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COSPECTRALITY MEASURES OF GRAPHS WITH AT MOST SIX VERTICES

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ABSTRACT. Cospectrality of two graphs measures the differences between the ordered spectrum of these graphs in various ways. Actually, the origin of this concept came back to Richard Brualdi's problems that are proposed in [1]:

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Let G_n and G'_n be two nonisomorphic simple graphs on n vertices with spectra

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \quad \text{and} \quad \lambda'_1 \geq \lambda'_2 \geq \cdots \geq \lambda'_n,$$

respectively. Define the distance between the spectra of G_n and G'_n as

$$\lambda(G_n, G'_n) = \sum_{i=1}^{n} (\lambda_i - \lambda'_i)^2 \quad \Big(\text{or use } \sum_{i=1}^{n} |\lambda_i - \lambda'_i|\Big).$$

Define the cospectrality of G_n by $\mathrm{cs}(G_n) = \min\{\lambda(G_n, G'_n) \ : \ G'_n \ \text{ not isomorphic to } G_n\}$. Let $\mathrm{cs}_n = \max\{\mathrm{cs}(G_n) \ : \ G_n \ \text{ a graph on } n \text{ vertices}\}$. Investigation of $\mathrm{cs}(G_n)$ for special classes of graphs and finding a good upper bound on cs_n are two main questions in this subject. In this paper, we briefly give some important results in this direction and then we collect all cospectrality measures of graphs with at most six vertices with respect to three norms. Also, we give the shape of all graphs that are closest (with respect to cospectrality measure) to a given graph G.

1. **Introduction**

Throughout all graphs are supposed to be finite, undirected and without loops and multiple edges, or simply all graphs are simple. Let G_n and G'_n be two nonisomorphic graphs on n vertices with spectra

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \quad \text{and} \quad \lambda'_1 \geq \lambda'_2 \geq \cdots \geq \lambda'_n,$$

respectively. Let $\Lambda = (\lambda_1, \ldots, \lambda_n)$ and $\Lambda' = (\lambda'_1, \ldots, \lambda'_n)$ be the ordered n-tuple of the spectrum of G_n and G'_n respectively. Cospectrality of two graphs measures the differences between the ordered spectrum of these graphs in various ways. Actually, the origin of this concept came back to Richard Brualdi's problems that are proposed in [1]. Define the distance spectra between G_n and G'_n by

$$\lambda^{(p)}(G_n, G'_n) = \begin{cases} \big(||\Lambda - \Lambda'||_{\ell_n^p}\big)^p & \text{if } 1 \leq p < \infty \\ ||\Lambda - \Lambda'||_{\ell_n^\infty} & \text{if } p = \infty \end{cases},$$

where the above norms are well known (for example see [4]) and are defined as follows:

$$||(a_1, \ldots, a_n)||_{\ell_n^p} = \begin{cases} \big(\sum_{i=1}^{n} |a_i|^p\big)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty \\ \max\{|a_i| \ : \ i = 1, \ldots, n\} & \text{if } p = \infty \end{cases}.$$

Also, as a generalization of the definitions in [1], we define

$$cs^{(p)}(G_n) = \min\{\lambda^{(p)}(G_n, G'_n) \ : \ G'_n \ \text{ not isomorphic to } G_n\},$$

and

$$\mathrm{cs}_n^{(p)} = \max\{\mathrm{cs}^{(p)}(G_n) \ : \ G_n \ \text{ a graph on } n \text{ vertices}\}.$$

There are two following problems:

Problem A. Investigate $\mathrm{cs}(G_n)$ for special classes of graphs.

Problem B. Find a good upper bound on cs_n.

Recently, A. Abdollahi and M. R. Oboudi studied these problems and obtained some new results and proposed a few new questions in this area [2]. Also, in [3] the authors completely solved Problem B in general case (with respect to any norms).

The following results are known.

