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THE ORDER GRAPHS OF GROUPS

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Abstract. Let G be a group. The order graph of G is the (undirected) graph $\Gamma(G)$, those whose vertices are non-trivial subgroups of G and two distinct vertices H and K are adjacent if and only if either $o(H)|o(K)$ or $o(K)|o(H)$. In this paper, we investigate the interplay between the group-theoretic properties of G and the graph-theoretic properties of $\Gamma(G)$. For a finite group G, we show that $\Gamma(G)$ is a connected graph with diameter at most two, and $\Gamma(G)$ is a complete graph if and only if G is a p-group for some prime number p. Furthermore, it is shown that $\Gamma(G) = K_5$ if and only if either $G \cong C_{p^5}, C_3 \times C_3$, $C_2 \times C_4$ or $G \cong Q_8$.

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1. Introduction

We will begin by introducing the necessary notations and terminology that we will use later. Throughout this paper, G is a group and $o(G)$ denotes the order of G. For $n \geq 0$, C_n denotes a cyclic group with order n. For a graph Γ, $V(\Gamma)$ and $E(\Gamma)$ denote the sets of vertices and edges of Γ respectively, and two elements x, $y \in V(\Gamma)$ are said to be adjacent and denoted $x - y$, if there exists an edge between them. A path between two elements $x, y \in V(\Gamma)$ is an ordered sequence of distinct vertices of Γ such as $x = a_1$, a_2, $\cdots$, $y = a_n$, where a_{i-1} is adjacent to a_i, for all i with $2 \leq i \leq n$. The length of a path between x and y is number of edges crossed to get from x to y in the path. The distance between x, $y \in V(\Gamma)$, denoted $d(x, y)$, is the length of a shortest path between x and y, if such a path exists otherwise, $d(x, y) = \infty$. For the purposes of this paper, we define $d(x, x) = 0$. The diameter of a graph Γ is $\mathrm{diam}(\Gamma) = \max\{d(x, y) : x, y \in V(\Gamma)\}$. A cycle is a non trivial path in a graph from a vertex to itself. The girth of Γ, denote by $\mathrm{gr}(\Gamma)$ is defined as the length of the shortest cycle in Γ. $\mathrm{gr}(\Gamma) = \infty$ if Γ contains no cycles. A complete graph, any two vertices are adjacent, with n vertices denotes by K_n. A graph Γ is called complete bipartite if there exist disjoint subsets A, B of $V(\Gamma)$ such that $A \cup B = V(\Gamma)$ and $x - y$ for any $x \in A$ and $y \in B$. Finite complete bipartite graphs are denoted by $K_{m,n}$, where $|A| = m$ and $|B| = n$, a graph is a star graph if $\Gamma = K_{1,n}$. For notations and terminologies not given in this article, the reader is referred to [2], [7].

Definition 1.1. Let G be a group. The order graph of G is the (undirected) graph $\Gamma(G)$, those whose vertices are non-trivial subgroups of G and two distinct vertices H and K are adjacent if and only if either $o(H)|o(K)$ or $o(K)|o(H)$.

The main object of this paper is to study the interplay of group-theoretic properties of G with graph-theoretic properties of $\Gamma(G)$. In section 2, for a finite group G, it is shown that $\Gamma(G)$ is always connected with $\mathrm{diam}\,\Gamma(G) \leq 2$, and $\Gamma(G)$ is a complete graph if and only if G is a p-group for some prime number p. Moreover, $\Gamma(G)$ is a star graph if and only if $G = C_{pq}$, where p, q are prime numbers. In section 3, we study a subgraph of $\Gamma(G)$ denotes by $\Gamma^*(G)$ those whose vertices are non-trivial proper subgroups of G, and for a finite abelian group G we show that $\Gamma^*(G)$ is connected but is not a star graph.

2. Properties of orders graph

We start with some examples.

