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STABILIZER TOPOLOGY OF HOOPS

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Abstract. In this paper, we introduce the concepts of right, left and product stabilizers on hoops and study some properties and the relation between them. And we try to find that how they can be equal and investigate that under what condition they can be filter, implicative filter, fantastic and positive implicative filter. Also, we prove that right and product stabilizers are filters and if they are proper, then they are prime filters. Then by using the right stabilizers produce a basis for a topology on hoops. We show that the generated topology by this basis is Baire, connected, locally connected and separable and we investigate the other properties of this topology. Also, by the similar way, we introduce the right, left and product stabilizers on quotient hoops and introduce the quotient topology that is generated by them and investigate that under what condition this topology is Hausdorff space, T_0 or T_1 spaces.

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1. Introduction

Hoop-algebras are naturally ordered commutative residuated integral monoids, introduced by B. Bosbach in [2, 3] then study by J. R. Bü̈chi and T. M. Owens in [6], a paper never published. In the last years, hoops theory was enriched with deep structure theorems(see [1, 2, 3]). Many of these results have a strong impact with fuzzy logic. Particularly, from the structure theorem of finite basic hoops([1, Corollary 2.10]) one obtains an elegant short proof of the completeness theorem for propositional basic logic(see [1, Theorem 3.8]), introduced by Hájek in [8]. The algebraic structures corresponding to Hájek's propositional (fuzzy) basic logic, BL-algebras, are particular cases of hoops. The main example of BL-algebras in interval $[0, 1]$ endowed with the structure induced by a t-norm. MV-algebras, product algebras and Gödel algebras are the most known classes of BL-algebras. Recent investigations are concerned with non-commutative generalizations for these structures.

A stabilizer is a part of a monoid acting on a set. Specifically, let A be a monoid operating on a set A and let X be a subset of S. The *stabilizer* of X, sometimes denoted $St(X)$ is the set of elements of a of A for which $a(X) \subseteq X$. The *strict stabilizer* is the set of $a \in A$ for which $a(X) = X$. In the other words, the stabilizer of X is the transporter of X to itself.

In this paper, we introduce the concept of right, left and product stabilizers on hoops. Then we use the right stabilizer of a hoop A and produce a basis for topology on A. We show that the generated topology by this basis is Baire, connected, locally connected and separable and we investigate the other properties of this topology.

2. Preliminaries

In this section, we gather some basic notions relevant to hoop which will need in the next sections.

Definition 2.1. [1] A *hoop* is an algebra $(A, \odot, \to, 1)$ of type $(2, 2, 0)$ such that, for all $x, y, z \in A$:

$(HP1)$ $(A, \odot, 1)$ is a commutative monoid,

$(HP2)$ $x \to x = 1$,

$(HP3)$ $(x \odot y) \to z = x \to (y \to z)$,

$(HP4)$ $x \odot (x \to y) = y \odot (y \to x)$.

On hoop A we define $x \leq y$ if and only if $x \to y = 1$. It is easy to see that " $\leq$ " is a partial order relation on A. A hoop A is *bounded* if there is an element $0 \in A$ such that $0 \leq x$, for all $x \in A$. We let $x^0 = 1$, $x^n = x^{n-1} \odot x$, for any $n \in \mathbb{N}$. Let A be a bounded hoop. We define a negation " $'$ " on A by, $x^{'} = x \to 0$, for all $x \in A$. If $(x^{'})^{'} = x$, for all $x \in A$, then the bounded hoop A is said to have the *double negation property*, or (DNP) for short.

The following proposition provides some properties of hoop.

Proposition 2.2. [2, 3] *Let* $(A, \odot, \to, 1)$ *be a hoop. Then the following condition hold, for all* $x, y, z, a \in A$*:*

(i) $(A,\leq)$ *is a* $\wedge$*-semilattice with* $x\wedge y = x\odot(x\to y)$,
(ii) $x\odot y\leq z$ *if and only if* $x\leq y\to z$,
(iii) $x\odot y\leq x,y$,
(iv) $x\leq y\to x$,
(v) $x\to 1=1$,
(vi) $1\to x=x$,
(vii) $x\odot(x\to y)\leq y$,
$(viii)$ $x\leq y$ *implies* $x\odot a\leq y\odot a$,
(ix) $x\leq y$ *implies* $z\to x\leq z\to y$,
(x) $x\leq y$ *implies* $y\to z\leq x\to z$,
(xi) $((x\to y)\to y)\to y = x\to y$,
(xii) $(x\to y)\odot(y\to z)\leq(x\to z)$.

Proposition 2.3. [7] *Let* A *be a bounded hoop. Then the following conditions hold, for all* $x,y\in A$*:*
(i) $x\leq x^{''}$,
(ii) $x\odot x^{'}=0$,
(iii) *if* $x=x^{''}$*, then* $x\to y=y^{'}\to x^{'}$,
(iv) $x=x^{''}$ *if and only if* $(x\to y)\to y=(y\to x)\to x$.

Proposition 2.4. [7] *Let* A *be a hoop and for any* $x,y\in A$*, we define,* $x\sqcup y=((x\to y)\to y)\wedge((y\to x)\to x)$*. Then the following conditions are equivalent:*
(i) $\sqcup$ *is associative operation on* A,
(ii) $x\leq y$ *implies* $x\sqcup z\leq y\sqcup z$*, for all* $x,y,z\in A$,
(iii) $x\sqcup(y\wedge z)\leq(x\sqcup y)\wedge(x\sqcup z)$*, for all* $x,y,z\in A$,
(iv) $\sqcup$ *is the join operation on* A.

Definition 2.5. A hoop A is called a $\sqcup$-*hoop*, if $\sqcup$ is a join operation on A.

Remark 2.6. [7, Remark 2.4] $\sqcup$-hoop $(A,\sqcup,\wedge)$ is a distributive lattice.

Proposition 2.7. [7] *Let* A *be a* $\sqcup$*-hoop. Then, for all* $x,y,z\in A$,
(i) $(x\sqcup y)\to z=(x\to z)\wedge(y\to z)$,
(ii) $x\odot(y\sqcup z)=(x\odot y)\sqcup(x\odot z)$.