Fact 1.1. [2, Theorem 1.1] For every integer $n \geq 2$, $\mathrm{cs}(nK_1) = 2$. In particular, $\lambda(nK_1, G) = \mathrm{cs}(nK_1)$ for some graph G if and only if $G \cong K_2 + (n-2)K_1$.

Fact 1.2. [2, Theorem 1.2] For every integer $n \geq 3$, $\mathrm{cs}(K_2 + (n-2)K_1) = 2(\sqrt{2}-1)^2$. Also, $\mathrm{cs}(K_2) = \lambda(K_2, 2K_1) = 2$. In particular, $\lambda(K_2 + (n-2)K_1, G) = \mathrm{cs}(K_2 + (n-2)K_1)$ for some graph G if and only if $G \cong P_3 + (n-3)K_1$.

Fact 1.3. [2, Theorem 1.3] For every integer $n \geq 2$, $\mathrm{cs}(K_n) = n^2 + n - n\sqrt{n^2+2n-7} - 2$. In particular, $\lambda(K_n, G) = \mathrm{cs}(K_n)$ for some graph G if and only if $G \cong K_n \setminus e$, where $K_n \setminus e$ is the graph obtaining from K_n by deletion one edge e.

Fact 1.4. [2, Theorem 1.4] Let $n \geq 2$ be an integer. Then $\mathrm{cs}(K_{n,n}) = 2(n - \sqrt{n^2-1})^2$. In particular, $\lambda(K_{n,n}, G) = \mathrm{cs}(K_{n,n})$ for some graph G if and only if $G \cong K_{n-1,n+1}$.

Fact 1.5. [3, Theorem 1.1] Let $n \geq 2$ be an integer. Then $\mathrm{cs}_n^{(p)} = \begin{cases} 2 & \text{if } 1 \leq p < \infty \\ 1 & \text{if } p = \infty \end{cases}$. In particular, we solved the Problem B, that is, $\mathrm{cs}_n = \mathrm{cs}_n^{(2)} = 2$ for all $n \geq 2$.

Fact 1.6. [3, Theorem 1.2] Let m and n be two possitive integers. If $m+2 \leq n < m-1+2\sqrt{m-1}$, then $\lambda(K_{m,n}, H) = \mathrm{cs}(K_{m,n})$ for some graph H if and only if $H \cong K_{m+1,n-1}$.

Fact 1.7. [3, Theorem 1.5] Let G and G' be two graphs such that $\mathrm{cs}(G) = \lambda(G, G')$ and let e and e' be the number of edges of G and G', respectively. Then $|\sqrt{e} - \sqrt{e'}| \leq 1$. In particular, $|e - e'| \leq 3e$.

In the next section we give the tables of cospectrality of graphs with at most six vertices.

2. Tables

In the following, we give the tables of cospectrality of graphs with at most six vertices. This parameter calculated for three well known norms that are shown in the top row of each table. The computations were performed by using Maple, Maple is a trademark of Waterloo Maple Inc. In Table 1, the cospectrality is determined for all graphs with at most 5 vertices. Also, we have 156 non-isomorphic graphs with 6 vertices that their cospectralities are determined in Table 2.

