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HYPERTRANSCENDENTAL FORMAL POWER SERIES OVER FIELDS OF POSITIVE CHARACTERISTIC

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Abstract. Let K be a field of characteristic $p > 0$, $K[[x]]$, the ring of formal power series over K, $K((x))$, the quotient field of $K[[x]]$, and $K(x)$ the field of rational functions over K. We shall give some characterizations of an algebraic function $f \in K((x))$ over K. Let L be a field of characteristic zero. The power series $f \in L[[x]]$ is called differentially algebraic, if it satisfies a differential equation of the form $P(x, y, y', ...) = 0$, where P is a non-trivial polynomial. This notion is defined over fields of characteristic zero and is not so significant over fields of characteristic $p > 0$, since $f^{(p)} = 0$. We shall define an analogue of the concept of a differentially algebraic power series over K and we shall find some more related results.

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Formal power series have been studied by many mathematicians from different points of view, for example, analytically, formally, etc. We have also studied some especial formal power series, such as, rational, algebraic, differentiably finite, differentially algebraic, E-algebraic, etc., depending on the context as belonging to the ring of formal power series $K[[x]]$ (all such series with coefficients in a field K), $K((x))$, the field of fractions of $K[[x]]$ or $K(x)$, the field of rational functions over K. (See [**8** − **14**]).

Let L be a field of characteristic zero. The power series $f \in L[[x]]$ is called differentially algebraic (D- algebraic, for short), if it satisfies a differential equation of the form $P(x, y, y', ...) = 0$, where P is a non-trivial polynomial. Equivalently, f is called D-algebraic if $tr.deg._{L(x)}L(x, f, f', ..., f^{(n)}, ...)$ is finite, where $tr.deg._B A$ means the transcendence degree of A over B. A function which is not D-algebraic is called transcendentally transcendental (TT, for short).

This notion is only defined over fields of characteristic zero and is not so significant over fields of characteristic $p > 0$, since $f^{(p)} = 0$.

In [**12**] we gave an analogue of the concept of a D-algebraic formal power series over a *perfect* field of positive characteristic in terms of some "E" operators. (By a perfect field F of characteristic $p > 0$, we mean that it coincides with its subfield F^p).

In this article, we shall give some more characterizations of algebraic formal power series over an *arbitrary* field of positive characteristic in terms of a "differential" operator. Then we shall give an analogue of the concept of a differentially algebraic power series over such fields in terms of the above operator and we shall study the connection of this new concept with the one given before in [**12**].

Notations. From now on K will denote a perfect field of characteristic $p > 0$, L will be a field of characteristic zero, $\mathbf{F}_p$, the Galois field of order p and $\mathbf{Z}_p$, the ring of p-adic integers.

1. Algebraic Formal Power Series

Some characterizations of algebraic formal power series have been given in [**1, 3** and **14**]. In this section we shall give some more characterizations of such series in terms of some "differential" operators. Recall that by an algebraic function (or a series) over a field K, we mean an element $f \in K((x))$, which is algebraic over $K(x)$. A function which is not algebraic is called transcendental (over K).

Example 2.1. *i*) The series $\sum_{n=0}^{\infty} \binom{2n}{n} x^n = (1 - 4x)^{-1/2}$ is algebraic with respect to any field.

ii) The series $\sum_{n=0}^{\infty} x^{p^n}$ is algebraic over any field of characteristic $p > 0$, since $f(x) = x + f(x^p) = x + f(x)^p$.

iii) The series $\sum\limits_{n=0}^{\infty} x^{n!}$ is transcendental with respect to any field. (See, [**17**, p. 220].)

In order to give the above mentioned characterizations of formal power series over a perfect field K of characteristic $p > 0$, we used a splitting process and defined an operator, which is in fact a differential operator. First, we explain the motivation of using a differential operator.