Definition 2.8. [7] Let A be a hoop. A non-empty subset F of A is called a *filter* of A if,
$(F1)$ $x,y\in F$ implies $x\odot y\in F$,
$(F2)$ $x\leq y$ and $x\in F$ imply $y\in F$, for any $x,y\in A$.
We use $\mathcal{F}(A)$ to denote the set of all filters of A. Clearly, $1\in F$, for all $F\in\mathcal{F}(A)$. $F\in\mathcal{F}(A)$ is called a *proper filter* if $F\neq A$. It can be easily seen that, if A is a bounded hoop, then a filter is proper if and only if it is not containing 0.

Proposition 2.9. [7] *Let A be a hoop and F be a non-empty subset of A. Then $F \in \mathcal{F}(A)$ if and only if $1 \in F$ and if $x,\ x \to y \in F$, then $y \in F$, for any $x, y \in A$.*

Let A be a hoop and $F \in \mathcal{F}(A)$. We define a binary relation $\sim_F$ on A by $x \sim_F y$ if and only if $x \to y,\ y \to x \in F$, for any $x, y \in A$. Then $\sim_F$ is a congruence relation on A. Let $A/F = \{\overline{x} \mid x \in A\}$, where $\overline{x} = \{y \in A \mid x \sim_F y\}$. Then the binary relation $\leq$ on A/F which is defined by:

$$\overline{x} \leq \overline{y} \text{ if and only if } x \to y \in F.$$

is an order relation on A/F(see [15]). Now, let A be a hoop and $F \in \mathcal{F}(A)$. Then $(A/F, \otimes, \rightsquigarrow, 1_{A/F})$ is a hoop, where for any $x, y \in A$:

$$1_{A/F} = \overline{1},\ \overline{x} \otimes \overline{y} = \overline{x \odot y},\ \overline{x} \rightsquigarrow \overline{y} = \overline{x \to y}.$$

Definition 2.10. [9, 13] Consider X as a non-empty set. A mapping $\varphi : \mathcal{P}(X) \to \mathcal{P}(X)$ is called a *closure operator* on X, if the following holds, for all $Y, Z \in \mathcal{P}(X)$:
$(C1)$ $Y \subseteq \varphi(Y)$,
$(C2)$ $\varphi^2(Y) = \varphi(Y)$,
$(C3)$ $Y \subseteq Z$ implies $\varphi(Y) \subseteq \varphi(Z)$.

Definition 2.11. [9] A topological space $(X, \mathcal{T})$ is called *Baire space*, if for each countable collection of open dense sets U_n, their intersection $\bigcap U_n$ is dense.

Note: From now one, in this paper, we let A be a hoop, unless otherwise state.

3. Stabilizer of hoops

In this section, we introduce the concept of right, left and product stabilizers on hoops and investigate some properties of them.

Definition 3.1. Let $X \subseteq A$. Then

- $St_r(X) = \{a \in A \mid a \to x = x,\ \forall x \in X\}$ is called the *right stabilizer* of X.
- $St_l(X) = \{a \in A \mid x \to a = a,\ \forall x \in X\}$ is called the *left stabilizer* of X.
- $St_\odot(X) = \{a \in A \mid a \odot x = x,\ \forall x \in X\}$ is called the *product stabilizer* of X.

Example 3.2. (i) Let $A = \{0, a, b, 1\}$ and operations $\odot$ and $\to$ on A are defined as follows:

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	0	a
b	0	0	a	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	b	1	1
1	0	a	b	1

Routine calculations show that A with these operations is a bounded hoop and $St_r(\{a\}) = St_\odot(\{b\}) = St_l(\{b\}) = \{1\}$.

(ii) Let $A = \{0, a, b, 1\}$. Define $\odot$ and $\rightarrow$ on A as follows:

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

Then $(A, \odot, \rightarrow, 1)$ is a bounded hoop. Let $X_1 = \{b\}$ and $X_2 = \{0, a, 1\}$. Then $St_l(X_1) = \{0, a, 1\}$, $St_\odot(X_1) = \{b, 1\}$ and $St_r(X_2) = \{b, 1\}$.

Note: $St_r(St_l(X))$ is called *right-left stabilizer of* X and we show it by $(St_l(X))_r$, for short. The following proposition provides some properties of stabilizers of hoops.

Proposition 3.3. *For all $x, y \in A$ and $X, Y \subseteq A$, the following statements hold:*

(i) $1 \in St_r(X) \cap St_l(X) \cap St_\odot(X)$,

(ii) *if* $X \subseteq Y$, *then* $St_r(Y) \subseteq St_r(X)$, $St_l(Y) \subseteq St_l(X)$ *and* $St_\odot(Y) \subseteq St_\odot(X)$,

(iii) $X \subseteq (St_r(X))_l \cap (St_l(X))_r$,

(iv) $St_r(X) = ((St_r(X))_l)_r$ *and* $St_l(X) = ((St_l(X))_r)_l$,

(v) $St_r(\bigcup_{i\in I} X_i) = \bigcap_{i\in I}(St_r(X_i))$, $St_l(\bigcup_{i\in I} X_i) = \bigcap_{i\in I}(St_l(X_i))$ *and* $St_\odot(\bigcup_{i\in I} X_i) = \bigcap_{i\in I}(St_\odot(X_i))$,

(vi) $St_r(A) = St_l(A) = St_\odot(A) = \{1\}$,

(vii) $St_r(\{1\}) = St_l(\{1\}) = A$,

(viii) $\{x\} \subseteq St_\odot(\{x\})$ *if and only if* $x^2 = x$, *for all* $x \in A$,

(ix) *if* $F \in \mathcal{F}(A)$, *then* $St_\odot(F) \subseteq F$,

(x) *if* $h : A \rightarrow A$ *is a homomorphism and* $x \in A$, *then*

$$h(St_r(\{x\})) \subseteq St_r(\{h(x)\}),\ h(St_l(\{x\})) \subseteq St_l(\{h(x)\}),\ h(St_\odot(\{x\})) \subseteq St_\odot(\{h(x)\}).$$

Proof. The proofs of (i), (ii), (iii), (vii) and (viii) are clear by Proposition 2.2 and some modifications.

