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CONDITIONAL EXPECTATION IN THE KÔPKA'S D -POSETS

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ABSTRACT. The notion of a D -poset was introduced in a connection with quantum mechanical models. In this paper, we introduce the conditional expectation of random variables on the Kôpka's D -Poset and prove the basic properties of conditional expectation on this structure.

1. INTRODUCTION

D -posets play an important role in quantum structures. They were described independently 25 years ago by Slovak authors as D -posets [5] and by American authors as effect algebras [3, 4]. Although the definitions of D -posets and effect algebras are formally different, actually their axioms and corresponding theories are equivalent. Very important examples of D -posets are MV-algebras with a

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corresponding sophisticated probability theory [9, 10]. The important tool in this theory is the notion of a product that was introduced independently by different authors with equivalent definitions, but from the distinct point of view. Therefore, it was of interest to follow D -posets with product [6, 7]. This notion is allowed to break new grounds in D -posets with promising probability applications. In [8], a new possibility for probability applications on a certain type of D -posets was shown. The main idea was to work only with a distribution function. Chovanec *et al.* [2] studied a new approach to a conditional probability in D -posets. They go into the inner structure of a conditional system which is a crucial notion for the existence of a conditional state. Samuelčík and Hollá [11] studied the conditional probability on the Kôpka's D -posets. They only proved three basic properties for conditional probability on this structure. In this paper, we introduce the notion of the conditional expectation on the Kôpka's D -posets and prove the basic properties of conditional expectation for binary random variables. All results can be easily extended to n -ary random variables.

2. PRELIMINARIES

Definition 2.1. A D -poset is an algebraic structure $D = (D, \leq, -, 0_D, 1_D)$ such that

- 1) \leq is a partial ordering on D with the least element 0_D and the greatest element 1_D ;
- 2) $- : D \times D \rightarrow D$ is a partial binary operation, where $b - a$ is defined if $a \leq b$ and there holds
 - i) $b - a \leq b$,
 - ii) $b - (b - a) = a$,
 - iii) $a \leq b \leq c \implies c - b \leq c - a, (c - a) - (c - b) = b - a$.

Definition 2.2. Let an algebraic structure $D = (D, \leq, -, 0_D, 1_D)$ be a D -poset. The algebraic system $D = (D, \leq, -, *, 0_D, 1_D)$ is called the Kôpka's D -poset, if a binary operation $* : D \times D \rightarrow D$ is commutative, associative and there holds

- i) $a * 1_D = a$,
- ii) $a \leq b \implies a * c \leq b * c, a, b, c \in D$,
- iii) $a - (a * b) \leq 1_D - b, a, b, c \in D$.

Definition 2.3. A state on a Kôpka's D -poset D is a mapping $m : D \rightarrow \langle 0, 1 \rangle$ satisfying the following properties:

- 1) $m(0_D) = 0, m(1_D) = 1$,
- 2) if $a_n \uparrow a$, then $m(a_n) \uparrow m(a)$ for $a, a_n \in D, n = 1, 2, 3, \dots$,
- 3) if $a_n \downarrow a$, then $m(a_n) \downarrow m(a)$ for $a, a_n \in D, n = 1, 2, 3, \dots$

Definition 2.4. Suppose $J = \{(-\infty, t); t \in \mathbb{R}\}$. An observable on a Kôpka's D -poset D is a mapping $x : J \rightarrow D$ satisfying the following conditions:

- 1) if $A_n \uparrow \mathbb{R}$, then $x(A_n) \uparrow 1_D$,
- 2) if $A_n \uparrow A$, then $x(A_n) \uparrow x(A)$ for $A_n \in J, n = 1, 2, 3, \dots$,
- 3) if $A_n \downarrow \varphi$, then $x(A_n) \downarrow 0_D$ for $A_n \in J, n = 1, 2, 3, \dots$

Definition 2.5. A D -additive state on a Kôpka's D -poset D is a state $m : D \rightarrow \langle 0, 1 \rangle$ with the following property:

$$\text{if } a \leq b, \text{ then } m(b) = m(b - a) + m(a), \quad a, b \in D.$$

3. CONDITIONAL PROBABILITY

Proposition 3.1. [11] *Let D be a Kôpka's D -poset, $m : D \rightarrow \langle 0, 1 \rangle$ be a state and $x : J \rightarrow D$ be an observable. If $F : \mathbb{R} \rightarrow \langle 0, 1 \rangle$ is the function defined as $F(t) = m(x((-\infty, t)))$, then*