Suppose that $\sum\limits_{n=-N}^{\infty} a_n x^n \in \mathbf{F}_2((x))$ and $f' = \sum a_n x^{n-1}$ is the formal derivative of f.Then $f'' = 0$. Thus dealing with those concepts which involve differentiating continuously, we can get nothing, since everything collapse, when the base field is of positive characteristic. To avoid this situation, we take the p-th root and then differentiate again. More precisely, for $i = 0, 1$ define the operator

$$E_i : \mathbf{F}_2((x)) \longrightarrow \mathbf{F}_2((x))$$

by $E_0(f) = \sqrt{xf'} = \sum\limits_{n\ even} \sqrt{a_n} x^{n/2}$ and $E_1(f) = \sqrt{f'} = \sum\limits_{n\ odd} \sqrt{a_n} x^{(n-1)/2}$. So we have

$$\begin{aligned} f(x) &= \sum_{n=-N}^{\infty} a_n x^n \\ &= \left(\sum_{n\ even} \sqrt{a_n} x^{n/2} \right)^2 + x \left(\sum_{n\ odd} \sqrt{a_n} x^{(n-1)/2} \right)^2 \\ &= E_0(f)^2 + xE_1(f)^2 \\ &= (xf)' + xf'. \end{aligned}$$

We used the above idea and defined the "E" operator and used it in a series of articles ([see, [**11-14**]) to get our results.

For completion, we explain the above splitting process in terms of the "E" operators briefly.

Lemma 2.2. *If $f(x) \in K[[x]]$ (respectively $K((x))$), then f can be written uniquely as*

$$\sum_{i=0}^{p-1} x^i f_i^p \qquad (1)$$

for some $f_i \in K[[x]]$ (respectively $K((x))$).

Proof. See [**14**]. (Note that $f_i = \sum_{n=0}^{\infty} a_{pn+i}^{\frac{1}{p}} x^n$.) $\square$

For each $i \in \{0, 1, 2, ..., p-1\}$ define $E_i : K((x)) \longrightarrow K((x))$ by $E_i(f) = f_i$. Now for $f \in K((x))$, by Lemma 2.2 we have

$$f = \sum_{i=0}^{p-1} x^i [E_i(f)]^p \qquad (2)$$

The operator E_i is semilinear; that is, if $f, g \in K((x))$ and $\lambda \in K$, then

$$E_i(\lambda f + g) = \lambda^{\frac{1}{p}} E_i(f) + E_i(g).$$

Moreover, $E_i(g^p f) = gE_i(f)$. (See also [**1-3**, **14**.])

Let Ω be the semigroup generated by the identity operator and the E_i for $i \in \{0, 1, 2, ..., p-1\}$, with ordinary composition as multiplication. To each $f \in K((x))$ we associate its orbit $\Omega(f) = \{E(f) : E \in \Omega\}$. Then we have the following theorem.

Theorem 2.3. *Let $f \in K((x))$ and $F = K(x,\ \Omega(f))$. Then $[F : F^p] = p$ if and only if F is invariant under each operator E_i, $0 \le i \le p-1$.*

Proof. $\Leftarrow$) Since

$$f = \sum_{i=0}^{p-1} x^i [E_i(f)]^p$$

and $E_i(F)^p \in F^p$, we conclude that $\{1,\ x,\ x^2, \cdots, x^{p-1}\}$ is a basis of F over F^p. Thus $[F : F^p] = p$.

$\Rightarrow$) Let $[F : F^p] = p$. Since $K(x) \subseteq F$ and $x \in F \setminus F^p$, $\{1,\ x,\ x^2, \cdots, x^{p-1}\}$ is a basis for F over F^p. Thus for any $g \in F$, there exist $g_0^p,\ g_1^p, \cdots, g_{p-1}^p \in F^p$ such that $g = g_0^p + xg_1^p + \cdots + x^{p-1}g_{p-1}^p$, where $g_i \in F$. By (2) each $E_i(g) = g_i \in F$. Thus F is invariant under each operator E_i. $\square$

Now, let $f \in K((x))$ and V be the $K(x)$-subspace of $K((x))$ spanned by $\Omega(f)$. Then we have the following theorem.

Theorem 2.4. *Let $f \in K((x))$. Then the following are equivalent:*

i) f is algebraic over K.

ii) $dim_{K(x)} V < \infty$.

iii) $F = K(x,\ \Omega(f))$ is a finitely generated field extension of $K(x)$.

