]/(X^2)$.

By Theorem 3.2 we have the following theorem:

Theorem 3.3. *If $\Gamma(R)$ has no cycles, then $\alpha(\Gamma(R))$ is either $|Z^*(R)|-1$ or 3.*

Theorem 3.4. (i) *Let R be a finite ring. If $\Gamma(R)$ is a regular graph of degree r, then $\alpha(\Gamma(R))$ is either 1 or r.*

(ii) *Let R be a finite decomposable ring. If $\Gamma(R)$ is a Hamiltonian graph, then $\alpha(\Gamma(R)) = \frac{|Z^*(R)|}{2}$.*

(iii) *Let R be a finite principal ideal ring and not decomposable. If $\Gamma(R)$ is Hamiltonian, then $\alpha(\Gamma(R)) = 1$*

Proof.

(i) Since $\Gamma(R)$ is a regular graph of degree r, $\Gamma(R)$ is a complete graph K_{r+1} or a complete bipartite graph $K_{r,r}$. Consequently, $\alpha(\Gamma(R))$ is either 1 or r.

(ii) In this case $\Gamma(R)$ is $K_{n,n}$ for some natural number n. So, $\alpha(\Gamma(R)) = n$.

(iii) If R is not decomposable then in this case $\Gamma(R)$ is a complete graph. Therefore we have the result. □

Corollary 3.5. *The graph $\Gamma(\mathbb{Z}_n)$ is a Hamiltonian graph if and only if $\alpha(\Gamma(\mathbb{Z}_n)) = 1$.*

Proof. By Corollary 2 of [2], we know that the graph $\Gamma(\mathbb{Z}_n)$ is a Hamiltonian graph if and only if $n = p^2$, where p is a prime larger than 3 and $\Gamma(\mathbb{Z}_n)$ is isomorphic to K_{p-1}. Thus, we have the result. □

Here we state a notation which is useful for the study of the independence number of more graphs associated to commutative rings.
Let $R = F_1 \times \ldots \times F_n$, where F_i is an integral domain, for every i, and $|F_i| \geq |F_{i+1}|$. We set

$$E_{i_1 \ldots i_k} = \{(x_1, \ldots, x_n) \in R | \forall i \in \{i_1, \ldots, i_k\}, x_i \neq 0 \text{ and } \forall i \notin \{i_1, \ldots, i_k\}, x_i = 0\}.$$

By this notation we have $|E_{i_1 \ldots i_k}| = |F^*_{i_1}||F^*_{i_2}| \ldots |F^*_{i_k}|$.

Theorem 3.6. *Suppose that for a fixed integer $n \geq 2$, $R = R_1 \times \cdots \times R_n$, where R_i is an integral domain for each $i = 1, \ldots, n$. We have*

(i) $\alpha(\Gamma(R)) = \infty$ *if one of R_i is infinity,*

(ii)

$$\alpha(\Gamma(R)) \geq \left(\sum_{2 \leq i_2 \leq \ldots \leq i_{\lfloor \frac{k-1}{2} \rfloor} \leq n} n_1 n_{i_2} \ldots n_{i_{\lfloor \frac{k-1}{2} \rfloor}} \right) + \sum_{\lfloor \frac{k-1}{2} \rfloor + 1}^{n-1} \left(\sum_{1 \leq i_1 \leq \ldots \leq i_l \leq n} n_{i_1} \ldots n_{i_l} \right).$$

Proof. (i) We can suppose that $|R_1|$ is infinity. So $S = \{(x, 0, \ldots, 0) | x \in R^*_1\}$ is an independent set and therefore $\alpha(\Gamma(R)) = \infty$.

(ii) Let $|R_1| \geq |R_2| \geq \ldots \geq |R_n|$. It is easy to see that

$$A = \left(\bigcup_{2 \leq i_2 \leq \ldots \leq i_{\lfloor \frac{k-1}{2} \rfloor} \leq n} E_{1 i_2 \ldots i_{\lfloor \frac{k-1}{2} \rfloor}} \right) \bigcup \left(\bigcup_{\lfloor \frac{k-1}{2} \rfloor + 1}^{n-1} \left(\bigcup_{1 \leq i_1 \leq \ldots \leq i_l \leq n} E_{i_1 \ldots i_l} \right) \right)$$

is an independent set of $\Gamma(R)$. So

$$\alpha(\Gamma(R)) \geq |A| = \sum_{\lfloor \frac{k-1}{2} \rfloor + 1}^{n-1} \left(\sum_{1 \leq i_1 \leq \ldots \leq i_l \leq n} n_{i_1} \ldots n_{i_l} \right) + \left(\sum_{2 \leq i_2 \leq \ldots \leq i_{\lfloor \frac{k-1}{2} \rfloor} \leq n} n_1 n_{i_2} \ldots n_{i_{\lfloor \frac{k-1}{2} \rfloor}} \right) \quad \square$$

Theorem 3.7. *Suppose that $n_1 \geq n_2 \geq n_3$ and $|F_i^*| = n_i$ for $i = 1, 2, 3$. If $R = F_1 \times F_2 \times F_3$, then*

$$\alpha(\Gamma(R)) = n_1 n_2 + n_1 n_3 + \max\{n_1, n_2 n_3\}.$$

Proof. It is not difficult to see that one of the following sets is a maximum independent set in the zero-divisor graph of $F_1 \times F_2 \times F_3$:

$$A_1 = E_{12} \cup E_{13} \cup E_{23},$$
$$A_2 = E_{12} \cup E_{13} \cup E_1.$$

So $\alpha(\Gamma(R)) = \max\{|A_1|, |A_2|\} = n_1 n_2 + n_1 n_3 + \max\{n_1, n_2 n_3\}$. $\square$

Let us to state two examples for the above theorem:

Example 3.8. Let $R = \mathbb{Z}_5 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Here $A_2 = E_{12} \cup E_{13} \cup E_1$ is a α-set of graph $\Gamma(R)$ and so $\alpha(\Gamma(R)) = n_1 n_2 + n_1 n_3 + n_1 = 9$.

Example 3.9. Let $R = \mathbb{Z}_7 \times \mathbb{Z}_5 \times \mathbb{Z}_5$. Here $A_1 = E_{12} \cup E_{13} \cup E_{23}$ is a α-set and $\alpha(\Gamma(R)) = n_1 n_2 + n_1 n_3 + n_2 n_3 = 64$.