- 1) F is non-decreasing,
- 2) F is left continuous in any point $t_0 \in \mathbb{R}$,
- 3) $\lim_{t \rightarrow -\infty} F(t) = 0$ and $\lim_{t \rightarrow +\infty} F(t) = 1$,
- 4) there exists exactly one measure $\lambda_F : \mathcal{B}(\mathbb{R}) \rightarrow \langle 0, 1 \rangle$ such that

$$\lambda_F(\langle u, v \rangle) = F(v) - F(u) = m(x((-\infty, v)) - x((-\infty, u))).$$

Definition 3.2. A Kôpka's D -poset D with the following property:

$$\text{if } k \leq l, \text{ then } a * (l - k) = a * l - a * k, \quad k, l, a \in D,$$

is called a strong Kôpka's D -poset.

Proposition 3.3. [11] *Let D be a strong Kôpka's D -poset, $m : D \rightarrow \langle 0, 1 \rangle$ be a state, $x : J \rightarrow D$ be an observable and $a \in D$. Define $G : \mathbb{R} \rightarrow \langle 0, 1 \rangle$ by the formula*

$$G(t) = m(a * x((-\infty, t))), \quad a \neq 0_D.$$

Then

- 1) G is non-decreasing,
- 2) G is left continuous in any point $t_1 \in \mathbb{R}$,
- 3) $\lim_{t \rightarrow -\infty} G(t) = 0$ and $\lim_{t \rightarrow +\infty} G(t) = 1$,
- 4) there exists exactly one measure $\lambda_G : \mathcal{B}(\mathbb{R}) \rightarrow \langle 0, 1 \rangle$ such that

$$\lambda_G(\langle u, v \rangle) = G(v) - G(u).$$

Theorem 3.4. [11] *Let D be a strong Kôpka's D -poset, $m : D \rightarrow \langle 0, 1 \rangle$ be a D -additive state and $x : J \rightarrow D$ be an observable. Then there exists a function $p(a|x) : D \rightarrow \mathbb{R}$, such that*

$$\int_{(-\infty, t)} p(a|x) d\lambda_F = m(a * x((-\infty, t))).$$

The function $p(a|x)$ is a version of conditional probability on the strong Kôpka's D -poset such that there almost everywhere holds:

- 1) $p(0_D|x) = 0, \quad p(1_D|x) = 1,$
- 2) $0 \leq p(a|x) \leq 1$ for any $a \in D,$
- 3) if $a_n \uparrow a,$ then $p(a_n|x) \uparrow p(a|x)$ for $a, a_n \in D, \quad n = 1, 2, 3, \dots$

4. CONDITIONAL EXPECTATION

For every family \mathcal{M} , there exists a unique σ -algebra $\sigma(\mathcal{M})$, the σ -algebra generated by \mathcal{M} , such that

- a) $\mathcal{M} \subseteq \sigma(\mathcal{M})$,
- b) for any σ -algebra \mathcal{H} with $\mathcal{M} \subseteq \mathcal{H}$, $\sigma(\mathcal{M}) \subseteq \mathcal{H}$ [1].

In this section, we first describe the following definitions:

Definition 4.1. Let D be a Kôpka's D -poset, $m : D \rightarrow \langle 0, 1 \rangle$ be a state, $x : J \rightarrow D$ be an observable. A Kôpka's D -poset D with the following properties:

- 1) $m((a * b) * x(B)) = m(a * x(B)) = m(b * x(B))$, $a, b \in D, B \in \sigma(J)$,
- 2) $m((a \vee b) * x(B)) = m(a * x(B)) + m(b * x(B))$,

is called an m -strong Kôpka's D -poset.

As mentioned earlier, the important tool in this theory is the notion of a product. In passing, we introduce a new version of product on D -poset D similar to [6]. Let $\otimes : D \times \mathbb{R} \rightarrow \mathbb{R}$ be an operation with the following conditions:

- 1) $1_D \otimes r = r$, $r \in \mathbb{R}$,
- 2) $(a * b) \otimes r = a \otimes (b \otimes r)$, $a, b \in D, r \in \mathbb{R}$.
- 3) a and r are commutative with respect to λ_F . i.e.,

$$a \otimes \int_B r d\lambda_F = \int_B (a \otimes r) d\lambda_F, \quad a \in D, r \in \mathbb{R}, B \in \sigma(J).$$

Definition 4.2. A random variable Y on a Kôpka's D -poset D is a function $Y : D \rightarrow \mathbb{R}$ such that $Y^{-1}(C) \in D$ for every $C \in \mathcal{B}(\mathbb{R})$.