Example 2.1. (i) The diagram in Figure 1 illustrates the order graphs for Klein Vieregruppe $V_4 = \{e, a, b, c\}$ and $\mathbb{Z}_{30}$. For $G = V_4$ we have $V(\Gamma(G)) = \{H_1 = \{e, a\}, H_2 = \{e, b\}, H_3 = \{e, c\}, G\}$ so that $\Gamma(G) = K_4$. For $G = \mathbb{Z}_{30}$ we have $V(\Gamma(G)) = \{H'_1 =< 2 >, H'_2 =< 3 >, H'_3 =< 5 >, H'_4 =< 6 >, H'_5 =< 10 >, H'_6 =< 15 >, G\}$.

FIGURE 1. $\Gamma(V_4)$, $\Gamma(\mathbb{Z}_{30})$

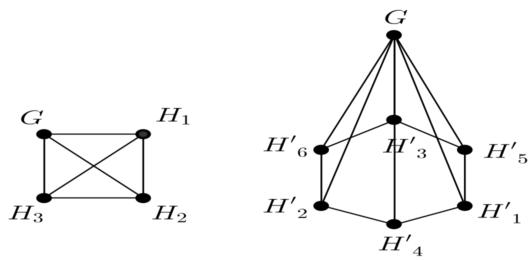


FIGURE 2. $\Gamma(G = \mathbb{Z}_6)$, $\Gamma(G' = \mathbb{Z}_4)$, $\Gamma(G'' = \mathbb{Z}_{27})$

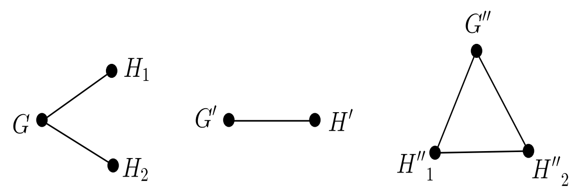
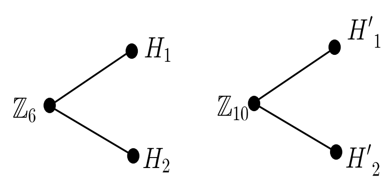


FIGURE 3. $\Gamma(\mathbb{Z}_6)$, $\Gamma(\mathbb{Z}_{10})$



(ii) The diagram in Figure 2 illustrates the order graphs for $G = \mathbb{Z}_6, G' = \mathbb{Z}_4$ and $G'' = \mathbb{Z}_{27}$, where $V(\Gamma(G)) = \{H_1 =< 2 >,\ H_2 =< 3 >,\ G\}$, $V(\Gamma(G')) = \{H'_1 =< 2 >,\ G'\}$ and $V(\Gamma(G'')) = \{H''_1 =< 3 >,\ H''_2 =< 9 >,\ G''\}$.

Assume that G and G' are two finite group. If $G \cong G'$, then it is obvious that $\Gamma(G) \cong \Gamma(G')$. The following example shows that the converse is not true.

Example 2.2. The diagram in Figure 3 illustrates the order graphs for $\mathbb{Z}_6, \mathbb{Z}_{10}$. We have $V(\Gamma(\mathbb{Z}_6)) = \{H_1 =< 2 >, H_2 =< 3 >, \mathbb{Z}_6\}$, $V(\Gamma(\mathbb{Z}_{10})) = \{H'_1 =< 2 >, H'_2 =< 5 >, \mathbb{Z}_{10}\}$. It is clear that $\mathbb{Z}_6 \ncong \mathbb{Z}_{10}$ but $\Gamma(\mathbb{Z}_6) \cong \Gamma(\mathbb{Z}_{10})$.

In the following we show that $\Gamma(G)$ is always connected and has small diameter and girth.

Lemma 2.3. *Let G be a finite group. Then there is a vertex of $\Gamma(G)$ which is adjacent to every other vertices.*

Proof. Let H be a subgroup of G, by Lagrange's Theorem [6, Corollary 4.6], $o(H)|o(G)$ so G is a vertex of $\Gamma(G)$ which is adjacent to every other vertices. □

Corollary 2.4. *Let G be a finite group. Then $\Gamma(G) = \emptyset$ if and only if G is a simple group. Furthermore, $\Gamma(G) \neq \emptyset$ if and only if $\Gamma(G)$ has not a single vertex.*

Corollary 2.5. *Let G be a finite group with $|V(\Gamma(G))| = n$. Then the following statements are true:*