Theorem 3.10. *Suppose that $n_1 \geq n_2 \geq n_3 \geq n_4$ and $|F_i^*| = n_i$ for $i = 1, 2, 3, 4$. Let $R = F_1 \times F_2 \times F_3 \times F_4$.*

(i) *If $n_1 \geq n_2 n_3 n_4$, then $\alpha(\Gamma(R)) = n_1(n_2 n_3 + n_2 n_4 + n_3 n_4 + n_2 + n_3 + n_4 + 1)$.*

(ii) *If $n_1 \leq n_2 n_3 n_4$ and $n_1 n_4 \geq n_2 n_3$, then $\alpha(\Gamma(R)) = n_1(n_2 n_3 + n_2 n_4 + n_3 n_4 + n_2 + n_3 + n_4) + n_2 n_3 n_4$.*

(iii) *If $n_1 n_4 \leq n_2 n_3$, then $\alpha(\Gamma(R)) = n_1(n_2 n_3 + n_2 n_4 + n_3 n_4 + n_2 + n_3) + n_2 n_3 + n_2 n_3 n_4$.*

Proof. Since $n_1 \geq n_2 \geq n_3 \geq n_4$, it is easy to check that one of the following sets is a α-set of the graph $\Gamma(R)$:

$$I_1 = E_{123} \cup E_{124} \cup E_{134} \cup E_{12} \cup E_{13} \cup E_{14} \cup E_1,$$
$$I_2 = E_{123} \cup E_{124} \cup E_{134} \cup E_{12} \cup E_{13} \cup E_{14} \cup E_{234},$$
$$I_3 = E_{123} \cup E_{124} \cup E_{134} \cup E_{12} \cup E_{13} \cup E_{23} \cup E_{234},$$

(i) If $n_1 \geq n_2n_3n_4$ then $n_1n_4 \geq n_2n_3n_4$, and I_1 is a α-set in the graph. Therefore $\alpha(\Gamma(R)) = n_1(n_2n_3 + n_2n_4 + n_3n_4 + n_2 + n_3 + n_4 + 1)$.

(ii) If $n_1 \leq n_2n_3n_4$ and $n_1n_4 \geq n_2n_3$, then I_2 is a α-set in the graph. Therefore $\alpha(\Gamma(R)) = n_1(n_2n_3 + n_2n_4 + n_3n_4 + n_2 + n_3 + n_4) + n_2n_3n_4$.

(iii) If $n_1n_4 \leq n_2n_3$ then $n_1 \leq n_2n_3n_4$ and I_3 is a α-set in the graph. So $\alpha(\Gamma(R)) = n_1(n_2n_3 + n_2n_4 + n_3n_4 + n_2 + n_3) + n_2n_3 + n_2n_3n_4$. □

The following corollary is an immediate consequence of Theorem 3.10.

Corollary 3.11. *Suppose that $n_1 \geq n_2 \geq n_3 \geq n_4$ and $|F_i^*| = n_i$ for $i = 1, 2, 3, 4$. If $R = F_1 \times F_2 \times F_3 \times F_4$, then*

$$\alpha(\Gamma(R)) = n_1(n_2n_3 + n_2n_4 + n_3n_4 + n_2 + n_3) + \max\{n_1 + n_1n_4, n_2n_3 + n_2n_3n_4, n_1n_4 + n_2n_3n_4\}.$$

Here we bring up some examples for Theorem 3.10.

Example 3.12. Let $R = \mathbb{Z}_5 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. The set I_1 in Theorem 3.10, is a α-set in the graph and so $\alpha(\Gamma(R)) = 28$.

Example 3.13. Let $R = \mathbb{Z}_5 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. The set I_2 in Theorem 3, is a α-set in the graph and so $\alpha(\Gamma(R)) = 80$.

Example 3.14. Let $R = \mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_3 \times \mathbb{Z}_2$. The set I_3 in Theorem 3, is a α-set in the graph and so $\alpha(\Gamma(R)) = 88$.

Theorem 3.15. *Suppose that $|F_i^*| = n_i$, where $n_i \geq n_j$ and $i \geq j$ for $i, j = 1, \ldots, 5$. Let $R = F_1 \times \ldots \times F_5$. If $t = n_1(\sum_{2\leq i<j<k\leq 5} n_in_jn_k) + n_1(\sum_{\substack{2\leq i<j\leq 5\\(i,j)\neq(4,5)}} n_in_j)$, then*

(i) *If $n_1 \geq n_2n_3n_4n_5$, then $\alpha(\Gamma(R)) = t + n_1(n_4n_5 + n_2 + n_3 + n_4 + n_5 + 1)$.*

(ii) *If $n_2n_3 \geq n_1n_4n_5$, then $\alpha(\Gamma(R)) = t + n_2(n_3n_4n_5 + n_3n_4 + n_3n_5 + n_1 + n_3) + n_1n_3$.*

(iii) *If $n_1n_5 \geq n_2n_3n_4$, then $\alpha(\Gamma(R)) = t + n_1(n_4n_5 + n_2 + n_3 + n_4 + n_5) + n_2n_3n_4n_5$.*

(iv) *If $n_1n_5 \leq n_2n_3n_4$ and $n_1n_4 \geq n_2n_3n_5$, then $\alpha(\Gamma(R)) = t + n_1(n_4n_5 + n_2 + n_3 + n_4) + n_2(n_3n_4n_5 + n_3n_4)$.*

(v) *If $n_1n_4 \leq n_2n_3n_5$ and $n_1n_3 \geq n_2n_4n_5$, then $\alpha(\Gamma(R)) = t + n_1(n_4n_5 + n_2 + n_3) + n_2(n_3n_4n_5 + n_3n_4 + n_3n_5)$.*

(vi) *If $n_1n_3 \leq n_2n_4n_5$ and $n_1n_2 \geq n_3n_4n_5$, then $\alpha(\Gamma(R)) = t + n_1(n_4n_5 + n_2) + n_2(n_3n_4n_5 + n_3n_4 + n_3n_5 + n_4n_5)$.*

(vii) *If $n_1n_2 \leq n_3n_4n_5$, then $\alpha(\Gamma(R)) = t + (n_1 + n_3)n_4n_5 + n_2(n_3n_4n_5 + n_3n_4 + n_3n_5 + n_4n_5)$.*

proof. We put $A = (\bigcup_{2\le i<j<k\le 5} E_{1ijk})\bigcup(\bigcup_{\substack{2\le i<j\le 5\\(i,j)\neq(4,5)}} E_{1ij})$. Consider the sets A_i and B_i for $i = 1,\dots,6$ as shown in the following table.

	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$
A_i	E_1	E_{23}	E_{12}	E_{13}	E_{14}	E_{15}
B_i	E_{2345}	E_{145}	E_{345}	E_{245}	E_{235}	E_{234}

We have:

(i) If $n_1 \ge n_2n_3n_4n_5$, then by the above table $|A_1| \ge |B_1|$ and this implies $|B_2| \ge |A_2|$ and for $i = 3,4,5,6$, $|A_i| \ge |B_i|$. So $A \cup A_1 \cup B_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6$ has the size of a maximum independent set in the graph and $\alpha(\Gamma(R)) = t + n_1(n_4n_5 + n_2 + n_3 + n_4 + n_5 + 1)$.

(ii) If $n_2n_3 \ge n_1n_4n_5$ then $|A_2| \ge |B_2|$ and this implies $|B_1| \ge |A_1|$,$|A_3| \ge |B_3|$,$|A_4| \ge |B_4|$,$|B_5| \ge |A_5|$ and $|B_6| \ge |A_6|$, so $A\cup B_1\cup A_2\cup A_3\cup A_4\cup B_5\cup B_6$ has the size of a maximum independent set in the graph and $\alpha(\Gamma(R)) = t + n_2(n_3n_4n_5 + n_3n_4 + n_3n_5 + n_1 + n_3) + n_1n_3$.