Proof. *i) $\Rightarrow$ ii)* Suppose that f is algebraic over K. Then by Theorem 5.3 of [**14**], $dim_K < \Omega(f) >$ is finite. So $dim_{K(x)} V\ <\ dim_K < \Omega(f) > < \infty$.

ii) $\Rightarrow$ iii) Let $dim_{K(x)} V < \infty$ and $B = \{\beta_1, \beta_2, \cdots, \beta_n\}$ be a basis for V over $K(x)$. Since $f \in V$, f is a linear combination of elements of B with coefficients in $K(x)$. By (2),

$$\beta_j = \sum_{i=0}^{p-1} x^i [E_i(\beta_j)]^p,\ j = 1,\ 2, \cdots,\ n.$$

Since $< \Omega(f) >$ is invariant under each E_i, so is V and hence $E_i(\beta_j) \in V$, for each $i = 0,\ 1, \cdots,\ p-1$ and $j = 1,\ 2, \cdots,\ n$. Hence we can write $E_i(\beta_j) = \sum_{t=1}^{n} \alpha_{ijt}\beta_t$, $\alpha_{ijt} \in K(x)$. Thus

$$\beta_j = \sum_{i=0}^{p-1} x^i (\sum_{t=1}^{n} \alpha_{ijt}\beta_t)^p = \sum_{t=1}^{n} (\sum_{i=0}^{p-1} \alpha_{ijt}^p x^i) \beta_t^p.$$

Hence β_j is algebraic over K by [**3**] for each $j = 1,\ 2, \cdots,\ n$ and so is f. Now, by Lemma 5.2 of [**14**], $f = \sum_{i=1}^{N} a_i f^{p^i}$ for some $a_1,\ a_2,\ \cdots, a_N$ in $K[x]$. Applying the operators E_i to the above equation and using the properties of the E_i, we get that $K(x,\ \Omega(f)) = K(x,\ f)$ and so a finitely generated field extension of $K(x)$.

iii) $\Rightarrow$ *i)* Let $F = K(x,\ \Omega(f))$ be a finitely generated field extension of $K(x)$ and $tr.deg._K F = r$. Let $\{x_1,\ x_2, \cdots,\ x_r\}$ be a transcendence basis for F over K such that

$$[F : K(x_1,\ x_2, \cdots,\ x_r)] = k.$$

Then

$$[F^p : K(x_1^p,\ x_2^p, \cdots,\ x_r^p)] = k.$$

Moreover,

$$[K(x_1,\ x_2, \cdots,\ x_r) : K(x_1^p,\ x_2^p, \cdots,\ x_r^p)] = p^r.$$

Since F is invariant under each operator $E \in \Omega$, by Theorem 2.3, $[F : F^p] = p$. Now, by the product rule of degrees, we have $r = 1$. So $tr.deg._K F = r = 1$. Thus F is an algebraic extension of K and hence f is algebraic over K. $\square$

Remark. In contrary with differentiating an equation in which we gain just one equation at each stage, in the splitting process in terms of the "E" operators, we shall have p equations at each stage. Despite of this behavior, the "E" operators have some nice advantages, e.g., "E" behaves well under the Hadamard product operation, the property which enables us to find some nice results (see, for example, [**8,11-14**]. Moreover, the characterizations of Theorem 2.4 are based on the "E" operators.

Now, we shall study another operator which was first introduced by Shikishima-Tsuji et. al. in [**15**]. This operator is again a differential operator but does not involve the splitting process which we discussed above. Then we shall give some more characterizations of algebraic power series over pefect fields of positive characteristics in terms of this operator.

From now on, K will denote a field of characteristic $p > 0$.

Definition 2.5. For $n \in \mathbf{N}$, define the n-th derivative operator

$$\delta_n : K[[x]] \longrightarrow K[[x]]$$

by

$$\delta_n(f) = \sum_{i=n}^{\infty} \binom{i}{n} a_i x^{i-n}.$$

One can easily shows that δ_n has the following properties.

