THE ORDER GRAPHS OF GROUPS

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Abstract. Let $G$ be a group. The order graph of $G$ is the (undirected) graph $\Gamma(G)$, those whose vertices are non-trivial subgroups of $G$ and two distinct vertices $H$ and $K$ are adjacent if and only if either $o(H)|o(K)$ or $o(K)|o(H)$. In this paper, we investigate the interplay between the group-theoretic properties of $G$ and the graph-theoretic properties of $\Gamma(G)$. For a finite group $G$, we show that $\Gamma(G)$ is a connected graph with diameter at most two, and $\Gamma(G)$ is a complete graph if and only if $G$ is a $p$-group for some prime number $p$. Furthermore, it is shown that $\Gamma(G) = K_5$ if and only if either $G \cong C_p^5, C_3 \times C_3, C_2 \times C_4$ or $G \cong Q_8$.

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1. Introduction

We will begin by introducing the necessary notations and terminology that we will use later. Throughout this paper, $G$ is a group and $o(G)$ denotes the order of $G$. For $n \geq 0$, $C_n$ denotes a cyclic group with order $n$. For a graph $\Gamma$, $V(\Gamma)$ and $E(\Gamma)$ denote the sets of vertices and edges of $\Gamma$ respectively, and two elements $x, y \in V(\Gamma)$ are said to be adjacent and denoted $x - y$, if there exists an edge between them. A path between two elements $x, y \in V(\Gamma)$ is an ordered sequence of distinct vertices of $\Gamma$ such as $x = a_1, a_2, \ldots, y = a_n$, where $a_{i-1}$ is adjacent to $a_i$, for all $i$ with $2 \leq i \leq n$. The length of a path between $x$ and $y$ is number of edges crossed to get from $x$ to $y$ in the path. The distance between $x, y \in V(\Gamma)$, denoted $d(x, y)$, is the length of a shortest path between $x$ and $y$, if such a path exists otherwise, $d(x, y) = \infty$. For the purposes of this paper, we define $d(x, x) = 0$. The diameter of a graph $\Gamma$ is $\text{diam}(\Gamma) = \max\{d(x, y) : x, y \in V(\Gamma)\}$. A cycle is a non trivial path in a graph from a vertex to itself. The girth of $\Gamma$, denote by $\text{gr}(\Gamma)$ is defined as the length of the shortest cycle in $\Gamma$. $\text{gr}(\Gamma) = \infty$ if $\Gamma$ contains no cycles. A complete graph, any two vertices are adjacent, with $n$ vertices denotes by $K_n$. A graph $\Gamma$ is called complete bipartite if there exist disjoint subsets $A, B$ of $V(\Gamma)$ such that $A \cup B = V(\Gamma)$ and $x - y$ for any $x \in A$ and $y \in B$. Finite complete bipartite graphs are denoted by $K_{m,n}$, where $|A| = m$ and $|B| = n$, a graph is a star graph if $\Gamma = K_{1,n}$. For notations and terminologies not given in this article, the reader is referred to [2], [7].

Definition 1.1. Let $G$ be a group. The order graph of $G$ is the (undirected) graph $\Gamma(G)$, those whose vertices are non-trivial subgroups of $G$ and two distinct vertices $H$ and $K$ are adjacent if and only if either $o(H)|o(K)$ or $o(K)|o(H)$.

The main object of this paper is to study the interplay of group-theoretic properties of $G$ with graph-theoretic properties of $\Gamma(G)$. In section 2, for a finite group $G$, it is shown that $\Gamma(G)$ is always connected with $\text{diam}(\Gamma(G)) \leq 2$, and $\Gamma(G)$ is a complete graph if and only if $G$ is a $p$-group for some prime number $p$. Moreover, $\Gamma(G)$ is a star graph if and only if $G = C_{pq}$, where $p, q$ are prime numbers. In section 3, we study a subgraph of $\Gamma(G)$ denotes by $\Gamma^*(G)$ those whose vertices are non-trivial proper subgroups of $G$, and for a finite abelian group $G$ we show that $\Gamma^*(G)$ is connected but is not a star graph.