(iii) If $n_1n_5 \ge n_2n_3n_4$ and $n_1 \le n_2n_3n_4n_5$ then $|A_6| \ge |B_6|$ and $|B_1| \ge |A_1|$, now $|B_2| \ge |A_2|$ and for $i = 3,4,5$, $|A_i| \ge |B_i|$, so $A \cup B_1 \cup B_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6$ has the size of a maximum independent set in the graph and $\alpha(\Gamma(R)) = t + n_1(n_4n_5 + n_2 + n_3 + n_4 + n_5) + n_2n_3n_4n_5$.

(iv) If $n_1n_5 \le n_2n_3n_4$ and $n_1n_4 \ge n_2n_3n_5$ then $|B_6| \ge |A_6|$ and $|A_5| \ge |B_5|$, now $|B_1| \ge |A_1|$, $|B_2| \ge |A_2|$ and for $i = 3,4$, $|A_i| \ge |B_i|$, so $A\cup B_1\cup B_2\cup A_3\cup A_4\cup A_5\cup B_6$ has the size of a maximum independent set in the graph and $\alpha(\Gamma(R)) = t+n_1(n_4n_5+n_2+n_3+n_4)+n_2(n_3n_4n_5+n_3n_4)$.

(v) If $n_1n_4 \le n_2n_3n_5$ and $n_1n_3 \ge n_2n_4n_5$ then $|B_5| \ge |A_5|$ and $|A_4| \ge |B_4|$, therefore $|A_3| \ge |B_3|$ and for $i = 1,2,6$, $|B_i| \ge |A_i|$, so $A \cup B_1 \cup B_2 \cup A_3 \cup A_4 \cup B_5 \cup B_6$ has the size of a maximum independent set in the graph and $\alpha(\Gamma(R)) = t+n_1(n_4n_5+n_2+n_3)+n_2(n_3n_4n_5+n_3n_4+n_3n_5)$.

(vi) If $n_1n_3 \le n_2n_4n_5$ and $n_1n_2 \ge n_3n_4n_5$ then $|B_4| \ge |A_4|$ and $|A_3| \ge |B_3|$, so for $i = 1,2,5,6$, $|B_i| \ge |A_i|$, hence $A \cup B_1 \cup B_2 \cup A_3 \cup B_4 \cup B_5 \cup B_6$ has the size of a maximum independent set in the graph and $\alpha(\Gamma(R)) = t + n_1(n_4n_5 + n_2) + n_2(n_3n_4n_5 + n_3n_4 + n_3n_5 + n_4n_5)$.

(vii) If $n_1n_2 \le n_3n_4n_5$ then $|B_3| \ge |A_3|$ and for $i = 1,2,4,5,6$, $|B_i| \ge |A_i|$, hence $A \cup B_1 \cup B_2 \cup B_3 \cup B_4 \cup B_5 \cup B_6$ has the size of a maximum independent set in the graph and $\alpha(\Gamma(R)) = t + (n_1 + n_3)n_4n_5 + n_2(n_3n_4n_5 + n_3n_4 + n_3n_5 + n_4n_5)$. $\square$

Corollary 3.16. *Let $R = F_1 \times \dots \times F_5$, $|F_i^*| = n_i$ and $n_i \ge n_j$, where $i,j = 1,\dots,5$ and $i \ge j$. Then*

$$\alpha(\Gamma(R)) = n_1(\sum_{2\le i<j<k\le 5} n_in_jn_k) + n_1(\sum_{\substack{2\le i<j\le 5\\(i,j)\neq(4,5)}} n_in_j) + \max_i\{\Delta_i\},$$

where

$$\Delta_1 = n_1(n_4n_5 + n_2 + n_3 + n_4 + n_5 + 1)$$
$$\Delta_2 = n_2(n_3n_4n_5 + n_3n_4 + n_3n_5 + n_1 + n_3) + n_1n_3$$
$$\Delta_3 = n_1(n_4n_5 + n_2 + n_3 + n_4 + n_5) + n_2n_3n_4n_5$$
$$\Delta_4 = n_1(n_4n_5 + n_2 + n_3 + n_4) + n_2(n_3n_4n_5 + n_3n_4)$$
$$\Delta_5 = n_1(n_4n_5 + n_2 + n_3) + n_2(n_3n_4n_5 + n_3n_4 + n_3n_5)$$
$$\Delta_6 = n_1(n_4n_5 + n_2) + n_2(n_3n_4n_5 + n_3n_4 + n_3n_5 + n_4n_5)$$
$$\Delta_7 = (n_1 + n_3)n_4n_5 + n_2(n_3n_4n_5 + n_3n_4 + n_3n_5 + n_4n_5)$$

Example 3.17. (i) Let $R = \mathbb{Z}_5 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Then in Theorem 3.15, $\alpha(\Gamma(R)) = t + \Delta_1$,

(ii) Let $R = \mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Then in Theorem 3.15, $\alpha(\Gamma(R)) = t + \Delta_2$,

(iii) Let $R = \mathbb{Z}_7 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. Then in Theorem 3.15, $\alpha(\Gamma(R)) = t + \Delta_3$,

(iv) Let $R = \mathbb{Z}_7 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_2$. Then in Theorem 3.15, $\alpha(\Gamma(R)) = t + \Delta_4$,

(v) Let $R = \mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Then in Theorem 3.15, $\alpha(\Gamma(R)) = t + \Delta_5$,

(vi) Let $R = \mathbb{Z}_7 \times \mathbb{Z}_7 \times \mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Then in Theorem 3.15, $\alpha(\Gamma(R)) = t + \Delta_6$,

(vii) Let $R = \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. Then in Theorem 3.15, $\alpha(\Gamma(R)) = t + \Delta_7$.

Theorem 3.18. *Let $(R, \mathfrak{m})$ be a finite local ring and $\mathfrak{m} \neq \{0\}$.*

(i) *If $\mathfrak{m}^2 = \{0\}$, then $\alpha(\Gamma(R)) = 1$.*

(ii) *If $\mathfrak{m}^2 \neq \{0\}$, then $2 \leq \alpha(\Gamma(R)) \leq |Z^*(R)| - |Ann(Z(R))^*|$.*

Proof. If R is a finite local ring, then the Jacobson radical of R equals $Z(R)$ and $Z(R) = \mathtt{m}$. Thus $Z(R)$ is a nilpotent ideal and since R is not a field, then $Ann(Z(R)) \neq \{0\}$. Moreover, each element of $Ann(Z(R))$ is adjacent to each other vertex of $Z^*(R)$.

(i) If $\mathtt{m}^2 = \{0\}$ then $Ann(Z(R)) = Z^*(R)$ and $\Gamma(R)$ is a complete graph.