(iv) Since by (iii), $X \subseteq (St_l(X))_r$ and by (ii), $((St_l(X))_r)_l \subseteq St_l(X)$. Moreover, by (iii), $St_l(X) \subseteq ((St_l(X))_r)_l$. So, $St_l(X) = ((St_l(X))_r)_l$. By the similar way, we have, $St_r(X) = ((St_r(X))_l)_r$.

(v) For all $i \in I$, since $X_i \subseteq \bigcup_{i\in I} X_i$, by (ii), $St_l(\bigcup_{i\in I} X_i) \subseteq St_l(X_i)$, then $St_l(\bigcup_{i\in I} X_i) \subseteq \bigcap_{i\in I}(St_l(X_i))$. Now, suppose that $a \in \bigcap_{i\in I}(St_l(X_i))$. Then $a \in St_l(X_i)$, for all $i \in I$. Thus, $x_i \rightarrow a = a$, for all $x_i \in X_i$, and so $x \rightarrow a = a$, for all $x \in \bigcup_{i\in I} X_i$. Then $a \in St_l(\bigcup_{i\in I} X_i)$. Therefore, $St_l(\bigcup_{i\in I} X_i) = \bigcap_{i\in I}(St_l(X_i))$. By the similar way, we can prove the other statements.

(vi) Let $a \in St_r(A)$. Then $a \rightarrow x = x$, for all $x \in A$. By considering, $x = a$, and by (HP2),

$1 = a \to a = a$. Hence, $St_r(A) = \{1\}$. By the similar way, the other statements are clear.
(ix) Let $F \in \mathcal{F}(A)$ and $a \in St_\odot(F)$. Then, for all $x \in F$, $a \odot x = x$, and so $a \odot x \in F$. Since by Proposition 2.2(iii), $a \odot x \leq a$, and $F \in \mathcal{F}(A)$, we have $a \in F$. Hence, $St_\odot(F) \subseteq F$.
(x) Let $x \in A$, $h : A \to A$ be a homomorphism and $y \in h(St_r(\{x\}))$. Then there exists $a \in St_r(\{x\})$ such that $y = h(a)$. Since $a \in St_r(\{x\})$, $a \to x = x$. Moreover, since h is a homomorphism, we have $h(a) \to h(x) = h(a \to x) = h(x)$, and so $y \to h(x) = h(x)$. Hence, $y \in St_r(\{h(x)\})$.
By the similar way, we can prove the other inclusion. □

Example 3.4. In Example 3.2(ii), let $F = \{b, 1\}$. It is clear that $F \in \mathcal{F}(A)$. By routine calculations, $St_\odot(F) = \{1\}$. Thus, $F \nsubseteq St_\odot(F)$. So, if $F \in \mathcal{F}(A)$ and $F \neq \{1\}$, then $St_\odot(F) \neq F$.

Proposition 3.5. *Let A be a $\sqcup$-hoop, $X \subseteq A$ and $a \in A$ such that $a \sqcup x = 1$, for all $x \in X$. Then $a \in St_r(X) \cap St_l(X)$.*

Proof. Let $a \in A$ such that $a \sqcup x = 1$, for all $x \in X$. Then by Proposition 2.4, $1 = a \sqcup x = ((x \to a) \to a) \wedge ((a \to x) \to x)$, and so $(x \to a) \to a = 1$ and $(a \to x) \to x = 1$. Suppose that $(x \to a) \to a = 1$. Then by Proposition 2.2(iv), $a \leq x \to a$, and so $a \to (x \to a) = 1$. Hence, $x \to a = a$, for all $x \in X$. Thus, $a \in St_l(X)$. By the similar way, we can see that $a \to x = x$, for all $x \in X$, and so $a \in St_r(X)$. Therefore, $a \in St_r(X) \cap St_l(X)$. □

Theorem 3.6. *Let $X \subseteq A$. Then $St_r(X)$ and $St_\odot(X)$ are filters of A.*

Proof. Let $X \subseteq A$. Then by Proposition 3.3(i), $1 \in St_r(X)$, and so $St_r(X) \neq \emptyset$. Now, let $a \leq b$ and $a \in St_r(X)$, for some $a, b \in A$. Then by Proposition 2.2(x), $b \to x \leq a \to x$, for all $x \in X$. Since $a \in St_r(X)$, we have $a \to x = x$, and so $b \to x \leq x$, for all $x \in X$. Moreover, since by Proposition 2.2(iv), $x \leq b \to x$ and so $b \to x = x$, for all $x \in X$. Hence, $b \in St_r(X)$. Suppose that $a, b \in St_r(X)$, for some $a, b \in A$. Since $a \to x = b \to x = x$, for all $x \in X$, by (HP3) we have, $(a \odot b) \to x = a \to (b \to x) = a \to x = x$, for all $x \in X$. So, $a \odot b \in St_r(X)$. Therefore, $St_r(X) \in \mathcal{F}(A)$. Now, we prove that $St_\odot(X) \in \mathcal{F}(A)$, too. Let $a, b \in St_\odot(X)$, for some $a, b \in A$. Then $a \odot x = b \odot x = x$, for all $x \in X$. Hence, by (HP1), $(a \odot b) \odot x = a \odot (b \odot x) = a \odot x = x$, for all $x \in X$. So, $a \odot b \in St_\odot(X)$. Now, suppose that $a \leq b$ and $a \in St_\odot(X)$. Then $a \odot x = x$, for all $x \in X$. Hence, by Proposition 2.2(iii) and (viii), $x = a \odot x \leq b \odot x \leq x$, and so $b \odot x = x$, for all $x \in X$, which implies that $b \in St_\odot(X)$. Therefore, $St_\odot(X) \in \mathcal{F}(A)$. □

Example 3.7. In Example 3.2(ii), let $X = \{b\}$. Since $St_l(X) = \{0, a, 1\}$, it is clear that $St_l(X)$ is not a filter.

Definition 3.8. [12] Let $F \subseteq A$ such that $1 \in F$. Then

- F is called an *implicative filter* of A if $x \to (y \to z) \in F$ and $x \to y \in F$, then $x \to z \in F$, for all $x, y, z \in A$.
- F is called a *positive implicative filter* of A if $((x \to y) \to x) \to x \in F$, for all $x, y \in A$.
- F is called a *fantastic filter* of A if $z \to (y \to x) \in F$ and $z \in F$, then $((x \to y) \to y) \to x \in F$, for all $x, y, z \in A$.