2. Properties of orders graph

We start with some examples.

Example 2.1. (i) The diagram in Figure 1 illustrates the order graphs for Klein Vieregruppe $V_4 = \{e, a, b, c\}$ and $\mathbb{Z}_{30}$. For $G = V_4$ we have $V(\Gamma(G)) = \{H_1 = \{e, a\}, H_2 = \{e, b\}, H_3 = \{e, c\}, G\}$ so that $\Gamma(G) = K_4$. For $G = \mathbb{Z}_{30}$ we have $V(\Gamma(G)) = \{H_1 = \langle 2 \rangle, H_2 = \langle 3 \rangle, H_3 = \langle 5 \rangle, H_4 = \langle 6 \rangle, H_5 = \langle 10 \rangle, H_6 = \langle 15 \rangle, G\}$. 

...
(ii) The diagram in Figure 2 illustrates the order graphs for $G = \mathbb{Z}_6, G' = \mathbb{Z}_4$ and $G'' = \mathbb{Z}_{27}$, where $V(\Gamma(G)) = \{H_1 = <2>, H_2 = <3>, G\}$, $V(\Gamma(G')) = \{H'_1 = <2>, G'\}$ and $V(\Gamma(G'')) = \{H''_1 = <3>, H''_2 = <9>, G''\}$.

Assume that $G$ and $G'$ are two finite group. If $G \cong G'$, then it is obvious that $\Gamma(G) \cong \Gamma(G')$. The following example shows that the converse is not true.

**Example 2.2.** The diagram in Figure 3 illustrates the order graphs for $\mathbb{Z}_6, \mathbb{Z}_{10}$. We have $V(\Gamma(\mathbb{Z}_6)) = \{H_1 = <2>, H_2 = <3>, \mathbb{Z}_6\}$, $V(\Gamma(\mathbb{Z}_{10})) = \{H'_1 = <2>, H'_2 = <5>, \mathbb{Z}_{10}\}$. It is clear that $\mathbb{Z}_6 \not\cong \mathbb{Z}_{10}$ but $\Gamma(\mathbb{Z}_6) \cong \Gamma(\mathbb{Z}_{10})$.

In the following we show that $\Gamma(G)$ is always connected and has small diameter and girth.
**Lemma 2.3.** Let $G$ be a finite group. Then there is a vertex of $\Gamma(G)$ which is adjacent to every other vertices.

**Proof.** Let $H$ be a subgroup of $G$, by Lagrange’s Theorem [6, Corollary 4.6], $o(H)|o(G)$ so $G$ is a vertex of $\Gamma(G)$ which is adjacent to every other vertices. 

**Corollary 2.4.** Let $G$ be a finite group. Then $\Gamma(G) = \emptyset$ if and only if $G$ is a simple group. Furthermore, $\Gamma(G) \neq \emptyset$ if and only if $\Gamma(G)$ has not a single vertex.

**Corollary 2.5.** Let $G$ be a finite group with $|V(\Gamma(G))| = n$. Then the following statements are true:

(i) The maximum degree of $\Gamma(G)$, $\Delta(\Gamma(G))$, is $n - 1$.

(ii) If $n \geq 4$, then $\Gamma(G)$ is not a path and an $n$-gon.

**Proof.** (i) In view of Lemma 2.3 we have $\deg(G) = n - 1$. Thus $\Delta(\Gamma(G)) = n - 1$.

(ii) If $n \geq 4$, then $\deg(G) \geq 3$ which shows $\Gamma(G)$ is not a path and an $n$-gon.

**Corollary 2.6.** Let $G$ be a finite group. Then $\Gamma(G)$ is a regular graph if and only if it is complete.

**Proof.** Let $\Gamma(G)$ be a graph with $n$ vertices. Then $\Delta(\Gamma(G)) = n - 1$. Hence, $\Gamma(G)$ is regular if the degree of any vertices is equal to $n - 1$ which implies that $\Gamma(G)$ is complete.