Lemma 2.6. *For any $f, g \in K[[x]], n \in \mathbf{N}$ and $\lambda \in K$, we have*
i) $\delta_n(f+g) = \delta_n(f) + \delta_n(g)$.
ii) $\delta_n(\lambda f) = \lambda \delta_n(f)$.
iii) $\delta_n(fg) = \frac{1}{n!}\sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) f^{(n-k)} g^{(k)}$.
iv) $\delta_n(f^p) = 0$.
v) $\delta_n(g^p f) = g^p \delta_n(f)$. □

Let Δ be the semigroup generated by the identity operator and the δ_n, $n \in \mathbf{N}$ with ordinary composition as multiplication. To each $f \in K((x))$, we associate its orbit

$$\Delta(f) = \{\delta(f) : \delta \in \Delta\}$$

.

Now we give the following characterization of algebraic formal power series in terms of the "δ_n". First, we state the following easy lemma.

Lemma 2.7. *$< \Delta(f) >$, the $K(x)$- linear space spanned by $\Delta(f)$, is the smallest $K(x)$-subspace of $K((x))$ containing f and is invariant under each $\delta_n, n \in \mathbf{N}$.* □

Note that, in the above results, K need not be perfect. However, when dealing with the "E" operators we shall need this condition.

Theorem 2.8. *Let $f \in K((x))$. Then the following are equivalent:*
i) f is algebraic over K.
ii) $dim_{K(x)} < \Delta(f) >$ is finite.

Proof. By equation (1) we have $\sum_{i=0}^{p-1} x^i f_i^p$ and also by Lemma 2.6, we have $\delta_n(f_i^p) = 0$. Hence, using equation (2) one can see that the K-linear space spanned by f, $x^i\delta(f)$, $i \in \{1, 2, \cdots, p-1\}$ and the K-linear space spanned by f, $x^i E(f), i \in \{1, 2, \cdots, p-1\}$ are the same. Hence the proof follows by Theorem 5.3 of [**14**]. □

Corollary 2.9. *Let $f \in K((x))$. Then the following are equivalent:*
i) f is algebraic over K.
ii) $K(x, \Delta(f))$ is a finitely generated extension of $K(x)$. □

Example 2.10. *i*) Let $\sum_{n=0}^{\infty} x^{p^n}$. Then $f(x) = x + f(x^p) = x + f(x)^p$. So $f' = 1$ and hence $< \Delta(f) >=< f >$ is a 1-dimensional $K(x)$-linear space.

ii) Let p be a prime number and $S_p(n)$ be the sum of the digits of n in its p-adic expansion. Suppose that $f(x) = \sum_{n=0}^{\infty} S_p(n)x^n$. Then one can show that

$$f(x) = \frac{x^p - 1}{x - 1} f(x)^p + \frac{x}{(x-1)^2}$$

(see, [**4**]). Thus $f'(x) \in K(x)$. Therefore, $< \Delta(f) >=< f >$ is a 1-dimensional $K(x)$-linear space.

2. δ-Algebraic Formal Power Series

Let L be a field of characteristic zero. The power series $f \in L[[x]]$ is called differentially algebraic (D- algebraic, for short), if it satisfies a differential equation of the form $P(x, y, y', ...) = 0$, where P is a non-trivial polynomial. Equivalently, f is called D-algebraic if $tr.deg._{L(x)}L(x, f, f', ..., f^{(n)}, ...)$ is finite, where $tr.deg._B A$ means the transcendence degree of A over B. A function which is not D-algebraic is called transcendentally transcendental (TT, for short).

This notion is only defined over fields of characteristic zero and is not so significant over fields of characteristic $p > 0$, since $f^{(p)} = 0$.

In [**12**] we gave an analogue of the concept of a D-algebraic formal power series over *perfect* fields of positive characteristic in terms of the "E" operators. Now we shall deal with the "δ" operator over arbitrary fields of positive characteristic.

Example 3.1. *i*) The series $\sum_{n=0}^{\infty} n!x^n$ is D-algebraic, since $x^2 f'(x) = (x-1)f(x) + 1 = 0$.

ii) The series $\sum_{n=0}^{\infty} x^{2^n}$ is TT (see [**6**]).

In [**12**] we defined an E-algebraic function over $K(x)$ as follows.

Definition 3.2. Suppose that $f \in K((x))$. We say that f is an E-algebraic function (over $K(x)$) if $tr.deg._{K(x)}K(x, \Omega(f)) < \infty$.