(i) *The maximum degree of $\Gamma(G)$, $\Delta(\Gamma(G))$, is $n-1$.*
(ii) *If $n \geq 4$, then $\Gamma(G)$ is not a path and an n-gon.*

Proof. (i) In view of Lemma 2.3 we have $\deg(G) = n-1$. Thus $\Delta(\Gamma(G)) = n-1$.
(ii) If $n \geq 4$, then $\deg(G) \geq 3$ which shows $\Gamma(G)$ is not a path and an n-gon. □

Corollary 2.6. *Let G be a finite group. Then $\Gamma(G)$ is a regular graph if and only if it is complete.*

Proof. Let $\Gamma(G)$ be a graph with n vertices. Then $\Delta(\Gamma(G)) = n-1$. Hence, $\Gamma(G)$ is regular if the degree of any vertices is equal to $n-1$ which implies that $\Gamma(G)$ is complete. □

Theorem 2.7. *Let G be a finite group. Then $\Gamma(G)$ is a connected graph with $\operatorname{diam}\Gamma(G) \leq 2$. Furthermore, if $\Gamma(G)$ contains a cycle, then $\operatorname{gr}\Gamma(G) = 3$.*

Proof. In view of Lemma 2.3, $\Gamma(G)$ has a vertex adjacent to every other vertices. So that $\Gamma(G)$ is connected and then $\operatorname{diam}\Gamma(G) \leq 2$. Now, assume that $\Gamma(G)$ contains a cycle. If $\Gamma(G)$ is an n-gon, then using Lemma 2.3 again we have $n = 3$. Let $\Gamma(G)$ consist an n-gon, where $n > 3$. It is known that $\Gamma(G)$ has a vertex that is adjacent to every other vertices so there are two cases. Case 1. If this vertex is a vertex of n-gon, then there are $n-2$ triangles thus $\operatorname{gr}\Gamma(G) = 3$. Case 2. If this vertex is not a vertex of the n-gon, then by connecting this vertex to the vertices of n-gon we have n triangles, thus $\operatorname{gr}\Gamma(G) = 3$. □

In the following we determine when $\Gamma(G)$ is a complete graph. By using definition $\Gamma(G)$ is complete if and only if $o(H)|o(K)$ or $o(K)|o(H)$ for all distinct subgroup H and K of G.

Lemma 2.8. *Let G be a finite group. Then $\Gamma(G)$ is complete if and only if G is a p-group, where p is a prime number.*

Proof. $\Rightarrow$ Let $\Gamma(G)$ be a complete graph and $o(G) = pqm$, where p, q are prime numbers and m is a natural number. By Cauchy's Theorem [6, Theorem 5.2], there are subgroups H and K of G with $o(H) = p$ and $o(K) = q$. By assumption we must have $p|q$, which implies that $p = q$. Thus G is a p-group.

$\Leftarrow$ Let G be a p-group. Then the orders of all subgroups of G is a power of p so that all vertices of $\Gamma(G)$ are adjacent with each others. Thus $\Gamma(G)$ is complete. □

In the following we determine all groups for which the order graph is complete with at most five vertices.

Theorem 2.9. *Let G be a finite group. Then $\Gamma(G) = K_2$ if and only if $G = C_{p^2}$, where p is a prime number.*

Proof. $\Rightarrow$ Let $\Gamma(G) = K_2$. Then by Lemma 2.8 it follows that G is a p-group for some prime number p . It is clear that $o(G) \leq p^2$ by Sylow Theorem [6, Theorem 5.7]. For $o(G) = p$ we have $\Gamma(G) = \emptyset$. So assume that $o(G) = p^2$. In this case G is an abelian group and so $G \cong C_{p^2}$ or $C_p \times C_p$. If $G \cong C_p \times C_p$, then it has at least three subgroups with orders p, which shows that $\Gamma(G) \neq K_2$. Hence, G is a cyclic group with order p^2 that is $G \cong C_{p^2}$.