Table 1: The cospectrality of graphs up to 5 vertices with three norms

Graph	$\|.\|_{\ell 2}$	$\|.\|_{\ell 1}$	$\|.\|_{\ell \infty}$	Graph	$\|.\|_{\ell 2}$	$\|.\|_{\ell 1}$	$\|.\|_{\ell \infty}$
	2.000000000	2.000000000	1.000000000		0.3431457498	0.828427124	0.414213562
	0.3431457498	0.828427124	0.414213562		1.514718626	2.000000000	1.000000000
	2.000000000	2.000000000	1.000000000		0.3431457498	0.828427124	0.414213562
	1.055728091	2.000000000	0.618033989		0.2020410298	0.635674492	0.317837246
	0.5635887683	1.377784366	0.552052498		0.3572654454	0.9623886085	0.481194304
	0.1435935390	0.535898384	0.267949192		0.1435935390	0.535898384	0.267949192
	0.2564914485	0.7829326525	0.391466326		0.2564914485	0.7829326525	0.391466326
	1.507577498	2.000000000	1.000000000		2.000000000	2.000000000	1.000000000
	0.3431457498	0.828427124	0.414213562		0.3431457498	0.828427124	0.414213562
	0.2020410298	0.635674492	0.317837246		0.2020410298	0.635674492	0.317837246
	0.1489611738	0.7541159040	0.229725076		0.3572654454	0.9623886085	0.481194304
	0.1435935390	0.535898384	0.267949192		0.1368822178	0.635674492	0.2346331353
	0.000000000	0.000000000	0.000000000		0.6306831236	1.123105626	0.561552813
	0.1368822178	0.7006827846	0.229725076		0.1658347318	0.6648243974	0.3069261712
	0.000000000	0.000000000	0.000000000		0.5730269261	1.236067978	0.6180339887
	0.1872172214	0.7824671156	0.288020140		0.2711530460	0.8537334657	0.426866733
	0.1532457026	0.6858461658	0.302775638		0.2363351767	0.7829326525	0.3349039854
	0.1304059435	0.6648243974	0.2182087626		0.1304059435	0.7129599676	0.2182087626
	0.2972681648	0.9216222552	0.4107842569		0.4040820582	0.898979486	0.449489743
	0.2711530460	0.8537334657	0.4107842569		0.2136040967	0.8142196095	0.3349039854
	0.4549449552	1.028273859	0.514136929		0.1557262365	0.6432743483	0.3216371743
	0.3622213098	1.157084086	0.4625984230		0.1114561795	0.472135954	0.236067977
	0.08916398388	0.4745481572	0.237274078		0.1114561795	0.472135954	0.236067977
	0.08916398388	0.4745481572	0.237274078		0.2332847845	0.7158527355	0.3579263675

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Graph	$\|\cdot\|_{\ell 2}$	$\|\cdot\|_{\ell 1}$	$\|\cdot\|_{\ell \infty}$	Graph	$\|\cdot\|_{\ell 2}$	$\|\cdot\|_{\ell 1}$	$\|\cdot\|_{\ell \infty}$
	1.542486890	2.000000000	1.000000000				

Table 2: The cospectrality of graphs with 6 vertices by three norms

Graph	$\|\cdot\|_{\ell 2}$	$\|\cdot\|_{\ell 1}$	$\|\cdot\|_{\ell \infty}$	Graph	$\|\cdot\|_{\ell 2}$	$\|\cdot\|_{\ell 1}$	$\|\cdot\|_{\ell \infty}$
	2.000000000	2.000000000	1.000000000		0.3431457498	0.828427124	0.414213562
	0.3431457498	0.828427124	0.414213562		0.2020410298	0.635674492	0.317837246
	1.055728091	2.000000000	0.618033989		0.2020410298	0.635674492	0.317837246
	0.1489611738	0.7541159040	0.229725076		0.3572654454	0.9623886085	0.481194304
	0.1435935390	0.535898384	0.267949192		0.3431457498	0.828427124	0.414213562
	0.2171217253	0.8284271250	0.246979604		0.000000000	0.000000000	0.000000000
	0.000000000	0.000000000	0.000000000		0.3431457498	0.828427124	0.414213562
	0.000000000	0.000000000	0.000000000		0.1123510771	0.5923739628	0.226554228
	0.1658347318	0.6648243974	0.3069261712		0.000000000	0.000000000	0.000000000
	0.1662935242	0.8989794866	0.246979604		0.2826076228	0.8284271244	0.3819660113
	0.000000000	0.000000000	0.000000000		0.2194769217	0.828427124	0.360409338
	0.08081372108	0.546914944	0.175570505		0.1662935242	0.8284271250	0.246979604
	0.06730771448	0.4686241664	0.1728461712		0.000000000	0.000000000	0.000000000