Theorem 4.3. Let D be a strong Kôpka's D -poset, $m : D \rightarrow \langle 0, 1 \rangle$ be a state and $x : J \rightarrow D$ be an observable. Let Y be a binary random variable with support $S_Y = \{a_1, a_2\}$, where $a_1, a_2 \in D$. Then there exists a function $\mathbb{E}(Y|x) : D \rightarrow \mathbb{R}$ such that

$$\int_{(-\infty, t)} \mathbb{E}(Y|x) d\lambda_F = (a_1 \otimes m(a_1 * x((-\infty, t)))) + (a_2 \otimes m(a_2 * x((-\infty, t)))).$$

The function $\mathbb{E}(Y|x)$ is a version of conditional expectation on the strong Kôpka's D -poset such that $\mathbb{E}(1_D|x) = 1$ (a.e.) and $\mathbb{E}(0_D|x) = 0$ (a.e.).

Proof. We have $\lambda_G(B) \leq \lambda_F(B)$ for every $B \in \sigma(J)$. Assume $\lambda_F(B) = 0$, then there holds

$$0 \leq m(a_1 * x(B)) = m(a_2 * x(B)) = \lambda_G(B) \leq \lambda_F(B) = 0.$$

Thus

$$0 \leq (a_1 \otimes m(a_1 * x(B))) + (a_2 \otimes m(a_2 * x(B))) \leq 0$$

and according to Radon-Nikodym theorem [1] there exists function f such that

$$\lambda_G(B) = \int_B f d\lambda_F.$$

For every $B \in \sigma(J)$,

$$\begin{aligned} \int_B \mathbb{E}(1_D|x)d\lambda_F &= 1_D \otimes m(1_D * x(B)) \\ &= m(x(B)) \\ &= \int_B p(1_D|x)d\lambda_F \\ &= \int_B 1 d\lambda_F. \end{aligned}$$

Also

$$\int_B \mathbb{E}(0_D|x)d\lambda_F = 0_D \otimes m(0_D * x(B)) = 0.$$

□

Theorem 4.4. *Let D be an m -strong Kôpka's D -poset and $x : J \rightarrow D$ be an observable. Let Y_1 be a binary random variable with support $S_{Y_1} = \{a_1, a_2\}$, where $a_1, a_2 \in D$ and Y_2 be a binary random variable with support $S_{Y_2} = \{b_1, b_2\}$, where $b_1, b_2 \in D$ and $a_i \leq b_i$ for $i = 1, 2$. Then*

- 1) $\mathbb{E}(Y_1|x) \leq \mathbb{E}(Y_2|x)$ (a.e.)
- 2) $0 \leq \mathbb{E}(Y|x) \leq 1$ (a.e.).

Proof. 1) For every $B \in \sigma(J)$,

$$\begin{aligned} \int_B \mathbb{E}(Y_1|x)d\lambda_F &= (a_1 \otimes m(a_1 * x(B))) + (a_2 \otimes m(a_2 * x(B))) \\ &\leq (b_1 \otimes m(b_1 * x(B))) + (b_2 \otimes m(b_2 * x(B))) \\ &= \int_B \mathbb{E}(Y_2|x)d\lambda_F. \end{aligned}$$

2) We have

$$\begin{aligned} 0 = 0_D \otimes m(0_D * x(\varphi)) &\leq (a_1 \otimes m(a_1 * x(B))) + (a_2 \otimes m(a_2 * x(B))) \\ &= \int_B \mathbb{E}(Y|x)d\lambda_F \\ &\leq \int_{\mathbb{R}} \mathbb{E}(1_D|x)d\lambda_F \\ &= 1. \end{aligned}$$

□

Theorem 4.5. *Let D be the m -strong Kôpka's D -poset and $d \in D$. If Y is a random variable and $x : J \rightarrow D$ be an observable, then*

$$\mathbb{E}(Y * d|x) = d \otimes \mathbb{E}(Y|x) \text{ (a.e.)}.$$

Proof. By definition 4.1,

$$\begin{aligned} \int_B \mathbb{E}(Y * d|x)d\lambda_F &= (a_1 * d) \otimes m((a_1 * d) * x(B)) \\ &+ (a_2 * d) \otimes m((a_2 * d) * x(B)) \\ &= \int_B d \otimes \mathbb{E}(Y|x)d\lambda_F. \end{aligned}$$