(ii) If $\mathtt{m}^2 \neq \{0\}$, then every element of $Ann(Z(R))^*$ is adjacent to each other vertex of $Z^*(R)$ and this implies $2 \leq \alpha(\Gamma(R)) \leq |Z^*(R)| - |Ann(Z(R))^*|$. □

Example 3.19. Let $R = \mathbb{Z}_{p^3}$ then $Z^*(R) = \{pk|(p,k) = 1\} \cup \{p^2k|(p^2,k) = 1\}$. We have $Ann(Z(R))^* = \{p^2k|(p^2,k) = 1\}$ and $\{pk|(p,k) = 1\}$ is an independent set in the $\Gamma(R)$ of maximum size. So $\alpha(\Gamma(R)) = |\{pk|(p,k) = 1\}| = |Z^*(R)| - |Ann(Z(R))^*|$.

4. The independence number of an ideal-based zero-divisor graph

In this section, we study the relationship between the independence numbers of $\Gamma_I(R)$ and $\Gamma(R/I)$. Suppose that R is a commutative ring with nonzero identity, and I is an ideal of R. The ideal-based zero-divisor graph of R, denoted by $\Gamma_I(R)$, is the graph which vertices are the set $\{x \in R\backslash I | xy \in I \quad for some \quad y \in R\backslash I\}$ and two distinct vertices x and y are adjacent if and only if $xy \in I$, see [16]. In the case $I = 0$, $\Gamma_0(R)$ is denoted by $\Gamma(R)$. Also, $\Gamma_I(R)$ is empty if and only if I is prime. Note that

Proposition 2.2(b) of [16] is equivalent to saying $\Gamma_I(R) = \emptyset$ if and only if R/I is an integral domain. That is, $\Gamma_I(R) = \emptyset$ if and only if $\Gamma(R/I) = \emptyset$.

This naturally raises the question: If R is a commutative ring with ideal I, whether $\alpha(\Gamma_I(R))$ is equal to $\alpha(\Gamma(R/I))$? We show that the answer is negative in general.

Lemma 4.1. *Let m be a composite natural number and p a prime number. Then*

$$\alpha(\Gamma_{m\mathbb{Z}}(\mathbb{Z})) = \begin{cases} \alpha(\mathbb{Z}/m\mathbb{Z}) = 1; & \text{if } m = p^2, \\ \\ \infty; & \text{otherwise.} \end{cases}$$

Note that for the second case $\alpha(\Gamma_{m\mathbb{Z}}(\mathbb{Z})) = \infty$ and $\alpha(\mathbb{Z}/m\mathbb{Z}) < \infty$.

Proof. If $m = p^2$ then for every $x \in \Gamma_{m\mathbb{Z}}(\mathbb{Z})$ we have $x = pk$, where $(p, k) = 1$. So $x, y \in \Gamma_{m\mathbb{Z}}(\mathbb{Z})$ are adjacent in $\Gamma_I(R)$ and $\Gamma_I(R)$ is a complete graph. Also $\mathbb{Z}/m\mathbb{Z} \cong \mathbb{Z}_{p^2}$ and $\Gamma(\mathbb{Z}/m\mathbb{Z})$ is a complete graph.

Now let m be a non-prime number and for every prime number p, $m \neq p^2$. Then we have $m = p^i n$, p is prime, $n \neq 1$ and $(n, p) = 1$, or $m = p^l$, p is prime and $l \geq 3$.

If $m = p^l$ then $S = \{kp|(k, p) = 1\}$ is an independent set and therefore $\alpha(\Gamma_{m\mathbb{Z}}(\mathbb{Z})) = \infty$.

If $m = p^i n$ then $S = \{kp|(k, p) = 1 \text{ and } n|k\}$ is an independent set and therefore $\alpha(\Gamma_{m\mathbb{Z}}(\mathbb{Z})) = \infty$. But, we have $\mathbb{Z}/m\mathbb{Z}$ is a finite ring and $\alpha(\Gamma(\mathbb{Z}/m\mathbb{Z}))$ is finite. □

Now we state the following results of [16].

Lemma 4.2. ([16]) *Let I be an ideal of a ring R, and x, y be in $R \backslash I$. Then:*

(i) *If $x + I$ is adjacent to $y + I$ in $\Gamma(R/I)$, then x is adjacent to y in $\Gamma_I(R)$;*

(ii) *If x is adjacent to y in $\Gamma_I(R)$ and $x + I \neq y + I$, then $x + I$ is adjacent to $y + I$ in $\Gamma(R/I)$;*

(iii) *If x is adjacent to y in $\Gamma_I(R)$ and $x + I = y + I$, then $x^2, y^2 \in I$.*

Lemma 4.3. ([16]) *If x and y are (distinct) adjacent vertices in $\Gamma_I(R)$, then all (distinct) elements $x + I$ and $y + I$ are adjacent in $\Gamma_I(R)$. If $x^2 \in I$, then all the distinct elements of $x + I$ are adjacent in $\Gamma_I(R)$.*

Theorem 4.4. *Let S be a nonempty subset of $R \backslash I$. If $S + I = \{s + I|s \in S\}$ is an independent set of $\Gamma(R/I)$, then S is a independent set of $\Gamma_I(R)$.*

Proof. Let S be a nonempty subset of $R \backslash I$ and $S + I = \{s + I|s \in S\}$ be an independent set of $\Gamma(R/I)$. If $x, y \in S$, then $x + I$ and $y + I$ are not adjacent in $\Gamma(R/I)$ and by Lemma 4.2(i), x and y are not adjacent in $\Gamma_I(R)$. □

The following corollary is an immediate consequence of the above theorem:

Corollary 4.5. $\alpha(\Gamma(R/I)) \leq \alpha(\Gamma_I(R))$.

Theorem 4.6. *Let $S+I$ be an independent set with cardinality $\alpha(\Gamma(R/I))$ and $A = \{s+I \in S+I | s^2+I = I\}$. Then $\alpha(\Gamma_I(R)) = |A| + |I|(\alpha(\Gamma(R/I)) - |A|)$.*

Proof. Suppose that $s \in S$, $x \in s+I$ and $y \in s+I$. If $s^2 \in I$ then $x \in s+I$ and $y \in s+I$ are adjacent vertices in $\Gamma_I(R)$. If $s^2 \notin I$ then $x \in s+I$ and $y \in s+I$ are not adjacent in $\Gamma_I(R)$. Therefore $T = \{s | s^2 \in I\} \cup \{s+i | i \in I, s^2 \notin I\}$ is an independent set with maximum cardinality. $\square$

Corollary 4.7. $\alpha(\Gamma(R/I)) \leq \alpha(\Gamma_I(R)) \leq |I|\alpha(\Gamma(R/I))$

Corollary 4.8. *If S is an independent set with cardinality $\alpha(\Gamma_I(R))$, and $s^2 \in I$ for every $s \in S$, then $\alpha(\Gamma_I(R)) = \alpha(\Gamma(R/I))$.*

Corollary 4.9. *If S is an independent set with cardinallity $\alpha(\Gamma_I(R))$, and $s^2 \notin I$ for every $s \in S$, then $\alpha(\Gamma_I(R)) = |I|\alpha(\Gamma(R/I))$.*

We state two following examples for corollaries:

Example 4.10. Let $R = \mathbb{Z}_6 \times \mathbb{Z}_3$ and $I = 0 \times \mathbb{Z}_3$ be an ideal of R. Then it easy to see that $\Gamma_I(R) = \{(2,0),(2,1),(2,2),(3,0),(3,1),(3,2),(4,0),(4,1),(4,2)\}$ and $\Gamma(R/I) = \{(2,0)+I,(3,0)+I,(4,0)+I\}$. The set $T = \{(2,0),(2,1),(2,2),(4,0),(4,1),(4,2)\}$ is an independent set of $\Gamma_I(R)$ and so $\alpha(\Gamma_I(R)) = 6$. On the other hand $S+I = \{(2,0)+I,(4,0)+I\}$ is an independent set of $\Gamma(R/I)$ and $\alpha(\Gamma(R/I)) = 2$. Therefore $\alpha(\Gamma_I(R)) = |I|\alpha(\Gamma(R/I))$.