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Graph	$\|\cdot\|_{\ell^2}$	$\|\cdot\|_{\ell^1}$	$\|\cdot\|_{\ell^\infty}$	Graph	$\|\cdot\|_{\ell^2}$	$\|\cdot\|_{\ell^1}$	$\|\cdot\|_{\ell^\infty}$
	0.2350679974	0.7829326525	0.360409337		0.1789335643	0.7630489480	0.2913668902
	0.1216171056	0.6435850622	0.254242224		0.2363351767	0.7829326525	0.3349039854
	0.06549518310	0.4686241664	0.1650003819		0.1138470308	0.5450473894	0.2182087626
	0.09109770040	0.426843532	0.213421766		0.4746344184	1.101020514	0.3819660113
	1.122413780	2.000000000	0.732050808		0.2826076228	0.9937471047	0.303442085
	0.03718086038	0.3562911702	0.1259679511		0.03914713370	0.373545462	0.126032501
	0.09109770040	0.426843532	0.213421766		0.1740097822	1.012081584	0.1980622642
	0.1397506917	0.6909220486	0.2953756312		0.1397506917	0.6909220486	0.2611437008
	0.2091053114	0.8537761732	0.3438289721		0.05109839675	0.4060582242	0.170086487
	0.3200561182	0.8284271244	0.339501261		0.000000000	0.000000000	0.000000000
	0.1552813202	0.7342444179	0.2762578774		0.4549449552	1.028273859	0.514136929
	0.03718086038	0.3562911702	0.1259679511		0.1666516688	0.6928123598	0.216172168
	0.000000000	0.000000000	0.000000000		0.06931024006	0.4712934404	0.200239458
	0.1222638001	0.6449503297	0.250882452		0.05109839675	0.4060582242	0.170086487
	0.1828429248	0.6993992011	0.303442085		0.1740097822	0.9711286236	0.1980622642
	0.02869800958	0.2966953136	0.1150730408		0.2546161992	0.8560442337	0.4142135624
	0.1773260086	0.7060092154	0.3438289721		0.1243928183	0.6268672527	0.216172168
	0.1075668917	0.6379013101	0.2031848841		0.08629743387	0.4955390875	0.210755881

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Graph	$\|\cdot\|_{\ell 2}$	$\|\cdot\|_{\ell 1}$	$\|\cdot\|_{\ell\infty}$	Graph	$\|\cdot\|_{\ell 2}$	$\|\cdot\|_{\ell 1}$	$\|\cdot\|_{\ell\infty}$
	0.000000000	0.000000000	0.000000000		0.1557262365	0.6432743483	0.3216371743
	0.5460744917	1.203358264	0.601679132		0.02869800958	0.2966953136	0.1150730408
	0.1493485990	0.7060054846	0.2611437008		0.1621005536	0.8179841269	0.247783626
	0.1475485670	0.6931317021	0.318761360		0.1192948139	0.5814492819	0.2611437008
	0.1374090196	0.6941532424	0.275349861		0.1225409292	0.7478642632	0.247783626
	0.1905297730	0.8736979817	0.2622259535		0.08916398388	0.4745481572	0.237274078
	0.09976782198	0.5948591236	0.208712153		0.1114561795	0.472135954	0.236067977
	0.000000000	0.000000000	0.000000000		0.09374780409	0.5315797816	0.240342107
	0.1276156744	0.5948591236	0.275110906		0.1847481671	0.7691905002	0.2679491924
	0.2357831869	0.7840787997	0.390058025		0.06687166768	0.4585505000	0.174438393
	0.1735578472	0.6993992011	0.312736332		0.1243928183	0.6268672527	0.271848797
	0.1276156744	0.5948591236	0.275110906		0.1213953842	0.5914564166	0.213797319
	0.09976782198	0.5814492815	0.208712153		0.1432745817	0.674692494	0.2679491924
	0.05887450288	0.343145750	0.171572875		0.1108513497	0.5796396451	0.257348571
	0.1110202750	0.5848031699	0.179490950		0.08709551248	0.5783562467	0.1966904942
	0.09374780409	0.5315797816	0.213797319		0.1110202750	0.5665901262	0.179490950
	0.1668290134	0.6926774377	0.3087774780		0.08709551248	0.5783562467	0.1966904942
	0.1230393715	0.7403032227	0.2301020235		0.1330577268	0.6169831571	0.288402487