**Theorem 2.7.** Let $G$ be a finite group. Then $\Gamma(G)$ is a connected graph with $\text{diam} \, \Gamma(G) \leq 2$. Furthermore, if $\Gamma(G)$ contains a cycle, then $\text{gr} \, \Gamma(G) = 3$.

**Proof.** In view of Lemma 2.3, $\Gamma(G)$ has a vertex adjacent to every other vertices. So that $\Gamma(G)$ is connected and then $\text{diam} \, \Gamma(G) \leq 2$. Now, assume that $\Gamma(G)$ contains a cycle. If $\Gamma(G)$ is an $n$-gon, then using Lemma 2.3 again we have $n = 3$. Let $\Gamma(G)$ consist an $n$-gon, where $n > 3$. It is known that $\Gamma(G)$ has a vertex that is adjacent to every other vertices so there are two cases. Case 1. If this vertex is a vertex of $n$-gon, then there are $n - 2$ triangles thus $\text{gr} \, \Gamma(G) = 3$. Case 2. If this vertex is not a vertex of the $n$-gon, then by connecting this vertex to the vertices of $n$-gon we have $n$ triangles, thus $\text{gr} \, \Gamma(G) = 3$.

In the following we determine when $\Gamma(G)$ is a complete graph. By using definition $\Gamma(G)$ is complete if and only if $o(H) | o(K)$ or $o(K) | o(H)$ for all distinct subgroup $H$ and $K$ of $G$.

**Lemma 2.8.** Let $G$ be a finite group. Then $\Gamma(G)$ is complete if and only if $G$ is a $p$-group, where $p$ is a prime number.
Hence, or number. By Cauchy’s Theorem [6, Theorem 5.2], there are subgroups $H$ and $K$ of $G$ with $o(H) = p$ and $o(K) = q$. By assumption we must have $p|q$, which implies that $p = q$. Thus $G$ is a $p$-group.

$\Leftrightarrow$ Let $G$ be a $p$-group. Then the orders of all subgroups of $G$ is a power of $p$ so that all vertices of $\Gamma(G)$ are adjacent with each others. Thus $\Gamma(G)$ is complete. $\square$

In the following we determine all groups for which the order graph is complete with at most five vertices.

**Theorem 2.9.** Let $G$ be a finite group. Then $\Gamma(G) = K_2$ if and only if $G = C_{p^2}$, where $p$ is a prime number.

**Proof.** $\Rightarrow$ Let $\Gamma(G) = K_2$. Then by Lemma 2.8 it follows that $G$ is a $p$-group for some prime number $p$. It is clear that $o(G) \leq p^2$ by Sylow Theorem [6, Theorem 5.7]. For $o(G) = p$ we have $\Gamma(G) = \emptyset$. So assume that $o(G) = p^2$. In this case $G$ is an abelian group and so $G \cong C_{p^2}$ or $C_p \times C_p$. If $G \cong C_p \times C_p$, then it has at least three subgroups with orders $p$, which shows that $\Gamma(G) \neq K_2$. Hence, $G$ is a cyclic group with order $p^2$ that is $G \cong C_{p^2}$.

$\Leftarrow$ It is obvious by Lemma 2.8. $\square$

**Theorem 2.10.** Let $G$ be a finite group. Then $\Gamma(G) = K_3$ if and only if $G = C_{p^3}$, where $p$ is a prime number.

**Proof.** Let $\Gamma(G) = K_3$. Then it follows that $G$ is a $p$-group with $o(G) \leq p^3$. By the proof of Theorem 2.9 it follows that $o(G) \neq p, p^2$. So that $o(G) = p^3$. If $G$ is an abelian group, then $G \cong C_{p^3}$ or $C_{p^2} \times C_p$ or $C_p \times C_p \times C_p$. It is obvious that $\Gamma(C_p \times C_p \times C_p) \neq K_3$. Also, if $G \cong C_{p^2} \times C_p$, then $G$ has at least three subgroups with orders $p$ and $p^2$ thus $\Gamma(C_{p^2} \times C_p) \neq K_3$. Hence, $G$ is a cyclic group with order $p^3$. Now, assume that $G$ is a non-abelian group. By hypothesis $G$ has only one subgroup with order $p$ and $p^2$ so that $G \cong Q_8$ by [3, Exercise 21, page 187], since any subgroup of $G$ is normal. But, $\Gamma(Q_8) \neq K_3$ since $Q_8$ has three subgroups with order four. $\square$