$\Leftarrow$ It is obvious by Lemma 2.8. □

Theorem 2.10. *Let G be a finite group. Then $\Gamma(G) = K_3$ if and only if $G = C_{p^3}$, where p is a prime number.*

Proof. Let $\Gamma(G) = K_3$. Then it follows that G is a p-group with $o(G) \leq p^3$. By the proof of Theorem 2.9 it follows that $o(G) \neq p, p^2$. So that $o(G) = p^3$. If G is an abelian group, then $G \cong C_{p^3}$ or $C_{p^2} \times C_p$ or $C_p \times C_p \times C_p$. It is obvious that $\Gamma(C_p \times C_p \times C_p) \neq K_3$. Also, if $G \cong C_{p^2} \times C_p$, then G has at least three subgroups with orders p and p^2 thus $\Gamma(C_{p^2} \times C_p) \neq K_3$. Hence, G is a cyclic group with order p^3. Now, assume that G is a non-abelian group. By hypothesis G has only one subgroup with order p and p^2 so that $G \cong Q_8$ by [3, Exercise 21, page 187], since any subgroup of G is normal. But, $\Gamma(Q_8) \neq K_3$ since Q_8 has three subgroups with order four. □

Theorem 2.11. *Let G be a finite group. Then $\Gamma(G) = K_4$ if and only if either $G = C_2 \times C_2$ or $G = C_{p^4}$, where p is a prime number.*

Proof. Let $\Gamma(G) = K_4$. Then it follows that G is a p-group with $p^2 \leq o(G) \leq p^4$. If $o(G) = p^2$, then by the proof of Theorem 2.9 we have $G \cong C_p \times C_p$. If $p \geq 3$, then G has at least four subgroups with orders p such as $H_1 =< (1, 0) >$, $H_2 =< (0, 1) >$, $H_3 =< (1, 1) >$, $H_4 =< (1, 2) >$ thus $\Gamma(C_p \times C_p) \neq K_4$. Hence, $p = 2$ and in this case Example 2.1(i) shows that $\Gamma(C_2 \times C_2) = K_4$. A same argument shows that $G \ncong C_p \times C_p \times C_p, C_p \times C_{p^2}$. Moreover, as above the non-abelian groups Q_8, D_8 can not have K_4 as its order graph. So assume that G is a non-abelian group with order p^3, where p is a odd prime number. By [3, Exercise 29, page 200], G has $p+1$ maximal subgroups so that it has at least six subgroups. Thus $\Gamma(G) \neq K_4$. Now, assume that G is a group with order p^4 and $\Gamma(G) = K_4$. Thus G must have only one maximal subgroup so it follows that G is a cyclic group, see Corollary 1.2, page 173 in [4]. □

Theorem 2.12. *Let G be a finite group. Then $\Gamma(G) = K_5$ if and only if either $G = C_3 \times C_3$, $C_2 \times C_4$, Q_8 or $G = C_{p^5}$, where p is a prime number.*

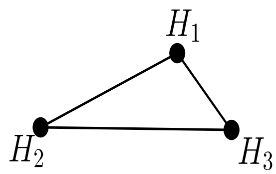
Proof. Let $\Gamma(G) = K_5$. Then G is a p group with $p^2 \leq o(G) \leq p^5$. If $o(G) = p^2$, then by the proof of Theorems 2.9, 2.11 , $G \ncong C_{p^2}, C_2 \times C_2$ also, it is easy to see that for $p \geq 5$ it has at least five subgroups with orders p such as $H_1 =< (1,0) >$, $H_2 =< (0,1) >$, $H_3 =< (1,1) >$, $H_4 =< (1,2) >$, $H_5 =< (1,3) >$ which implies that $\Gamma(C_p \times C_p) \neq K_5$. On the other hand, $\Gamma(C_3 \times C_3) = K_5$. Now, assume that $o(G) = p^3$. If G is an abelian group it is easy to show that $\Gamma(C_2 \times C_4) = K_5$ and $G \ncong C_p \times C_p \times C_p, C_p \times C_{p^2}, C_{p^3}$. Let G be a non-abelian group with order p^3. Then we have $\Gamma(Q_8) = K_5$ and $\Gamma(D_8) \neq K_5$. So assume that G is a non-abelian group with order p^3, where p is a odd prime number. By [3, Exercise 29, page 200] it has $p+1$ maximal subgroups so that it has at least six subgroups. Thus $\Gamma(G) \neq K_5$. As above with a same argument one can show that G can not be an abelian group with order p^4. Let G be a non-abelian group with order p^4. Then by assumption G has only four proper subgroup. If it has only one maximal subgroup, then $G \cong C_{p^4}$ it follows by [4, Corollary 1.2] which is a contradiction. Otherwise, G has only one subgroup of order p. In this case Proposition 1.3 in [1] shows that G is a generalized Quaternion group with order $o(G) = 16$ or 32. But by the gap software one can see that Q_{16} has eight proper subgroups and Q_{32} has eighteen proper subgroups. Hence, $G \ncong Q_{16}$ and $G \ncong Q_{32}$. If G is a group with order p^5 and $\Gamma(G) = K_5$, then G must have only one maximal subgroup so it follows that G is a cyclic group, see [4, Corollary 1.2].