Example 4.11. Let $R = \mathbb{Z}_{16}$ and $I = 4\mathbb{Z}_{16}$. Then $\Gamma_I(R) = \{2,6,10,14\}$ and $\Gamma(R/I) = \{2+I\}$. Then $T = \{2\}$ is an independent set of $\Gamma_I(R)$ and $\alpha(\Gamma_I(R)) = 1$. On the other hand $S+I = \{2+I\}$ is an independent set of $\Gamma(R/I)$ and $\alpha(\Gamma(R/I)) = 1$. So we have $\alpha(\Gamma_I(R)) = \alpha(\Gamma(R/I))$.

Example 4.12. Let $R = \mathbb{Z}_{16} \times \mathbb{Z}_3$ and $I = 0 \times \mathbb{Z}_3$ be an ideal of R. Then it easy to see that $\Gamma_I(R) = \{(x,y) | x = 2,4,\ldots,14, y = 0,1,2\}$ and $\Gamma(R/I) = \{(x,0)+I | x = 2,4,\ldots,14\}$. Then $T = \{(x,y) | x = 2,6,10,14, y = 0,1,2\} \cup \{(4,0)\}$ is an independent set of $\Gamma_I(R)$ and so $\alpha(\Gamma_I(R)) = 13$. On the other hand $S+I = \{(x,0)+I | x = 2,4,6,10,14\}$ is an independent set of $\Gamma(R/I)$ and $\alpha(\Gamma(R/I)) = 5$. Let A be the set defined in Theorem 4.6, then $A = \{4\}$. So we have $\alpha(\Gamma_I(R)) = 13 = 1 + 3(5-1) = |A| + |I|(\alpha(\Gamma(R/I)) - |A|)$.

5. Independence, domination and clique number of small finite commutative rings

In this section similar to [15], we list the tables for graphs associated to commutative ring R, and write independence, domination and clique number of $\Gamma(R)$. Note that the tables for $n = |V\Gamma| = 1,2,3,4$ can be found in [4]. The results for $n = 5$ can be found in [16]. In [15], all graphs on $6,7,\ldots,14$

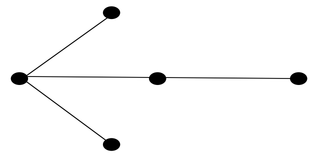

FIGURE 1. Graph for $\mathbb{Z}_2 \times \mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2)$

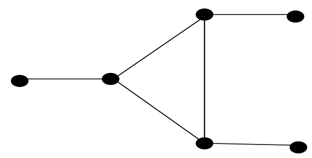

FIGURE 2. Graph for $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

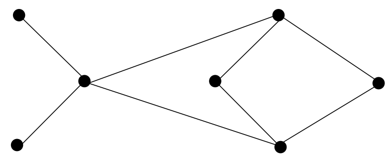

FIGURE 3. Graph for $\mathbb{Z}_3 \times \mathbb{Z}_4$ and $\mathbb{Z}_3 \times \mathbb{Z}_2[X]/(X^2)$

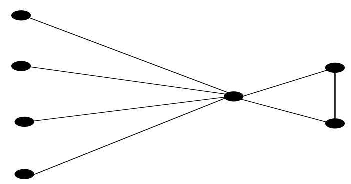

FIGURE 4. Graph for $\mathbb{Z}_4, \mathbb{Z}_2[X]/(X^4), \mathbb{Z}_4[X]/(X^2+2), \mathbb{Z}_4[X]/(X^2+3X)$ and $\mathbb{Z}_4[X]/(X^3-2, 2X^2, 2X)$

vertices which can be realized as the zero-divisor graphs of a commutative rings with 1, and the list of all rings (up to isomorphism) which produce these graphs, are given.

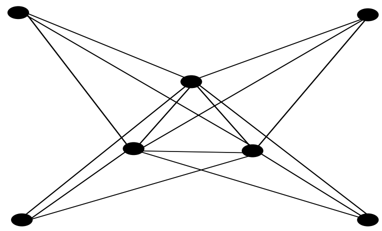

FIGURE 5. Graph for $\mathbb{Z}_2[X,Y]/(X^3, XY, Y^2), \mathbb{Z}_8[X]/(2X, X^2)$ and $\mathbb{Z}_4[X]/(X^3, 2X^2, 2X)$

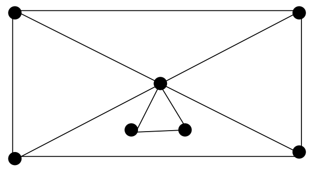

FIGURE 6. Graph for $\mathbb{Z}_4[X]/(X^2+2X), \mathbb{Z}_8[X]/(2X, X^2+4), \mathbb{Z}_2[X,Y]/(X^2, Y^2-XY)$ and $\mathbb{Z}_4[X]/(X^2, Y^2-XY, XY-2, 2X, 2Y)$

Vertices	R	$\|R\|$	Graph	$\alpha(\Gamma(R))$	$\gamma(\Gamma(R))$	$\omega(\Gamma(R))$
3	$\mathbb{Z}_6$	6	$K_{1,2}$	2	1	2
	$\mathbb{Z}_8$	8	$K_{1,2}$	2	1	2
	$\mathbb{Z}_2[X]/(X^3)$	8	$K_{1,2}$	2	1	2
	$\mathbb{Z}_4[X]/(2X, X^2-2)$	8	$K_{1,2}$	2	1	2
	$\mathbb{Z}_2[X,Y]/(X,Y)^2$	8	K_3	1	1	3
	$\mathbb{Z}_4[X]/(2,X)^2$	8	K_3	1	1	3
	$\mathbb{F}_4[X]/(X^2)$	16	K_3	1	1	3
	$\mathbb{Z}_4[X]/(X^2+X+1)$	16	K_3	1	1	3

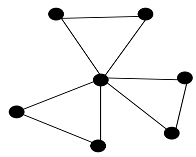

FIGURE 7. Graph for $\mathbb{Z}_4[X,Y]/(X^2, Y^2, XY-2, 2X, 2Y), \mathbb{Z}_2[X,Y]/(X^2, Y^2)$ and $\mathbb{Z}_4[X]/(X^2)$