We denoted by Γ_K, the set of all E-algebraic functions over K. (See [**12**] for some nice properties of Γ_K.)

Note that if $f \in K((x))$ is algebraic, then

$$tr.deg._{K(x)}K(x, \Omega(f)) = 0.$$

So every algebraic function is E-algebraic.

Now we shall give an analogue of the concept of a D-algebraic formal power series over an arbitrary field of positive characteristic in terms of the "differential" operator δ.

Note. From now on K will denote a field of characteristic $p > 0$ (except when we are dealing with the "E" operators, where the restriction of being perfect is necessary).

Definition 3.3. We say that $f \in K[[x]]$ is δ- algebraic over K if

$$tr.deg._{K(x)}K(x, f, \delta_1 f, \delta_2 f, ..., \delta_n f, ...) < \infty.$$

If f is not δ-algebraic, we say that f is hypertranscendental, which was first used by Shikishima-Tsuji and et. al. in [**15**]. In other words, f is hypertranscendental if for any $n \in \mathbf{N}$, $f, \delta_1 f, \delta_2 f, ..., \delta_n f$ are algebraically independent over $K(x)$.

Notation. We shall denote by Δ_K the set of all δ-algebraic formal power series.

Since any algebraic formal power series over K is obviously δ-algebraic, we have $\overline{K(x)} \subseteq \Delta_K$. We shall show that the above inclusion is strict

Example 3.4. Let $K = \mathbf{F}_2$ and $\alpha \in \mathbf{Z}_2$ be a 2-adic integer. Let $f_\alpha = (1+x)^\alpha \in \mathbf{F}_2[[x]]$. If α is rational, then f_α is algebraic ([**7**]). Hence $tr.deg._{\mathbf{F}_2(x)}\mathbf{F}_2(x, \Omega(f_\alpha)) = 0$. However, if α is not rational, then by [**7**], f_α is not algebraic over $\mathbf{F}_2$. However, we show that $f_\alpha \in \Delta_K$ and hence the set Δ_K strictly contains the set of all algebraic functions.

Let $\alpha = \sum_{i=0}^{\infty} \alpha_i 2^i$ be the 2-adic expansion of α. Then

$$\delta_1 f = \alpha(1+x)^{\alpha-1} = (\sum_{i=0}^{\infty} \alpha_i 2^i)(1+x)^{\alpha-1} = \frac{\alpha_0(1+x)^\alpha}{(1+x)} = \frac{\alpha_0 f_\alpha}{(1+x)} \in \mathbf{F}_2(x, f_\alpha).$$

Similarly, for any $n \in \mathbf{N}$ we have

$$\delta_n f_\alpha \in \mathbf{F}_2(x, f_\alpha),$$

Therefore,

$$tr.deg._{\mathbf{F}_2(x)}\mathbf{F}_2(x, \Delta(f_\alpha)) = 1,$$

since f_α is not algebraic. Hence f is δ-algebraic. Thus we have shown that

$$\overline{K(x)} \subsetneqq \Delta_K$$

.

Now we show that the set Δ_K is a field.

Theorem 3.5. *The set Δ_K with ordinary addition and multiplication of series is an infinite field of characteristic $p > 0$.*

Proof. Let $f,\ g \in \Delta_K$ and suppose that

$$tr.deg._{K(x)}K(x,\ f,\ \delta_1 f,\ \cdots, \delta_n f,\ \cdots) = r$$

and

$$tr.deg._{K(x)}K(x,\ g,\ \delta_1 g,\ \cdots,\delta_n g,\ \cdots) = s.$$

By Lemma 2.6, $\delta_n(f+g) = \delta_n(f) + \delta_n(g)$ and hence

$$K(x,\ f+g,\ \delta_1(f+g),\ \cdots,\delta_n(f+g),\ \cdots) \subseteq K(x,\ f,\ g,\ \delta_1 f,\ \delta_1 g,\ \delta_n f, \delta_n g\ ,\cdots).$$

So

$$tr.deg._{K(x)}K(x,\ f+g,\ \delta_1(f+g),\ \cdots,\delta_n(f+g),\ \cdots) \leq r+s.$$