**Theorem 2.11.** Let $G$ be a finite group. Then $\Gamma(G) = K_4$ if and only if either $G = C_2 \times C_2$ or $G = C_{p^4}$, where $p$ is a prime number.

**Proof.** Let $\Gamma(G) = K_4$. Then it follows that $G$ is a $p$-group with $p^2 \leq o(G) \leq p^4$. If $o(G) = p^2$, then by the proof of Theorem 2.9 we have $G \cong C_p \times C_p$. If $p \geq 3$, then $G$ has at least four subgroups with orders $p$ such as $H_1 = \langle (1, 0) \rangle$, $H_2 = \langle (0, 1) \rangle$, $H_3 = \langle (1, 1) \rangle$, $H_4 = \langle (1, 2) \rangle$ thus $\Gamma(C_p \times C_p) \neq K_4$. Hence, $p = 2$ and in this case Example 2.1(i) shows that $\Gamma(C_2 \times C_2) = K_4$. A same argument shows
that $G \not\cong C_p \times C_p \times C_p, C_p \times C_p$. Moreover, as above the non-abelian groups $Q_8, D_8$ can not have $K_4$ as its order graph. So assume that $G$ is a non-abelian group with order $p^3$, where $p$ is a odd prime number. By [3, Exercise 29, page 200], $G$ has $p + 1$ maximal subgroups so that it has at least six subgroups. Thus $\Gamma(G) \not\cong K_4$. Now, assume that $G$ is a group with order $p^4$ and $\Gamma(G) = K_4$. Thus $G$ must have only one maximal subgroup so it follows that $G$ is a cyclic group, see Corollary 1.2, page 173 in [4]. □

**Theorem 2.12.** Let $G$ be a finite group. Then $\Gamma(G) = K_5$ if and only if either $G = C_3 \times C_3, C_2 \times C_4, Q_8$ or $G = C_p \cdot 5$, where $p$ is a prime number.

**Proof.** Let $\Gamma(G) = K_5$. Then $G$ is a $p$ group with $p^2 \leq o(G) \leq p^5$. If $o(G) = p^2$, then by the proof of Theorems 2.9, 2.11, $G \not\cong C_p \cdot 2, C_3 \times C_2$ also, it is easy to see that for $p \geq 5$ it has at least five subgroups with orders $p$ such as $H_1 = \langle (1, 0) \rangle, H_2 = \langle (0, 1) \rangle, H_3 = \langle (1, 1) \rangle, H_4 = \langle (1, 2) \rangle, H_5 = \langle (1, 3) \rangle$ which implies that $\Gamma(C_p \times C_p) \not\cong K_5$. On the other hand, $\Gamma(C_3 \times C_3) = K_5$. Now, assume that $o(G) = p^3$. If $G$ is an abelian group it is easy to show that $\Gamma(C_2 \times C_4) = K_5$ and $G \not\cong C_p \times C_p \times C_p, C_p \times C_p \times C_p$. Let $G$ be a non-abelian group with order $p^3$. Then we have $\Gamma(Q_8) = K_5$ and $\Gamma(D_8) \not\cong K_5$. So assume that $G$ is a non-abelian group with order $p^3$, where $p$ is a odd prime number. By [3, Exercise 29, page 200] it has $p + 1$ maximal subgroups so that it has at least six subgroups. Thus $\Gamma(G) \not\cong K_5$. As above with a same argument one can show that $G$ can not be an abelian group with order $p^4$. Let $G$ be a non-abelian group with order $p^4$. Then by assumption $G$ has only four proper subgroup. If it has only one maximal subgroup, then $G \cong C_p \cdot 3$, it follows by [4, Corollary 1.2] which is a contradiction. Otherwise, $G$ has only one subgroup of order $p$. In this case Proposition 1.3 in [1] shows that $G$ is a generalized Quaternion group with order $o(G) = 16$ or $32$. But by the gap software one can see that $Q_{16}$ has eight proper subgroups and $Q_{32}$ has eighteen proper subgroups. Hence, $G \not\cong Q_{16}$ and $G \not\cong Q_{32}$. If $G$ is a group with order $p^5$ and $\Gamma(G) = K_5$, then $G$ must have only one maximal subgroup so it follows that $G$ is a cyclic group, see [4, Corollary 1.2]. □