Theorem 3.9. *Let $X \subseteq A$. Then the following statements hold.*
(i) If $a^2 = a$, for all $a \in A$, then $St_r(X)$ and $St_\odot(X)$ are implicative filters.
(ii) If A is a bounded hoop with (DNP), then $St_r(X)$ and $St_\odot(X)$ are fantastic filters.
(iii) If A is a bounded hoop with (DNP) and $a^2 = a$, for all $a \in A$, then $St_r(X)$ and $St_\odot(X)$ are positive implicative filters.

Proof. (i) Let $X \subseteq A$ such that $a^2 = a$, for all $a \in A$. Then by Theorem 3.6, $St_r(X) \in \mathcal{F}(A)$, thus, $1 \in St_r(X)$. Now, suppose that $a \to (b \to c)$ and $a \to b \in St_r(X)$, for some $a, b, c \in A$. Then, for all $x \in X$, $(a \to (b \to c)) \to x = x$ and $(a \to b) \to x = x$. Hence, $x = (a \to (b \to c)) \to x = (a \to (b \to c)) \to ((a \to b) \to x)$. By (HP1) and (HP3), we have

$$x = (a \to (b \to c)) \to ((a \to b) \to x) = (b \to (a \to c)) \to ((a \to b) \to x) = [(a \to b) \odot (b \to (a \to c))] \to x.$$

Then by Proposition 2.2(xii) and (x),

$$x = [(a \to b) \odot (b \to (a \to c))] \to x \geq (a \to (a \to c)) \to x = (a^2 \to c) \to x.$$

Moreover, since $a^2 = a$, for all $a \in A$, we have $x \geq (a \to c) \to x$. Also, by Proposition 2.2(iv), $x \leq (a \to c) \to x$. Hence, $(a \to c) \to x = x$, for all $x \in X$. Then $a \to c \in St_r(X)$. Therefore, $St_r(X)$ is an implicative filter. Now, we prove that $St_\odot(X)$ is an implicative filter. By Theorem 3.6, $St_\odot(X) \in \mathcal{F}(A)$, and so $1 \in St_\odot(X)$. Let $a \to (b \to c)$ and $a \to b \in St_\odot(X)$, for some $a, b, c \in A$. Then, for all $x \in X$, $(a \to (b \to c)) \odot x = x$ and $(a \to b) \odot x = x$. Thus, by (HP1), $((a \to b) \odot (a \to (b \to c))) \odot x = x$. Hence, by (HP3) and Proposition 2.2(xii), (viii) and (iii),

$$x = ((a \to b) \odot (b \to (a \to c))) \odot x \leq (a \to (a \to c)) \odot x = (a^2 \to c) \odot x \leq x.$$

Then $(a^2 \to c) \odot x = x$. Since $a^2 = a$, for all $a \in A$, we have $(a \to c) \odot x = x$, for all $x \in X$, and so $a \to c \in St_\odot(X)$. Therefore, $St_\odot(X)$ is an implicative filter.
(ii) Let A be a bounded hoop with (DNP) and $X \subseteq A$. Then by Theorem 3.6, $St_r(X) \in \mathcal{F}(A)$, and so $1 \in St_r(X)$. Now, suppose that $a \to (b \to c)$ and $a \in St_r(X)$, for some $a, b, c \in A$. Since $St_r(X) \in \mathcal{F}(A)$, by Proposition 2.9, $b \to c \in St_r(X)$. Hence, $(b \to c) \to x = x$, for all $x \in X$. Now, we prove that $((c \to b) \to b) \to c \in St_r(X)$. Since A has (DNP), by Proposition 2.3(iv), for all $x \in X$, $[((c \to b) \to b) \to c] \to x = [((b \to c) \to c) \to c] \to x$. Thus, by Proposition 2.2(xi), $[((c \to b) \to b) \to c] \to x = (b \to c) \to x$. Since $b \to c \in St_r(X)$, we have $(b \to c) \to x = x$, for all $x \in X$, and so $[((c \to b) \to b) \to c] \to x = x$. Hence, $((c \to b) \to b) \to c \in St_r(X)$.

Therefore, $St_r(X)$ is a fantastic filter. Now, we prove that $St_\odot(X)$ is a fantastic filter. By Theorem 3.6, $St_\odot(X) \in \mathcal{F}(A)$, and so $1 \in St_\odot(X)$. Suppose that $a \to (b \to c)$ and $a \in St_\odot(X)$, for some $a, b, c \in A$. Since $St_\odot(X) \in \mathcal{F}(A)$, by Proposition 2.9, $b \to c \in St_\odot(X)$. Hence, $(b \to c) \odot x = x$, for all $x \in X$. Now, we prove that $((c \to b) \to b) \to c \in St_\odot(X)$. Since A has (DNP), by Proposition 2.3(iv), for all $x \in X$, $[((c \to b) \to b) \to c] \odot x = [((b \to c) \to c) \to c] \odot x$. Thus, by Proposition 2.2(xi), $[((c \to b) \to b) \to c] \odot x = (b \to c) \odot x$. Since $b \to c \in St_\odot(X)$, we have $(b \to c) \odot x = x$, for all $x \in X$. So, $[((c \to b) \to b) \to c] \odot x = x$. Hence, $St_\odot(X)$ is a fantastic filter.
(iii) By Theorem 3.6, $St_r(X) \in \mathcal{F}(A)$. Then $1 \in St_r(X)$. Now, we prove that $((a \to b) \to a) \to a \in St_r(X)$. Since A has (DNP), by Proposition 2.3(iv) and (HP3), $((a \to b) \to a) \to a = (a \to (a \to b)) \to (a \to b) = (a^2 \to b) \to (a \to b)$. Moreover, since $a^2 = a$, for all $a \in A$, we have $[((a \to b) \to a) \to a] = (a \to b) \to (a \to b) = 1$. Thus, $[((a \to b) \to a) \to a] \to x = 1 \to x = x$. Hence, $St_r(X)$ is a positive implicative filter of A. By the similar way, we can prove that $St_\odot(X)$ is a positive implicative filter of A. □

Proposition 3.10. *Let A be bounded with (DNP) and $X \subseteq A$. Then $St_r(X) = St_l(X)$.*

Proof. Let A be bounded with (DNP), $X \subseteq A$ and $a \in St_r(X)$. Then $a \to x = x$, for all $x \in X$. Hence, $(a \to x) \to x = 1$. Since A has (DNP), by Proposition 2.3(iv), $(a \to x) \to x = (x \to a) \to a$, and so $(x \to a) \to a = 1$. Also, by Proposition 2.2(iv), $a \to (x \to a) = 1$, then $x \to a = a$. Thus, $a \in St_l(X)$. Hence, $St_r(X) \subseteq St_l(X)$. By the similar way, we have $St_l(X) \subseteq St_r(X)$. Therefore, $St_r(X) = St_l(X)$. □

Definition 3.11. [4] Let A be a $\sqcup$-hoop and F be a proper filter of A. Then F is called a *prime filter* of A, if $x \sqcup y \in F$, for some $x, y \in A$, then $x \in F$ or $y \in F$.