□

Theorem 4.6. *Let D be an m -strong Kôpka's D -poset. Let $(Y_n)_{n \geq 1}$ be a sequence of random variables with support $S_{Y_n} = \{a_{1n}, a_{2n}\}$, where $a_{1n}, a_{2n} \in D$ ($n = 1, 2, \dots$) and Y be a random variable with support $S_Y = \{a_1, a_2\}$, where $a_1, a_2 \in D$. If $a_{in} \uparrow a_i$ for each $i \in \{1, 2\}$, then $\mathbb{E}(Y_n|x) \uparrow \mathbb{E}(Y|x)$ (a.e.)*

Proof. This is an immediate consequence of the monotone convergence theorem (MCT) and the definition of a state m [1]. □

Theorem 4.7. *Let D be a m -strong Kôpka's D -poset. If $Y = d$ (a.e.) and there exists a real number r such that*

$$d \otimes \int_B 1d\lambda_F = \int_B rd\lambda_F \text{ (a.e.)}, \text{ for } d \in D,$$

then $\mathbb{E}(Y|x) = r$ (a.e.). Also, if

$$d \otimes \int_B 1d\lambda_F \leq \int_B rd\lambda_F \text{ (a.e.)}, \text{ for } d \in D,$$

then $\mathbb{E}(Y|x) \leq r$ (a.e.)

Proof. For every $B \in \sigma(J)$,

$$\begin{aligned} \int_B \mathbb{E}(Y|x)d\lambda_F &= d \otimes m(d * x(B)) \\ &= d \otimes m(x(B)) \\ &= d \otimes \int_B 1d\lambda_F \\ &= \int_B rd\lambda_F. \end{aligned}$$

Also

$$\begin{aligned} \int_B \mathbb{E}(Y|x)d\lambda_F &\leq d \otimes m(d * x(B)) + d \otimes m(d * x(B)) \\ &= d \otimes m(1_D * x(B)) \\ &\leq \int_B r d\lambda_F. \end{aligned}$$

□

Example 4.8. Consider the classical Kolmogorov probability space (Ω, S, P) . Let

$$F = \{\mu_A : \Omega \rightarrow \langle 0, 1 \rangle, \mu_A \text{ is } S\text{-measurable}\},$$

and define a partial operation on F as

$$\mu_A \leq \mu_B \iff \mu_A(w) \leq \mu_B(w), \forall w \in \Omega,$$

where the least element is 0_Ω and the greatest element is constant function 1_Ω . Also, define two partial binary operations “ $-$ ” and “ $*$ ” as

$$- : F \times F \rightarrow F, \text{ if } \mu_A \leq \mu_B, \text{ then } (\mu_B - \mu_A)(w) = \mu_B(w) - \mu_A(w)$$

and

$$* : F \times F \rightarrow F, \mu_A(w) * \mu_B(w) = \mu_B(w) \cdot \mu_A(w).$$

Now, using the scalar product

$$\otimes : F \times \mathbb{R} \rightarrow \mathbb{R}, \mu_A(w) \otimes r = r\mu_A(w),$$

the algebraic structure $F = (F, \leq, -, *, 0_\Omega, 1_\Omega)$ becomes a Kôpka's D -poset. Note that, on this structure we have

$$\mathbb{E}(rY|x) = r\mathbb{E}(Y|x) \text{ (a.e.)}$$

Example 4.9. MV-algebras are an examples of D -posets. We can obtain similar results by defining the appropriate operations on this structures (see [2]).

All of the mentioned results can be extended using a generalization of Theorem 4.3 as follows.

Theorem 4.10. Let D be a strong Kôpka's D -poset, $m : D \rightarrow \langle 0, 1 \rangle$ be a state and $x : J \rightarrow D$ be an observable. Let Y be a n -ary random variable with support $S_Y = \{a_1, a_2, \dots, a_n\}$, where $a_1, a_2, \dots, a_n \in D$. Then there exists a function $\mathbb{E}(Y|x) : D \rightarrow \mathbb{R}$ such that

$$\int_{(-\infty, t)} \mathbb{E}(Y|x)d\lambda_F = \sum_{i=1}^n a_i \otimes m(a_i * x((-\infty, t))).$$

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