Let $G$ be a finite group. Then the order graph $\Gamma(G)$ is a complete bipartite graph whenever $\Gamma(G)$ is a star graph, since it has a vertex is adjacent with other vertices. In the sequel, we determine when $\Gamma(G)$ is a star graph.

**Lemma 2.13.** Let $G$ be a finite group. Then $\Gamma(G) = K_{1n}(n \geq 2)$ if and only if any subgroups of $G$ has prime order and $G$ has only one subgroup with any orders. Furthermore, $o(G) = p_1 \cdots p_n$ for some prime numbers $p_1, \cdots, p_n$. 


The order graphs of groups

Figure 4. \( \Gamma^*(V_4) \)

![Diagram](https://via.placeholder.com/150)

**Proof.** \( \Rightarrow \) By assumption \(|V(\Gamma(G))| \geq 3 \) and in view of Lemma 2.3 we have \( \deg(G) = |V(\Gamma(G))| - 1 \). Let \( H \) be a subgroup of \( G \) with \( o(H) = p_1^{a_1} \cdots p_k^{a_k} \). By Cauchy’s Theorem the degree of \( H \) is at least \( k \) which implies that \( k = 1 \) and \( a_1 = 1 \). Moreover, \( G \) has precisely one subgroup with order \( p_1 \).

\( \Rightarrow \) It is obvious. \( \square \)

**Theorem 2.14.** Let \( G \) be a finite group. Then \( \Gamma(G) \) is a star graph if and only if \( G = C_{pq} \), where \( p, q \) are prime numbers.

**Proof.** In view of Lemma 2.13 we have \( \Gamma(G) \) is a star graph if and only if any subgroups of \( G \) has prime order and also \( G \) has only one subgroup with any orders. Furthermore, \( o(G) = p_1 \cdots p_k \) for some prime numbers \( p_1, \cdots, p_k \). For \( k = 2 \), if \( p < q \) and \( p \nmid q - 1 \), then \( G \) is a cyclic group and we are done, otherwise \( G \) has \( 3 \leq 1 + tp \) subgroups with orders \( p \) which shows that \( \Gamma(G) \neq K_{12} \). Let \( k \geq 3 \) and \( p_1 = 2 \). Then \( o(G) = 2n \), where \( n \) is an odd number. Thus \( G \) has a subgroup \( H \) with \( o(H) = n \), by [5, Theorem 1.35]. Hence, the order of \( H \) is not a prime number that contradict with Lemma 2.13. Now, suppose that for all \( i \) with \( 1 \leq i \leq k, p_i \) is an odd prime number. Then \( G \) is not a simple group, see [8, 1.12]. So that \( G \) has a normal subgroup such as \( H \). Assume that \( L \) is a subgroup of \( G \) with order \( p_1 \). If \( HL = G \), then \( o(H) = p_2 \cdots p_k \) which contradict with Lemma 2.13. Thus \( HL \neq G \). In this case \( HL \) is a subgroup of \( G \) with order multiplied more than one prime number which is a contradiction. Therefore, \( k = 2 \) and the proof is completed. \( \square \)

3. A Subgraph of a Order Graph

Let \( G \) be a group. By \( \Gamma^*(G) \) we mean a subgraph of \( \Gamma(G) \) with \( V(\Gamma^*(G)) = V(\Gamma(G)) - \{G\} \). So that \( \Gamma^*(G) \) is a simple graph with vertices \( V(\Gamma^*(G)) = \{H : H \text{ is a non-trivial proper subgroup of } G \} \) and two distinct vertices \( H \) and \( K \) are adjacent if and only if either \( o(H)|o(K) \) or \( o(K)|o(H) \).