Theorem 3.12. *Let A be a $\sqcup$-hoop and $a \in A$. If $St_r(\{a\})(St_\odot(\{a\}))$ is a proper subset of A, then it is a prime filter.*

Proof. Let $a \in A$. Then by Theorem 3.6, $St_r(\{a\}) \in \mathcal{F}(A)$. Thus, $St_r(\{a\})$ is a proper filter. Now, let, for $x, y \in A$, $x \sqcup y \in St_r(\{a\})$, but $x, y \notin St_r(\{a\})$. Then $x \to a \neq a$ and $y \to a \neq a$, and so by Proposition 2.2(iv), $a < x \to a$ and $a < y \to a$. Since $x \sqcup y \in St_r(\{a\})$, we have $(x \sqcup y) \to a = a$. Hence, by Proposition 2.7(i), $a < (x \to a) \wedge (y \to a) = (x \sqcup y) \to a = a$, which is a contradiction. Therefore, $St_r(\{a\})$ is a prime filter. Moreover, by considering $a \in A$ and Theorem 3.6, $St_\odot(\{a\}) \in \mathcal{F}(A)$, and so $St_r(\{a\})$ is a proper filter. Now, let $x, y \in A$, $x \sqcup y \in St_\odot(\{a\})$ but $x, y \notin St_\odot(\{a\})$. Then $x \odot a \neq a$ and $y \odot a \neq a$, and so by Proposition 2.2(iii), $x \odot a < a$ and $y \odot a < a$. Since $x \sqcup y \in St_\odot(\{a\})$, we have $(x \sqcup y) \odot a = a$. Hence, by Proposition 2.7(ii), $a = (x \sqcup y) \odot a = (x \odot a) \sqcup (y \odot a) < a$, which is a contradiction. Therefore, $St_\odot(\{a\})$ is a prime filter. □

Corollary 3.13. *Let A be a $\sqcup$-hoop and $X \subseteq A$. If $St_r(X)(St_{\odot}(X))$ is a proper subset of A, then it is a prime filter.*

Proof. The proof is similar to the proof of Proposition 3.12, by some modification. $\square$

Theorem 3.14. *Let A be a $\sqcup$-hoop, $X \subseteq A$ and $F_1, F_2 \in \mathcal{F}(A)$. If $St_r(X)$ is a proper subset of A such that $St_r(X) = F_1 \cap F_2$, then $St_r(X) = F_1$ or $St_r(X) = F_2$.*

Proof. Let $St_r(X)$ be a proper subset of A such that $St_r(X) = F_1 \cap F_2$, for some $F_1, F_2 \in \mathcal{F}(A)$. Since $St_r(X) = F_1 \cap F_2$, we have $St_r(X) \subseteq F_1, F_2$. Now, suppose that $St_r(X) = F_1 \cap F_2$ such that $F_1 \nsubseteq St_r(X)$ and $F_2 \nsubseteq St_r(X)$. Then there exist $a \in F_1$ and $b \in F_2$ such that $a, b \notin St_r(X)$. Since $F_1, F_2 \in \mathcal{F}(A)$ and $a, b \leq a \sqcup b$, $a \sqcup b \in F_1 \cap F_2 = St_r(X)$. Moreover, since $St_r(X)$ is a proper subset of A, by Corollary 3.13, $St_r(X)$ is a prime filter. Hence, $a \in St_r(X)$ or $b \in St_r(X)$, which is a contradiction. Thus, $F_1 \subseteq St_r(X)$ or $F_2 \subseteq St_r(X)$. Therefore, $St_r(X) = F_1$ or $St_r(X) = F_2$. $\square$

By the similar way, we can prove that the Theorem 3.14 holds for $St_{\odot}(X)$, for $X \subseteq A$.

Proposition 3.15. *Let $X \subseteq A$. Then*
(i) $A/St_r(X) = \{((a \to b) \odot (b \to a)) \to x = x, \text{ for all } x \in X\}$.
(ii) $A/St_{\odot}(X) = \{((a \to b) \odot (b \to a)) \odot x = x, \text{ for all } x \in X\}$.

Proof. (i) Since by Theorem 3.6, $St_r(X) \in \mathcal{F}(A)$, we have $A/St_r(X)$ is well-defined. Hence,

$$A/St_r(X) = \{\overline{b} \mid b \in A\} = \{a \in A \mid a \to b \text{ and } b \to a \in St_r(X)\}.$$

Thus, for all $x \in X$, $(a \to b) \to x = x$ and $(b \to a) \to x = x$. By (HP3), for all $x \in X$,

$$((a \to b) \odot (b \to a)) \to x = (a \to b) \to ((b \to a) \to x) = (a \to b) \to x = x.$$

Therefore, $A/St_r(X) = \{((a \to b) \odot (b \to a)) \to x = x, \text{ for all } x \in X\}$.
(ii) The proof is similar to the proof of (i). $\square$

Proposition 3.16. *Let $X \subseteq A$ and $F \in \mathcal{F}(A)$. Then the following statements hold.*
(i) *If $F \subseteq St_r(X)$ and $\overline{a} \in St_r(X)/F$, then $(a \to x) \to x \in F$, for all $x \in X$.*
(ii) *If $F \subseteq St_{\odot}(X)$ and $\overline{a} \in St_{\odot}(X)/F$, then $x \to (a \odot x) \in F$, for all $x \in X$.*