Vertices	R	$\|R\|$	Graph	$\alpha(\Gamma(R))$	$\gamma(\Gamma(R))$	$\omega(\Gamma(R))$
4	$\mathbb{Z}_2 \times \mathbb{F}_4$	8	$K_{1,3}$	3	1	2
	$\mathbb{Z}_3 \times \mathbb{Z}_3$	9	$K_{2,2}$	2	2	2
	$\mathbb{Z}_25$	25	K_4	1	1	4
	$\mathbb{Z}_5[X]/(X^2)$	25	K_4	1	1	4

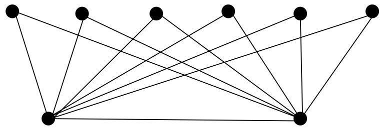

FIGURE 8. Graph for $\mathbb{Z}_4[X]/(X^3 - X^2 - 2, 2X^2, 2X)$

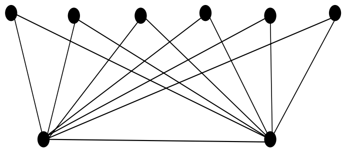

FIGURE 9. Graph for $\mathbb{Z}_9[X]/(3X, X^2 - 3), \mathbb{Z}_9[X]/(3X, X^2 - 6)$ and $\mathbb{Z}_3[X]/(X^3)$

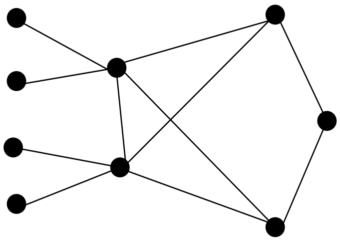

FIGURE 10. Graph for $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

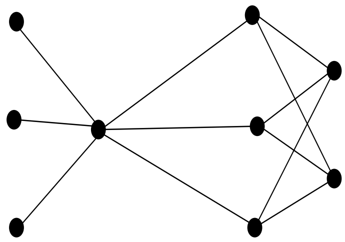

FIGURE 11. Graph for $\mathbb{Z}_4 \times \mathbb{F}_4, \mathbb{Z}_2[X]/(X^2) \times F_4$.

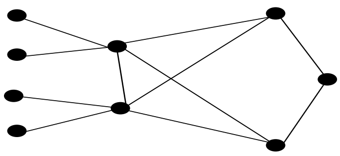

FIGURE 12. Graph for $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

Vertices	R	$\|R\|$	Graph	$\alpha(\Gamma(R))$	$\gamma(\Gamma(R))$	$\omega(\Gamma(R))$
5	$\mathbb{Z}_2 \times \mathbb{Z}_5$	10	$K_{1,4}$	4	1	2
	$\mathbb{Z}_3 \times \mathbb{F}_4$	12	$K_{2,3}$	3	2	2
	$\mathbb{Z}_2 \times \mathbb{Z}_4$	8	Fig. 1	3	2	2
	$\mathbb{Z}_2 \times Z_2[X]/(X^2)$	8	Fig. 1	2	1	2

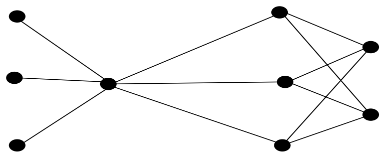

FIGURE 13. Graph for $\mathbb{Z}_4 \times \mathbb{F}_4, \mathbb{Z}_2[X]/(X^2) \times F_4$

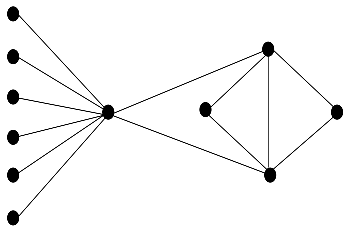

FIGURE 14. Graph for $\mathbb{Z}_2 \times \mathbb{Z}_9$ and $\mathbb{Z}_2 \times \mathbb{Z}_3[X]/(X^2)$

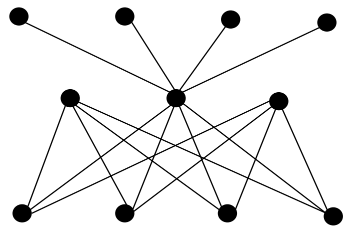

FIGURE 15. Graph for $\mathbb{Z}_5 \times \mathbb{Z}_4$ and $\mathbb{Z}_5 \times \mathbb{Z}_2[X]/(X^2)$

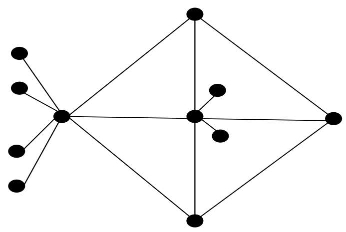

FIGURE 16. Graph for $\mathbb{Z}_2 \times \mathbb{Z}_8$ and $\mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^3)$ and $\mathbb{Z}_2 \times \mathbb{Z}_4[X]/(2X, X^2 - 2)$

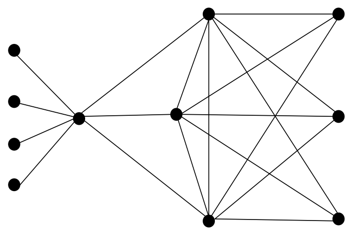

FIGURE 17. Graph for $\mathbb{Z}_2 \times \mathbb{Z}_2[X,Y]/(X,Y)^2$ and $\mathbb{Z}_2 \times \mathbb{Z}_4[X]/(2,X)^2$

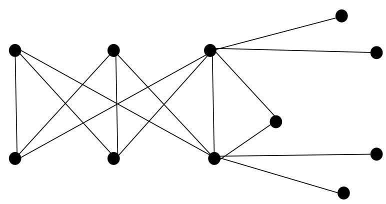

FIGURE 18. Graph for $\mathbb{Z}_4 \times \mathbb{Z}_4, \mathbb{Z}_4 \times \mathbb{Z}_2[X]/(X^2)$ and $\mathbb{Z}_2[X]/(X^2) \times \mathbb{Z}_2[X]/(X^2)$

Vertices	R	$\|R\|$	Graph	$\alpha(\Gamma(R))$	$\gamma(\Gamma(R))$	$\omega(\Gamma(R))$
6	$\mathbb{Z}_3 \times \mathbb{Z}_5$	15	$K_{2,4}$	4	2	2
	$\mathbb{F}_4 \times \mathbb{F}_4$	16	$K_{3,3}$	3	2	2
	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	8	Fig. 2	3	3	3
	$\mathbb{Z}_{49}$	49	K_6	1	1	6
	$\mathbb{Z}_7[X]/(X^2)$	49	K_6	1	1	6