**Example 3.1.** The diagram in Figure 4 illustrates \( \Gamma^*(G = V_4) \), where \( V(\Gamma^*(G)) = \{H_1 = \{e, a\}, H_2 = \{e, b\}, H_3 = \{e, c\}\} \).
**Example 3.2.** The diagram in Figure 5 illustrates $\Gamma^*(G)$ for $G = \mathbb{Z}_{30}$, $G = \mathbb{Z}_{27}$ and $G = \mathbb{Z}_9$, where $V(\Gamma^*(\mathbb{Z}_{30})) = \{H_1 = < 2 >, H_2 = < 3 >, H_3 = < 5 >, H_4 = < 6 >, H_5 = < 10 >, H_6 = < 15 >\}$; $V(\Gamma^*(\mathbb{Z}_{27})) = \{H'_1 = < 3 >, H'_2 = < 9 >\}$ and $V(\Gamma^*(\mathbb{Z}_9)) = \{H''_1 = < 3 >\}$.

**Theorem 3.3.** Let $G$ be a finite abelian group. Then $\Gamma^*(G)$ is a connected graph with diameter at most four.

**Proof.** Let $H$ and $K$ be two proper subgroups of $G$. We have to show that there is a path from $H$ to $K$ in $\Gamma^*(G)$. If $o(H)o(K) > o(G)$, then $o(H \cap K) = m > 1$ and $H - H \cap K - K$ is a path from $H$ to $K$. Let $o(G) > o(H)o(K)$. Then $HK$ is a proper subgroup of $G$ and $H - HK - H$ is a path from $H$ to $K$. Now, let $o(G) = o(H)o(K)$. If $o(H \cap K) = m > 1$, then we are done. Suppose that $o(H \cap K) = 1$. There are prime numbers $p, q$ such that $p|o(H)$ and $q|o(K)$. If $o(G) = pq$, then $\Gamma^*(G) = \emptyset$ and we are done. Assume that $o(G) > pq$. It is clear that $G$ has subgroups such as $L$, $L'$ and $L''$ with $o(L) = p$, $o(L') = q$ and $o(L'') = pq$. Hence, $H - L - L'' - L' - K$ is a path from $H$ to $K$. If $o(H) = o(L) = p$, then $H - L'' - L' - K$ is a path from $H$ to $K$, so $\Gamma^*(G)$ is a connected graph and $\text{diam} \Gamma^*(G) \leq 4$. □

**Corollary 3.4.** Let $G$ be a group and $H$ be a subgroup of $G$ with order $o(H) = p^n m$, where $p$ is a prime number, $n \geq 2$ and $m > 1$. Then $\text{gr}(\Gamma^*(G)) = 3$.

**Proof.** By Sylow Theorems $H$ has subgroups $H_1, \ldots, H_n$ with orders $o(H_i) = p^i$ for all $i$ with $1 \leq i \leq n$. If $n \geq 3$, then $\Gamma^*(G)$ has a complete subgraph with $n$ vertices so that $\text{gr}(\Gamma^*(G)) = 3$. Otherwise, $G$ has subgroups $H_1, H_2$ with $o(H_1) = p$, $o(H_2) = p^2$ thus $H_1 - H_2 - H - H_1$ is a triangle and $\text{gr}(\Gamma^*(G)) = 3$. □