Proof. (i) Let $F \subseteq St_r(X)$ and $\overline{a} \in St_r(X)/F$. Then there exists $b \in St_r(X)$ such that $\overline{a} = \overline{b}$. By definition of quotient, $a \to b,\ b \to a \in F$. Since $F \subseteq St_r(X)$, we have $a \to b,\ b \to a \in St_r(X)$. By Theorem 3.6, $St_r(X) \in \mathcal{F}(A)$ and since $b \to a$, $b \in St_r(X)$, by Proposition 2.9, $a \in St_r(X)$. Hence, for all $x \in X$, $a \to x = x$, and so $(a \to x) \to x = 1 \in F$. Therefore, $(a \to x) \to x \in F$.
(ii) Let $F \subseteq St_{\odot}(X)$ and $\overline{a} \in St_{\odot}(X)/F$. Then there exists $b \in St_{\odot}(X)$ such that $\overline{a} = \overline{b}$. By definition

of quotient, $a \to b,\ b \to a \in F$. Since $F \subseteq St_{\odot}(X)$, we have $a \to b,\ b \to a \in St_{\odot}(X)$. By Theorem 3.6, $St_{\odot}(X) \in \mathcal{F}(A)$ and since $b \to a$, $b \in St_{\odot}(X)$, by Proposition 2.9, $a \in St_{\odot}(X)$. Then, for all $x \in X$, $a \odot x = x$, and so $x \leq a \odot x$. Hence, $x \to (a \odot x) = 1 \in F$. Therefore, $x \to (a \odot x) \in F$. $\square$

Proposition 3.17. *Let $X \subseteq A$ and $F \in \mathcal{F}(A)$ such that $F \subseteq St_r(X)(F \subseteq St_{\odot}(X))$. Then $St_r(X)/F$ $(St_{\odot}(X)/F)$ is a filter of A/F.*

Proof. Let $X \subseteq A$ and $F \in \mathcal{F}(A)$ such that $F \subseteq St_r(X)$. Since $F \in \mathcal{F}(A)$, we have $St_r(X)/F$ and A/F are well-defined. Now, let $\overline{a}, \overline{b} \in St_r(X)/F$, for $a, b \in St_r(X)$. Then by Theorem 3.6, $St_r(X) \in \mathcal{F}(A)$, and so $a \odot b \in St_r(X)$. Hence, $\overline{a \odot b} = \overline{a} \odot \overline{b} = a \odot b/F \in St_r(X)/F$. Now, suppose that $\overline{a} \leq \overline{b}$ and $\overline{a} \in St_r(X)/F$. Since $\overline{a} \leq \overline{b}$, we have $a \to b \in F$. Moreover, since $F \subseteq St_r(X)$, $a \to b \in St_r(X)$. Also, $a \in St_r(X)$ and by Theorem 3.6, $St_r(X) \in \mathcal{F}(A)$, then by Proposition 2.9, $b \in St_r(X)$. Hence, $\overline{b} \in St_r(X)/F$. Therefore, $St_r(X)/F$ is a filter of A/F. $\square$

Proposition 3.18. (i) $\quad St_r(A/St_r(A)) = \overline{1}$.
(ii) $\quad St_{\odot}(A/St_{\odot}(A)) = \overline{1}$.

Proof. (i) Suppose that $\overline{a} \in St_r(A/St_r(A))$. Then, for all $\overline{x} \in A/St_r(A)$, $\overline{a} \to \overline{x} = \overline{x}$. By definition of quotient, $(a \to x) \to x \in St_r(A)$. So, for all $x \in A$, $((a \to x) \to x) \to x = x$. Since by Proposition 2.2(xi), $((a \to x) \to x) \to x = a \to x$, we have $a \to x = x$, and so $a \in St_r(A)$. Moreover, suppose that $x = 1$. Since $a \to 1 = 1 \in St_r(A)$ and $1 \to a = a \in St_r(A)$, $a \sim_{St_r(A)} 1$. This means that $\overline{a} = \overline{1}$. Therefore, $St_r(A/St_r(A)) = \overline{1}$.
(ii) Suppose that $\overline{a} \in St_{\odot}(A/St_{\odot}(A))$. Then, for all $\overline{x} \in A/St_{\odot}(A)$, $\overline{a} \odot \overline{x} = \overline{x}$. By definition of quotient, $x \to (a \odot x) \in St_{\odot}(A)$. So, for all $x \in A$, $(x \to (a \odot x)) \odot x = x$. By Proposition 2.2(vii) and (iii), $x = x \odot (x \to (a \odot x)) \leq a \odot x \leq x$. Then $a \odot x = x$, and so $a \in St_{\odot}(A)$. Hence, similar to the proof of (i), $\overline{a} = \overline{1}$. Therefore, $St_{\odot}(A/St_{\odot}(A)) = \overline{1}$. $\square$

4. Stabilizer topology

In this section, we use of the right and left stabilizers of a hoop and produce a basis for a topology on it. Then we show that the generated topology by this basis is Baire, connected, locally connected and separable and investigate the other properties of this topology.

Theorem 4.1. *Let $X \subseteq A$. Define $\alpha : \mathcal{P}(A) \to \mathcal{P}(A)$ such that $\alpha(X) = (St_l(X))_r$, for all $X \in \mathcal{P}(A)$. Then*
(i) $\quad$ *α is a closure map.*
(ii) $\quad$ *$X \subseteq \alpha(Y)$ if and only if $\alpha(X) \subseteq \alpha(Y)$, for all $Y \subseteq A$.*
(iii) $\quad$ *$\beta_{\alpha} = \{X \in \mathcal{P}(A) \mid \alpha(X) = X\}$ is a basis for a topology on A.*