Similarly, since

$$\delta_n(fg) = \frac{(fg)^{(n)}}{n!} \in K(x,\ f,\ g,\ \delta_1 f,\ \delta_1 g,\ \cdots,\delta_n f,\ \delta_n g,\ \cdots)$$

we have $fg \in \Delta_K$. Also for a nonzero $f \in \Delta_K$, $f^{-1} \in \Delta_K$, since $\delta_n(f^{-1}) = \frac{(f^{-1})^{(n)}}{n!}$. Therefore, Δ_K is an infinite field of characteristic $p > 0$. □

Shikishima-Tsuiji et. al. in [**15**] have proved the following theorm.

Theorem 3.6. *Let $f = \sum_{i=0}^{\infty} a_i x^{m^i} \in K[[x]]$ with nonzero coefficients and $m_0 < m_1 < m_2 < \cdots$ be natural numbers. If f satisfies the following conditions, then f is hypertranscendental over K.*

For every $e, s \in \mathbf{N}$, there exist natural numbers $i_0 < i_1 < i_2, \cdots$ such that $m_{i_j} \equiv s (mod\ p^e)$ and $\lim_{j\to\infty} \frac{m_{i_j}}{m_{i_j-1}} = \infty$. □

By theorem 3.6, a large class of hypertranscendental formal power series are determined.

Example 3.7. The series $f = \sum_{i=0}^{\infty} x^{p^{i^2}+i}$, $g = \sum_{i=0}^{\infty} x^{i^i p+i}$, and $h = \sum_{i=0}^{\infty} x^{i!+i}$ are hypertranscendental by Theorem 3.6. Hence f, g and h are not in Δ_K. Therefore, Δ_K is a proper subfield of $K((x))$. (See also the complicated Example 2.5 of [**12**], to show that $\Gamma_K \subset K((x))$.)

Thus, we have shown that

$$\overline{K(x)} \subset \Delta_K \subset K((x)).$$

Now, we show that over a perfect field of positive characteristic, a formal power series f is E-algebraic if and only if f is δ-algebraic.

Remark 3.8. One can show that each E_j is a derivative. In fact, for $f \in K((x))$, $E_j(f) = -\frac{d^{p-1}(x^{p-j-1}f)}{dx^{p-1}}$ (see [**14**]).

Theorem 3.9. *Suppose that K is a perfect field of characteristic $p > 0$ and $f \in K((x))$. Then f is E-algebraic if and only if f is δ-algebraic.*

Proof. Let $F = K(x, \Omega(f))$. Then by Remark 3.8 and equation (2) we have

$$F^p \subseteq K(x,\ \Delta(f)) \subseteq F.$$

Also by Theorem 2.3, we have $[F : F^p] = p$. Now, f is E-algebraic, if and only if, $tr.deg._{K(x)}K(x, \Omega(f)) < \infty$, if and only if, $tr.deg._{K(x)}K(x, \Delta(f)) < \infty$, if and only if, f is E-algebraic. $\square$

As we mentioned above, the "E" operator have some nice advantages, e.g., "E" behaves well under the Hadamard product operation, but the operation δ does not. In [**14**], we showed that Γ_K has the following properties:

i) Γ_K *with ordinary addition and multiplication of series is an infinite field of positive characteristic.*

ii) Γ_K *is closed under the* E *and the* D *(derivative) operators.*

iii) $K(x) \subseteq \Gamma_K$ *is a separable extension.*

iv) $tr.deg._{K(x)}\Gamma_K = \infty$.

v) Γ_K *is algebraically closed in* $K((x))$.

vi) Γ_K *is not closed under the Hadamard product operation.*

Now, when K is a perfect field of positive characteristic, Theorem 3.9 implies that Δ_K has the above properties too.

Remark. Although Theorem 3.6 introduces a large class of hypertranscendental formal power series, there are still a lot of formal power series which can not be measured by even the results which we discussed so far. For example, we do not know if the series $\sum_{n=0}^{\infty} x^{n^3}$ is hypertranscendental over K, even it is unknown to be TT over $L(x)$ (see [**6**]).

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