**Theorem 3.5.** Let $G$ be a finite abelian group such that $\Gamma^*(G) \neq \emptyset$. Then $\Gamma^*(G)$ is a complete graph if and only if $G$ is a $p$-group.
Proof. Suppose that \( o(G) = pqm \) for some prime numbers \( p, q \) and \( m \geq 1 \). Thus \( G \) has at least three proper subgroups with orders \( o(H) = p \), \( o(K) = q \) and \( o(L) = pq \) which show that \( H, K \) are not single vertices and \( H, K \) are not adjacent to each other. That is a contradiction with assumption that \( \Gamma^*(G) \) is a complete graph. So that either \( m = 1 \) or \( p = q \). If \( m = 1 \) and \( p \neq q \), then \( G \cong C_{pq} \) and so \( \Gamma^*(G) = \emptyset \) that contradict with assumption. Hence, \( p = q \), now by continuing this way one can shows that \( G \) is a \( p \)-group. \( \Box \)

The following example shows that the Theorem 3.5 is not true for non-abelian groups. It is well-known that \( S_3 \) is a non-abelian group with order six, but \( \Gamma^*(S_3) \) is a complete graph.

Example 3.6. Figure 6 displays \( \Gamma^*(S_3) \). Note that \( V(\Gamma^*(S_3)) = \{ H_1 = \langle 1 \, 2 \rangle, H_2 = \langle 1 \, 3 \rangle, H_3 = \langle 2 \, 3 \rangle, H_4 = \langle 1 \, 2 \, 3 \rangle \} \) and \( \Gamma^*(S_3) = K_3 \).

Theorem 3.7. Let \( G \) be a finite group with \( o(G) = pq \) (\( p < q \)) where \( p, q \) are distinct primes numbers, then either \( \Gamma^*(G) = \emptyset \) or \( \Gamma^*(G) = K_n \).

Proof. It is well known that \( G \) has only one subgroup of order \( q \). If \( p \nmid q - 1 \), then \( G \) has only one subgroup of order \( p \). So that \( G \) is a cyclic group and \( \Gamma^*(G) = \emptyset \). Now, assume that \( p|q - 1 \). In this case \( G \) has \( 1 + tp \) subgroups of order \( p \) and so \( \Gamma^*(G) \) is a complete graph. \( \Box \)

Theorem 3.8. Let \( G \) be a finite group. Then \( \Gamma^*(G) \) is not a star graph.

Proof. Let \( G \) be a finite group. We have to show that \( \Gamma^*(G) \neq K_{1n} \) \((n \geq 2) \). If \( H \) is the center vertex of \( \Gamma^*(G) \), then the proof of Corollary 3.4 shows that \( o(H) = p_1 \cdots p_n \), where \( p_1, \ldots, p_n \) are prime numbers. Case1. Assume that \( n \geq 3 \). If \( p_1 = 2 \), then \( o(H) = 2p_2 \cdots p_n \). Thus \( H \) has a subgroup \( K \) with \( o(K) = p_2 \cdots p_n \) by [5, Theorem 1.35]. Hence, \( \Gamma^*(G) \) is not a star graph. Now, suppose that \( p_i \) is an odd prime number, for all \( 1 \leq i \leq k \). Then \( H \) is not a simple group, see [8, 1.12]. So that \( H \) has a normal subgroup such as \( K \). Assume that \( L \) is a subgroup of \( H \) with order \( p_1 \). Thus \( KL \) is a subgroup of \( H \). If \( KL = H \), then the order of \( K \) is a multiplication of more than one prime
number this is a contradiction. If $KL \neq H$, then $KL$ is a subgroup of $H$ and the order of $KL$ is not a prime number that is a contradiction. So that $\Gamma^*(G)$ is not a star graph. Case2. Let $n = 2$. Then $o(H) = p_1p_2$. Since $o(H) < o(G)$ so there exists $m \geq 2$ such that $o(G) = p_1p_2m$. If $q^2|m$ for some prime number $q \neq p_1, p_2$, then $H$ is not the center vertex of $\Gamma^*(G)$. So assume that $o(G) = p_1p_2q$, where $p_1 < p_2 < q$. Then $G$ has a subgroup with order $p_1q$ which shows that $\Gamma^*(G)$ is not a star graph, the proof is completed. □

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