□

Let G be a finite group. Then the order graph $\Gamma(G)$ is a complete bipartite graph whenever $\Gamma(G)$ is a star graph, since it has a vertex is adjacent with other vertices. In the sequel, we determine when $\Gamma(G)$ is a star graph.

Lemma 2.13. *Let G be a finite group. Then $\Gamma(G) = K_{1n} (n \geq 2)$ if and only if any subgroups of G has prime order and G has only one subgroup with any orders. Furthermore, $o(G) = p_1 \cdots p_n$ for some prime numbers $p_1, \cdots, p_n$.*

FIGURE 4. $\Gamma^*(V_4)$



Proof. $\Rightarrow$ By assumption $|V(\Gamma(G))| \geq 3$ and in view of Lemma 2.3 we have $\deg(G) = |V(\Gamma(G))| - 1$. Let H be a subgroup of G with $o(H) = p_1^{\alpha_1} \ldots p_k^{\alpha_k}$. By Cauchy's Theorem the degree of H is at least k which implies that $k = 1$ and $\alpha_1 = 1$. Moreover, G has precisely one subgroup with order p_1.

$\Rightarrow$ It is obvious. □

Theorem 2.14. *Let G be a finite group. Then $\Gamma(G)$ is a star graph if and only if $G = C_{pq}$, where p, q are prime numbers.*

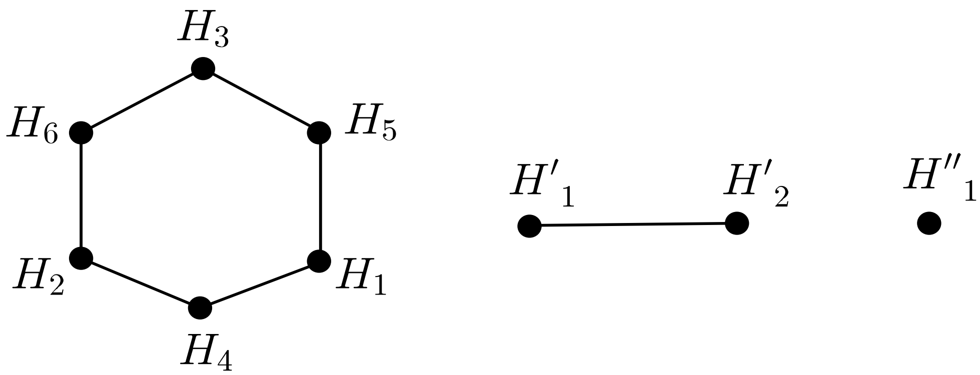
Proof. In view of Lemma 2.13 we have $\Gamma(G)$ is a star graph if and only if any subgroups of G has prime order and also G has only one subgroup with any orders. Furthermore, $o(G) = p_1 \cdots p_k$ for some prime numbers $p_1,\cdots,p_k$. For $k = 2$, if $p < q$ and $p \not| q - 1$, then G is a cyclic group and we are done, otherwise G has $3 \leq 1 + tp$ subgroups with orders p which shows that $\Gamma(G) \neq K_{12}$. Let $k \geq 3$ and $p_1 = 2$. Then $o(G) = 2n$, where n is an odd number. Thus G has a subgroup H with $o(H) = n$, by [5, Theorem 1.35]. Hence, the order of H is not a prime number that contradict with Lemma 2.13. Now, suppose that for all i with $1 \leq i \leq k$, p_i is an odd prime number. Then G is not a simple group, see [8, 1.12]. So that G has a normal subgroup such as H. Assume that L is a subgroup of G with order p_1. If $HL = G$, then $o(H) = p_2 \cdots p_k$ which contradict with Lemma 2.13. Thus $HL \neq G$. In this case HL is a subgroup of G with order multiplied more than one prime number which is a contradiction. Therefore, $k = 2$ and the proof is completed. □