Proof. (i) By Proposition 3.3(iii), (iv) and (ii), the proof is clear.
(ii) It is straightforward.
(iii) Let $\beta_\alpha = \{X \in \mathcal{P}(A) \mid \alpha(X) = X\}$. It is clear that $\emptyset \in \beta_\alpha$. Also, by Proposition 3.3(vi) and (vii), $\alpha(A) = (St_l(A))_r = St_r(\{1\}) = A$. Thus, $\alpha(A) = A$, and so $A \in \beta_\alpha$. Now, suppose that $X, Y \in \beta_\alpha$. Then $\alpha(X) = X$ and $\alpha(Y) = Y$. We prove that $X \cap Y \in \beta_\alpha$. Since $X \cap Y \subseteq X, Y$, by (i), $\alpha(X \cap Y) \subseteq \alpha(X)$ and $\alpha(Y)$. Thus, $\alpha(X \cap Y) \subseteq \alpha(X) \cap \alpha(Y)$. Also, since $X, Y \in \beta_\alpha$, we have $\alpha(X \cap Y) \subseteq X \cap Y$. Moreover, by Proposition 3.3(iii), $X \cap Y \subseteq \alpha(X \cap Y)$. Then $\alpha(X \cap Y) = X \cap Y$, and so $X \cap Y \in \beta_\alpha$. Therefore, β_α is a basis. □

Definition 4.2. According to Theorem 4.1, the topological space, $(A, \mathcal{T}_\alpha)$ is called a *stabilizer topology.*

Note: Since $St_r(X) \in \mathcal{F}(A)$, for any $X \subseteq A$, every element of β_α is a filter of A.

Example 4.3. In Example 3.2(ii), $(A, \odot, \rightarrow, 1)$ is a bounded hoop. By Proposition 3.3(i) and (vi), we see that $\{1\} \in \alpha(X)$, for all non-empty subset X of A. So, if $1 \notin X \subseteq A$, then $X \notin \beta_\alpha$. By some manipulations, we get that $\beta_\alpha = \{\emptyset, A, \{1, b\}, \{1\}\}$. Thus, $\mathcal{T}_\alpha = \{\emptyset, A, \{1\}, \{1, b\}\}$. By routine calculation, we can see that in this example, $\{1, a\} \notin \beta_\alpha$, because $St_l(\{1, a\}) = \{1\}$ and $St_r(\{1\}) = A$, then $\alpha(\{1, a\}) = A$, and so $\alpha(\{1, a\}) \neq \{1, a\}$.

Theorem 4.4. *The stabilizer topology $(A, \mathcal{T}_\alpha)$ is*
(i) *connected.*
(ii) *locally connected.*
(ii) *Hausdorff space if and only if $A = \{1\}$.*

Proof. It is straightforward. □

Theorem 4.5. *Let $(A, \mathcal{T}_\alpha)$ be a stabilizer topology. If $\emptyset \neq X \subseteq A$ such that $1 \in X$, then $\overline{X} = A$.*

Proof. Let $\emptyset \neq X \subseteq A$ such that $1 \in X$. It is enough to prove that $A \subseteq \overline{X}$. Let $x \in A$. If $x = 1$, then $x \in \overline{X}$. Hence, $\overline{X} = A$. Now, suppose that $1 \neq x \in A$. Then there exists an open subset $U \in \beta_\alpha$ such that $x \in U$. Since $U \in \mathcal{F}(A)$ and $1 \in U$, we have $U \cap (X - \{x\}) \neq \emptyset$. Hence, $x \in \overline{X}$, and so $\overline{X} = A$. □

Corollary 4.6. *$(A, \mathcal{T}_\alpha)$ is separable.*

Proof. Since $\{1\} \in \beta_\alpha$, by Theorem 4.5, $\overline{\{1\}} = A$. Hence, $(A, \mathcal{T}_\alpha)$ is separable. □

Theorem 4.7. *$(A, \mathcal{T}_\alpha)$ is Baire space.*

Proof. Let $U \in \mathcal{T}_\alpha$. Since $U \in \mathcal{F}(A)$, we have $1 \in U$. Then by Theorem 4.5, $\overline{U} = A$. Thus, every open set of $(A, \mathcal{T}_\alpha)$ is dense. On the other side, for each collection of open set U_n, $\bigcap U_n \in \mathcal{F}(A)$, and so $1 \in \bigcap U_n$. Thus, by Theorem 4.5, $\bigcap U_n$ is dense. Therefore, $(A, \mathcal{T}_\alpha)$ is Baire space. □

In the following example, we show that $(A, \mathcal{T}_\alpha)$ is not a T_0-space or T_1-space.

Example 4.8. In Example 3.2(i) and (ii), $\beta_\alpha = \{\emptyset, A, \{1\}\}$. Since $a \neq b$, for $a, b \in A$, there is not $U \in \beta_\alpha$ such that $a \in U$ and $b \notin U$. Therefore, $(A, \mathcal{T}_\alpha)$ is not a T_0-space. It is clear that $(A, \mathcal{T}_\alpha)$ is not a T_1-space.

Theorem 4.9. *Let A be bounded. If A has a cover of $U_i \in \beta_\alpha$, for $i \in I$, then there exists $i \in I$ such that $U_i = A$.*

Proof. Let A be bounded and $\{U_i\}_{i \in I}$ be a cover of A such that, for all $i \in I$, $U_i \in \beta_\alpha$ and $A \subseteq \bigcup_{i \in I} U_i$. Since, for all $i \in I$, $U_i \in \beta_\alpha$, we have $U_i \in \mathcal{F}(A)$. On the other side, A is bounded, then $0 \in A$, and so $0 \in \bigcup_{i \in I} U_i$. Thus, there exists $i \in I$ such that $0 \in U_i$. Since $U_i \in \mathcal{F}(A)$ and $0 \in U_i$, then $U_i = A$. Hence, there exists a finite family of $\{U_i\}_{i \in I}$ such that $A \subseteq \bigcup_{i=1}^{n} U_i$. □

Theorem 4.10. *Let $F \in \mathcal{F}(A)$ and $\overline{\alpha} : \mathcal{P}(A/F) \to \mathcal{P}(A/F)$ is defined by $\overline{\alpha}(X/F) = (St_l(X/F))_r$, for all $X/F \in \mathcal{P}(A/F)$. Then*
(i) $\overline{\alpha}$ *is a closure map.*
(ii) $\beta_{\overline{\alpha}} = \{X/F \mid \overline{\alpha}(X/F) = X/F,\ F \subseteq X \subseteq A\}$ *is a basis for a topology on A/F.*

Proof. The proof is similar to the proof of Theorem 4.1. □

Note: If $\mathcal{T}_{\overline{\alpha}}$ is a topology induced by $\beta_{\overline{\alpha}}$, then $(A/F, \mathcal{T}_{\overline{\alpha}})$ is called the *quotient stabilizer topology* or *QS-topology*, for short. It is clear that $\mathcal{T}_{\overline{\alpha}} = \{O \subseteq A/F \mid \pi^{-1}(O) \in \mathcal{T}_\alpha\}$, where $\pi : A \to A/F$ is the canonical epimorphism.