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Graph	$\lVert\cdot\rVert_{\ell^2}$	$\lVert\cdot\rVert_{\ell^1}$	$\lVert\cdot\rVert_{\ell^\infty}$	Graph	$\lVert\cdot\rVert_{\ell^2}$	$\lVert\cdot\rVert_{\ell^1}$	$\lVert\cdot\rVert_{\ell^\infty}$
	0.1456584356	0.7443658239	0.2271344422		0.1544239091	0.6408729162	0.257138128
	0.1114561795	0.472135954	0.236067977		0.08916398388	0.4745481572	0.237274078
	0.1315001400	0.5814492815	0.290724641		0.05887450288	0.343145750	0.171572875
	0.2322331477	0.9256276823	0.301301038		0.09439404028	0.6077179513	0.178650949
	0.09439404028	0.6558405928	0.178650949		0.3287701710	0.9821534182	0.4384471872
	0.2963130604	1.037305531	0.4064206546		0.1022069966	0.5365599624	0.268279981
	0.1169531422	0.5665901262	0.283295063		0.07660053718	0.5860410384	0.219295149
	0.1338798124	0.6077179513	0.245189059		0.09109770040	0.426843532	0.213421766
	0.09681440515	0.5020164591	0.251008230		0.08971256159	0.5229031956	0.232667198
	0.2128755489	0.6909195888	0.3442922790		0.06541003851	0.4153940551	0.207697028
	0.08971256159	0.5229031956	0.219270009		0.1929657793	0.7169933045	0.358496652
	0.1334239692	0.6323283852	0.2271344422		0.1334239692	0.6323283852	0.243875739
	0.2332847845	0.7158527355	0.3579263675		0.07660053718	0.5860410384	0.1872837251
	0.09109770040	0.426843532	0.213421766		0.1795559317	0.744562646	0.2700810714
	0.09681440515	0.5020164591	0.251008230		0.1229064662	0.7306492652	0.2147808815
	0.1383477514	0.6548336402	0.245189059		0.2066002936	0.7755415457	0.324532955
	0.06541003851	0.4153940551	0.207697028		0.1806376626	0.6774237346	0.338711868
	0.06698531493	0.4901463723	0.1872837251		0.1254666106	0.7032607564	0.233792760
	0.05978078527	0.3877826107	0.193891305		0.09579599548	0.5268931361	0.2465585217
	0.09872649624	0.5529875985	0.233037217		0.5208656730	1.068140394	0.534070197

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Graph	$\|\cdot\|_{\ell^2}$	$\|\cdot\|_{\ell^1}$	$\|\cdot\|_{\ell^\infty}$	Graph	$\|\cdot\|_{\ell^2}$	$\|\cdot\|_{\ell^1}$	$\|\cdot\|_{\ell^\infty}$
	0.09974661865	0.5358983834	0.2679491924		0.06698531493	0.4901463723	0.199523224
	0.1429090351	0.6270727693	0.248467928		0.2589406658	0.8431182815	0.4215591408
	0.2186710025	0.7229123274	0.3493257604		0.05978078527	0.3877826107	0.193891305
	0.1257492185	0.7116516375	0.260075241		0.06751280246	0.4408902178	0.184770869
	0.07157861390	0.4249960998	0.212498050		0.6874432168	1.353191448	0.6180339887
	0.06751280246	0.4408902178	0.184770869		0.1257492185	0.7116516375	0.260075241
	0.2000511372	0.7232416585	0.3616208295		0.07157861390	0.4249960998	0.212498050
	0.2090289609	0.744562646	0.3616208295		0.08321328306	0.4528130685	0.226406534
	0.2264256749	0.7513272473	0.3756636233		1.581254576	2.000000000	1.000000000

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