Vertices	R	$\|R\|$	Graph	$\alpha(\Gamma(R))$	$\gamma(\Gamma(R))$	$\omega(\Gamma(R))$
7	$\mathbb{Z}_2 \times \mathbb{Z}_7$	14	$K_{1,6}$	6	1	2
	$\mathbb{F}_4 \times \mathbb{Z}_5$	10	$K_{3,4}$	4	2	2
	$\mathbb{Z}_3 \times \mathbb{Z}_4$	12	Fig. 3	4	2	2
	$\mathbb{Z}_3 \times \mathbb{Z}_2[X]/(X^2)$	12	Fig. 3	4	2	2
	$\mathbb{Z}_{16}$	16	Fig. 4	5	1	3
	$\mathbb{Z}_2[X]/(X^4)$	16	Fig. 4	5	1	3
	$\mathbb{Z}_4[X]/(X^2+2)$	16	Fig. 4	5	1	3
	$\mathbb{Z}_4[X]/(X^2+3X)$	16	Fig. 4	5	1	3
	$\mathbb{Z}_4[X]/(X^3-2, 2X^2, 2X)$	16	Fig. 4	5	1	3
	$\mathbb{Z}_2[X,Y]/(X^3, XY, Y^2)$	16	Fig. 5	4	1	4
	$\mathbb{Z}_8[X]/(2X, X^2)$	16	Fig. 5	4	1	4
	$\mathbb{Z}_4[X]/(X^3, 2X^2, 2X)$	16	Fig. 5	4	1	4
	$\mathbb{Z}_4[X]/(X^2+2X)$	16	Fig. 6	3	1	3
	$\mathbb{Z}_8[X]/(2X, X^2+4)$	16	Fig. 6	3	1	3
	$\mathbb{Z}_2[X,Y]/(X^2, Y^2-XY)$	16	Fig. 6	3	1	3
	$\mathbb{Z}_4[X,Y]/(X^2, Y^2-XY, XY-2, 2X, 2Y)$	16	Fig. 6	3	1	3
	$\mathbb{Z}_4[X,Y]/(X^2, Y^2, XY-2, 2X, 2Y)$	16	Fig. 7	3	1	3
	$\mathbb{Z}_2[X,Y]/(X^2, Y^2)$	16	Fig. 7	3	1	3
	$\mathbb{Z}_4[X]/(X^2)$	16	Fig. 7	3	1	3
	$\mathbb{Z}_4[X]/(X^3-X^2-2, 2X^2, 2X)$	16	Fig. 8	4	1	3
	$\mathbb{Z}_2[X,Y,Z]/(X,Y,Z)^2$	16	K_7	1	1	7
	$\mathbb{Z}_4[X,Y]/(X^2, Y^2, XY, 2X, 2Y)$	16	K_7	1	1	7
	$\mathbb{F}_8[X]/(X^2)$	64	K_7	1	1	7
	$\mathbb{Z}_4[X]/(X^3+X+1)$	64	K_7	1	1	7

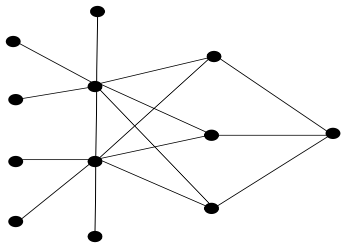

FIGURE 19. Graph for $\mathbb{Z}_2 \times \mathbb{Z}_2 \times F_4$

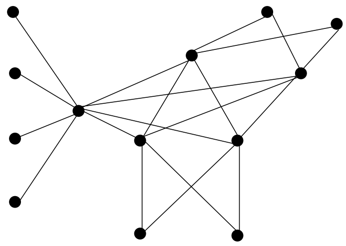

FIGURE 20. Graph for $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$.

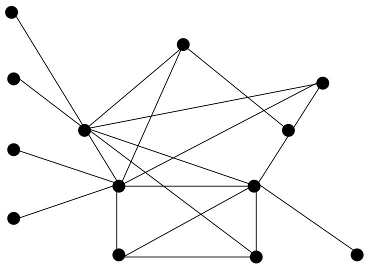

FIGURE 21. Graph for $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2)$

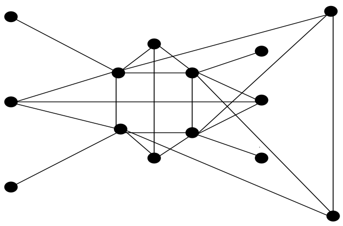

FIGURE 22. Graph for $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

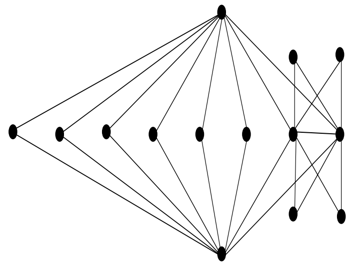

FIGURE 23. Graph for Graph for $\mathbb{Z}_3 \times \mathbb{Z}_9$ and $\mathbb{Z}_3 \times \mathbb{Z}_3[X]/(X^2)$.

Vertices	R	$\lvert R\rvert$	Graph	$\alpha(\Gamma(R))$	$\gamma(\Gamma(R))$	$\omega(\Gamma(R))$
8	$\mathbb{Z}_2 \times \mathbb{F}_8$	16	$K_{1,7}$	7	1	2
	$\mathbb{Z}_3 \times \mathbb{Z}_7$	21	$K_{2,6}$	6	2	2
	$\mathbb{Z}_5 \times \mathbb{Z}_5$	25	$K_{4,4}$	4	2	2
	$\mathbb{Z}_{27}$	27	Fig. 9	6	1	3
	$\mathbb{Z}_9[X]/(3X, X^2-3)$	27	Fig. 9	6	1	3
	$\mathbb{Z}_9[X]/(3X, X^2-6)$	27	Fig. 9	6	1	3
	$\mathbb{Z}_3[X]/(X^3)$	27	Fig. 9	6	1	3
	$\mathbb{Z}_3[X,Y]/(X,Y)^2$	27	K_8	1	1	8
	$\mathbb{Z}_9[X]/(3,X)^2$	27	K_8	1	1	8
	$\mathbb{F}_9[X]/(X^2)$	81	K_8	1	1	8
	$\mathbb{Z}_9[X]/(X^2+1)$	81	K_8	1	1	8

Vertices	R	$\|R\|$	Graph	$\alpha(\Gamma(R))$	$\gamma(\Gamma(R))$	$\omega(\Gamma(R))$
9	$\mathbb{Z}_2 \times F_9$	18	$K_{1,8}$	8	1	2
	$\mathbb{Z}_3 \times F_8$	24	$K_{2,7}$	7	2	2
	$F_4 \times \mathbb{Z}_7$	28	$K_{3,6}$	6	2	2
	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$	12	Fig. 10	6	3	3
	$\mathbb{Z}_4 \times F_4$	16	Fig. 11	6	2	2
	$\mathbb{Z}_2[X]/(X^2) \times F_4$	16	Fig. 11	6	2	2

Vertices	R	$\|R\|$	Graph	$\alpha(\Gamma(R))$	$\gamma(\Gamma(R))$	$\omega(\Gamma(R))$
10	$\mathbb{Z}_3 \times F_9$	27	$K_{2,8}$	8	2	2
	$\mathbb{F}_4 \times F_8$	32	$K_{3,7}$	7	2	2
	$\mathbb{Z}_5 \times \mathbb{Z}_7$	35	$K_{4,6}$	6	2	2
	$\mathbb{Z}_{121}$	121	K_{10}	1	1	10
	$\mathbb{Z}_{11}[X]/(X^2)$	121	K_{10}	1	1	10