3. A subgraph of a order graph

Let G be a group. By $\Gamma^*(G)$ we mean a subgraph of $\Gamma(G)$ with $V(\Gamma^*(G)) = V(\Gamma(G)) - \{G\}$. So that $\Gamma^*(G)$ is a simple graph with vertices $V(\Gamma^*(G)) = \{H : H \text{is a non-trivial proper subgroup of G }\}$ and two distinct vertices H and K are adjacent if and only if either $o(H)|o(K)$ or $o(K)|o(H)$.

Example 3.1. The diagram in Figure 4 illustrates $\Gamma^*(G = V_4)$, where $V(\Gamma^*(G)) = \{H_1 = \{e, a\}$, $H_2 = \{e, b\}$, $H_3 = \{e, c\}\}$.

FIGURE 5. $\Gamma^*(\mathbb{Z}_{30}), \Gamma^*(\mathbb{Z}_{27}), \Gamma^*(\mathbb{Z}_9)$



Example 3.2. The diagram in Figure 5 illustrates $\Gamma^*(G)$ for $G = \mathbb{Z}_{30}$, $G = \mathbb{Z}_{27}$ and $G = \mathbb{Z}_9$, where $V(\Gamma^*(\mathbb{Z}_{30})) = \{H_1 =< 2 >,\ H_2 =< 3 >,\ H_3 =< 5 >,\ H_4 =< 6 >,\ H_5 =< 10 >,\ H_6 =< 15 >\}$, $V(\Gamma^*(\mathbb{Z}_{27})) = \{H'_1 =< 3 >,\ H'_2 =< 9 >\}$ and $V(\Gamma^*(\mathbb{Z}_9)) = \{H''_1 =< 3 >\}$.

Theorem 3.3. *Let G be a finite abelian group. Then $\Gamma^*(G)$ is a connected graph with diameter at most four.*

Proof. Let H and K be two proper subgroups of G. We have to show that there is a path from H to K in $\Gamma^*(G)$. If $o(H)o(K) > o(G)$, then $o(H \cap K) = m > 1$ and $H - H \cap K - K$ is a path from H to K. Let $o(G) > o(H)o(K)$. Then HK is a proper subgroup of G and $H - HK - H$ is a path from H to K. Now, let $o(G) = o(H)o(K)$. If $o(H \cap K) = m > 1$, then we are done. Suppose that $o(H \cap K) = 1$. There are prime numbers p, q such that $p|o(H)$ and $q|o(K)$. If $o(G) = pq$, then $\Gamma^*(G) = \emptyset$ and we are done. Assume that $o(G) > pq$. It is clear that G has subgroups such as L, L' and L'' with $o(L) = p$, $o(L') = q$ and $o(L'') = pq$. Hence, $H - L - L'' - L' - K$ is a path from H to K. If $o(H) = o(L) = p$, then $H - L'' - L' - K$ is a path from H to K, so $\Gamma^*(G)$ is a connected graph and $\operatorname{diam}\Gamma^*(G) \leq 4$. □

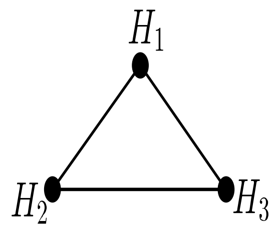
Corollary 3.4. *Let G be a group and H be a subgroup of G with order $o(H) = p^n m$, where p is a prime number, $n \geq 2$ and $m > 1$. Then* $\mathrm{gr}(\Gamma^*(G)) = 3$.