Theorem 4.11. *Let $F \in \mathcal{F}(A)$, $F \subseteq X \subseteq A$ and $\pi : A \to A/F$ be an open canonical epimorphism. Then X/F is an open set in the QS-topology, $\mathcal{T}_{\overline{\alpha}}$.*

Proof. Let $F \in \mathcal{F}(A)$ and $F \subseteq X \subseteq A$. Then $X/F \subseteq A/F$. Since π is an open epimorphism, there exists an open subset U of A such that $\pi(U) = X/F$. Thus, $U = \pi^{-1}(X/F)$. Since U is open, there is a $V \in \beta_\alpha$ such that $V \subseteq U = \pi^{-1}(X/F)$. Hence, $\pi(V) \subseteq X/F$, and so $V/F \subseteq X/F$. Therefore, X/F is an open subset in the QS-topology, $\mathcal{T}_{\overline{\alpha}}$. □

Theorem 4.12. *Let S be a topological subalgebra of A and $F \in \mathcal{F}(A)$ such that $F \subseteq S$. Then the QS-topology on S/F is stronger than the topology induced on the subalgebra S/F of A/F by the QS-topology on A/F.*

Proof. Let $\pi_S : S \to S/F$ be the canonical map and U be a set in the subspace topology of S/F. Then $U = W \cap S/F$, where W is an open set in A/F. We obtain that $\pi_S^{-1}(U) = \pi_S^{-1}(W) \cap S$. Since W and S/F are open sets and π_S is a continuous epimorphism, we have $\pi_S^{-1}(W) \cap S$ is an open set. Hence, $\pi^{-1}(U)$ is open in S. Therefore, U is an open set in the quotient topology of S/F. $\square$

Theorem 4.13. *Let $F \in \mathcal{F}(A)$ and $(A/F, \mathcal{T}_{\overline{\alpha}})$ be the QS-topology. Then $(A/F, \mathcal{T}_{\overline{\alpha}})$ is connected.*

Proof. The proof is similar to the proof of Theorems 4.4(i). $\square$

Theorem 4.14. *Let $F \in \mathcal{F}(A)$, $(A/F, \mathcal{T}_{\overline{\alpha}})$ be the QS-topology and $\pi : A \to A/F$ be an open continuous epimorphism. Then $(A/F, \mathcal{T}_{\overline{\alpha}})$ is Housdorff if and only if F is closed.*

Proof. $(\Rightarrow)$ We prove that F^c is open. Let $x \in F^c$, for some $x \in A$. Since $x \notin F$, then $\overline{x} \neq \overline{1}$. Since A/F is Hausdorff, there exist two open neighborhoods U and V of $\overline{x}$ and $\overline{1}$, respectively such that $U \cap V = \emptyset$. So, $\pi^{-1}(U)$ and $\pi^{-1}(V)$ are two open neighborhoods of x and 1, respectively. Let $z \in F$. Then $\pi(z) = \overline{z} = \overline{1} \in V$. Hence, $z \in \pi^{-1}(V)$ and then $F \subseteq \pi^{-1}(V)$. We show that $\pi^{-1}(U) \subseteq F^c$. Let $z \in \pi^{-1}(U)$. Then $\overline{z} \in U$. Since $U \cap V = \emptyset$, we have $z \notin \pi^{-1}(V)$. Moreover, since $F \subseteq \pi^{-1}(V)$, we have $z \notin F$. So, $z \in F^c$. Hence, $\pi^{-1}(U) \subseteq F^c$. Since $\pi^{-1}(U)$ is an open set, we get that F^c is open. Therefore, F is closed.
$(\Leftarrow)$ Suppose that F is closed and $\overline{x} \neq \overline{y}$ in A/F. Then $\overline{x} \to \overline{y} \neq \overline{1}$ or $\overline{y} \to \overline{x} \neq \overline{1}$. Let $\overline{x} \to \overline{y} \neq \overline{1}$. Then $x \to y \notin F$. So, $x \to y \in F^c$. Moreover, since F^c is an open, there exists an open subset U such that $x \to y \in U \subseteq F^c$ and $U \cap F = \emptyset$. Since π is an open continuous epimorphism, we have $\pi(U)$ is open neighborhood of $\overline{x} \neq \overline{y}$. If $\overline{1} \in \pi(U)$, then there exists $a \in U$ such that $\overline{a} = \overline{1}$, and so $a \to 1, 1 \to a \in F$. Hence, $a \in F$, which is a contradiction. Therefore, $\overline{1} \notin \pi(U)$. Now, since $\mathcal{T}_{\overline{\alpha}}$ is the QS-topology on A/F, there exist open neighborhoods $\overline{P}$ and $\overline{Q}$ of $\overline{x}$ and $\overline{y}$ such that

$$\overline{x} \to \overline{y} \in \overline{P} \to \overline{Q} \subseteq \overline{P \to Q} \subseteq \pi(P \to Q) \subseteq \pi(P) \to \pi(Q) \subseteq \pi(U).$$

Now, we prove that $\pi(P) \cap \pi(Q) = \emptyset$. Let $\overline{z} \in \pi(P) \cap \pi(Q)$. Then there are $a \in P$ and $b \in Q$ such that $\overline{a} = \overline{z} = \overline{b}$. Since $\overline{z} \to \overline{z} = \overline{a} \to \overline{b} \in \pi(P) \to \pi(Q) \subseteq \pi(U)$. Hence, $\overline{1} \in \pi(U)$, which is a contradiction. Therefore, $(A/F, \mathcal{T}_{\overline{\alpha}})$ is Hausdorff. $\square$

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5. Conclusions and future works

In this paper, we introduce the concepts of right, left and product stabilizers on hoops. Then by using the right stabilizers produce a basis for a topology on hoops. We show that the generated topology by this basis is Baire, connected, locally connected and separable. Finally, we investigate the other properties of this topology.

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