Vertices	R	$\|R\|$	Graph	$\alpha(\Gamma(R))$	$\gamma(\Gamma(R))$	$\omega(\Gamma(R))$
11	$\mathbb{Z}_2 \times \mathbb{Z}_{11}$	22	$K_{1,10}$	10	1	2
	$F_4 \times \mathbb{F}_9$	36	$K_{3,8}$	8	2	2
	$\mathbb{Z}_5 \times F_8$	40	$K_{4,7}$	7	2	2
	$\mathbb{Z}_2 \times \mathbb{Z}_9$	18	Fig. 12	8	3	3
	$\mathbb{Z}_2 \times \mathbb{Z}_3[X]/(X^2)$	18	Fig. 12	8	3	3
	$\mathbb{Z}_5 \times \mathbb{Z}_4$	20	Fig. 13	8	3	2
	$\mathbb{Z}_5 \times \mathbb{Z}_2[X]/(X^2)$	20	Fig. 13	8	3	2
	$\mathbb{Z}_2 \times \mathbb{Z}_8$	16	Fig. 14	8	2	3
	$\mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^3)$	16	Fig. 14	8	2	3
	$\mathbb{Z}_2 \times \mathbb{Z}_4[X]/(2X, X^2-2)$	16	Fig. 14	8	2	3
	$\mathbb{Z}_2 \times \mathbb{Z}_2[X,Y]/(X,Y)^2$	16	Fig. 15	7	2	4
	$\mathbb{Z}_2 \times \mathbb{Z}_4[X]/(2,X)^2$	16	Fig. 15	7	2	4
	$\mathbb{Z}_4 \times \mathbb{Z}_4$	16	Fig. 16	6	2	3
	$\mathbb{Z}_4 \times \mathbb{Z}_2[X]/(X^2)$	16	Fig. 16	6	2	3
	$\mathbb{Z}_2[X]/(X^2) \times \mathbb{Z}_2[X]/(X^2)$	16	Fig. 16	6	2	3

Vertices	R	$\|R\|$	Graph	$\alpha(\Gamma(R))$	$\gamma(\Gamma(R))$	$\omega(\Gamma(R))$
12	$\mathbb{Z}_3 \times \mathbb{Z}_{11}$	33	$K_{2,10}$	10	2	2
	$\mathbb{Z}_5 \times \mathbb{Z}_9$	45	$K_{4,8}$	8	2	2
	$\mathbb{Z}_7 \times \mathbb{Z}_7$	49	$K_{6,6}$	6	2	2
	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$	16	Fig. 17	6	2	2
	$\mathbb{Z}_{169}$	169	K_{12}	1	1	12
	$\mathbb{Z}_{13}[X]/(X^2)$	169	K_{12}	1	1	12

Vertices	R	$\|R\|$	Graph	$\alpha(\Gamma(R))$	$\gamma(\Gamma(R))$	$\omega(\Gamma(R))$
13	$\mathbb{Z}_2 \times \mathbb{Z}_{13}$	26	$K_{1,12}$	12	1	2
	$F_4 \times \mathbb{Z}_{11}$	44	$K_{3,10}$	10	2	2
	$\mathbb{Z}_7 \times F_8$	56	$K_{6,7}$	7	2	2
	$\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$	18	Fig. 18	8	3	3
	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$	16	Fig. 19	8	3	3
	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2)$	16	Fig. 19	8	3	3

Vertices	R	$\|R\|$	Graph	$\alpha(\Gamma(R))$	$\gamma(\Gamma(R))$	$\omega(\Gamma(R))$
14	$\mathbb{Z}_3 \times \mathbb{Z}_{13}$	39	$K_{2,12}$	12	2	2
	$\mathbb{Z}_5 \times \mathbb{Z}_{11}$	55	$K_{4,10}$	10	2	2
	$\mathbb{Z}_7 \times F_9$	63	$K_{6,8}$	8	2	2
	$\mathbb{F}_8 \times F_8$	64	$K_{7,7}$	7	2	2
	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times Z_2 \times \mathbb{Z}_2$	16	Fig. 20	7	4	3
	$\mathbb{Z}_3 \times \mathbb{Z}_9$	27	Fig. 21	10	2	3
	$\mathbb{Z}_3 \times \mathbb{Z}_3[X]/(X^2)$	27	Fig. 21	10	2	3

References

[1] S. Akbari, H.R. Maimani, S. Yassemi, *When a zero-divisor graph is planar or a complete r-partite graph*, J. Algebra 270 (2003) 169-180.

[2] S. Akbari, A. Mohammadian, *On the zero-divisor graph of a commutative ring*, J. Algebra 274 (2004) 847-855.

[3] S. Alikhani and Y.H. Peng, *Independence roots and independence fractals of certain graphs*, J. Appl. Math. Computing, vol. 36, no. 1-2, (2011) 89-100.

[4] D.F. Anderson, A. Frazier, A. Lauve, P.S. Livingston, *The zero-divisor graph of a commutative ring, II*, in: Lecture Notes in Pure and Appl. Math., vol. 220, Dekker, New York, (2001) 61-72.

[5] D.F. Anderson, P.S. Livingston, *The zero-divisor graph of a commutative ring*, J. Algebra 217 (1999) 434 447.

[6] M.F. Atiyah and I.G. MacDonald, *Introduction to Commutative Algebra*. Reading, MA: Addison-Wesley, 1969.

[7] M. Axtell, J. Coykendall, J. Stickles, *Zero-divisor graphs of polynomials and power series over commutative rings*, Comm. Algebra. 6 (2005) 2043-2050.

[8] I. Beck, *Coloring of commutative rings*, J. Algebra 116 (1988) 208-226.

[9] I. Gutman, F. Harary, *Generalization of the matching polynomial*, Utilitas Mathematica 24 (1983) 97-106.

[10] C. Hoede, and X. Li, *Clique polynomials and independent set polynomials of graphs*, Discrete Math. 25 (1994) 219-228.

[11] N. Jafari Rad, S. H. Jafari, D.A. Mojdeh, *On domination in zero-divisor graphs*, Canad. Math. Bull. DOI:10.4153/CMB-2011-156-1 (2012).

[12] I. Kaplansky, *Commutative Rings*, Chicago-London: The University of Chicago Press, 1974.

[13] D. A. Mojdeh and A. M. Rahimi, *Dominating Sets of Some Graphs Associated to Commutative Rings*, Comm. Alg., 40:9 (2012) 3389-3396.

[14] S.P. Redmond, *The zero-divisor graph of a non-commutative ring*, Internat. J. Commutative Rings 1 (4) (2002) 203-211.

[15] S.P. Redmond, *On zero-divisor graphs of small finite commutative rings*, Discrete Math. 307 (2007) 1155–1166.

[16] S.P. Redmond, *An ideal-based zero-divisor graph of a commutative ring*, Comm. Alg. 31, (2003) 4425-4443.

[17] D.B. West, *Introduction to Graph Theory*, 2nd ed. USA: Prentice Hal, (2001).

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