Proof. By Sylow Theorems H has subgroups $H_1,\cdots,H_n$ with orders $o(H_i) = p^i$ for all i with $1 \leq i \leq n$. If $n \geq 3$, then $\Gamma^*(G)$ has a complete subgraph with n vertices so that $\mathrm{gr}(\Gamma^*(G)) = 3$. Otherwise, G has subgroups H_1, H_2 with $o(H_1) = p$, $o(H_2) = p^2$ thus $H_1 - H_2 - H - H_1$ is a triangle and $\mathrm{gr}(\Gamma^*(G)) = 3$.

□

Theorem 3.5. *Let G be a finite abelian group such that $\Gamma^*(G) \neq \emptyset$. Then $\Gamma^*(G)$ is a complete graph if and only if G is a p-group.*

FIGURE 6. $\Gamma^*(S_3)$



Proof. Suppose that $o(G) = pqm$ for some prime numbers p, q and $m \geq 1$. Thus G has at least three proper subgroups with orders $o(H) = p, o(K) = q$ and $o(L) = pq$ which show that H, K are not single vertices and H, K are not adjacent to each other. That is a contradiction with assumption that $\Gamma^*(G)$ is a complete graph. So that either $m = 1$ or $p = q$. If $m = 1$ and $p \neq q$, then $G \cong C_{pq}$ and so $\Gamma^*(G) = \emptyset$ that contradict with assumption. Hence, $p = q$, now by continuing this way one can shows that G is a p-group. □

The following example shows that the Theorem 3.5 is not true for non-abelian groups. It is well-known that S_3 is a non-abelian group with order six, but $\Gamma^*(S_3)$ is a complete graph.

Example 3.6. Figure 6 displays $\Gamma^*(S_3)$. Note that $V(\Gamma^*(S_3)) = \{H_1 =< (1\ 2) >$, $H_2 =< (1\ 3) >$, $H_3 =< (2\ 3) >$, $H_4 =< (1\ 2\ 3) >\}$ and $\Gamma^*(S_3) = K_3$.

Theorem 3.7. *Let G be a finite group with $o(G) = pq$ $(p < q)$ where p, q are distinct primes numbers, then either $\Gamma^*(G) = \emptyset$ or $\Gamma^*(G) = K_n$.*

Proof. It is well known that G has only one subgroup of order q. If $p \not| q - 1$, then G has only one subgroup of order p. So that G is a cyclic group and $\Gamma^*(G) = \emptyset$. Now, assume that $p|q - 1$. In this case G has $1 + tp$ subgroups of order p and so $\Gamma^*(G)$ is a complete graph. □

Theorem 3.8. *Let G be a finite group. Then $\Gamma^*(G)$ is not a star graph.*

Proof. Let G be a finite group. We have to show that $\Gamma^*(G) \neq K_{1n}$ $(n \geq 2)$. If H is the center vertex of $\Gamma^*(G)$, then the proof of Corollary 3.4 shows that $o(H) = p_1 \cdots p_n$, where $p_1, \cdots, p_n$ are prime numbers. Case1. Assume that $n \geq 3$. If $p_1 = 2$, then $o(H) = 2p_2 \cdots p_n$. Thus H has a subgroup K with $o(K) = p_2 \cdots p_n$ by [5, Theorem 1.35]. Hence, $\Gamma^*(G)$ is not a star graph. Now, suppose that p_i is an odd prime number, for all $1 \leq i \leq k$. Then H is not a simple group, see [8, 1.12]. So that H has a normal subgroup such as K. Assume that L is a subgroup of H with order p_1. Thus KL is a subgroup of H. If $KL = H$, then the order of K is a multiplication of more than one prime number this is a contradiction. If $KL \neq H$, then KL is a subgroup of H and the order of KL is not a prime number that is a contradiction. So that $\Gamma^*(G)$ is not a star graph. Case2. Let $n = 2$. Then $o(H) = p_1p_2$. Since $o(H) < o(G)$ so there exists $m \geq 2$ such that $o(G) = p_1p_2m$. If $q^2|m$ for some prime number $q \neq p_1, p_2$, then H is not the center vertex of $\Gamma^*(G)$. So assume that $o(G) = p_1p_2q$, where $p_1 < p_2 < q$. Then G has a subgroup with order p_1q which shows that $\Gamma^*(G)$ is not a star graph, the proof is completed. □

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