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Research Paper

ON LEFT WEAKLY JOINTLY PRIME (R,S)-MODULES

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ABSTRACT. Let R and S be commutative rings and M an (R,S)-module. A proper (R,S)-submodule P of M is called left weakly jointly prime if for each (R,S)-submodule N of M and elements a,b of R such that $abNS \subseteq P$ implies either $aNS \subseteq P$ or $bNS \subseteq P$. This paper defines left weakly jointly prime (R,S)-modules and presents some of their properties. On the other hand, a ring R is called fully prime if each proper ideal of R is prime. We extend this fact to (R,S)-modules. An (R,S)-module M is called fully left weakly jointly prime if each proper (R,S)-submodule of M is left weakly jointly prime. Moreover, we present some properties of fully left weakly jointly prime (R,S)-modules. At the end of this paper, we present our main results about the necessary and sufficient conditions for an arbitrary (R,S)-module to be fully left weakly jointly prime.

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1. Introduction

The notion of prime submodules has been introduced by Dauns in [9]. A proper submodule P of an R-module M is said to be prime if for any element $r \in R$ and element $m \in M$ with $rm \in P$, we have $m \in P$ or $rM \subseteq P$. Moreover, a non-zero R-module M is considered prime if the zero submodules are prime. Some previous authors have studied the prime submodules and prime modules, for example, in papers [14, 4, 12, 10].

In module theory, an R-module has been generalized into an (R, S)-bimodule, where R and S are arbitrary rings. Khumprapussorn et al. in [13] have generalized the (R, S)-bimodule structure to an (R, S)-module structure. Let R and S be rings and M an abelian group under addition. Khumprapussorn et al. in [13] said M is an (R, S)-module if there exists a function $\_ \cdot \_ \cdot \_ : R \times M \times S \to M$ such that for all $r_1, r_2, r \in R$, $s_1, s_2, s \in S$, and $m, n \in M$ satisfied (1) $r \cdot (m + n) \cdot s = r \cdot m \cdot s + r \cdot m \cdot s$; (2) $(r_1 + r_2) \cdot m \cdot s = r_1 \cdot m \cdot s + r_2 \cdot m \cdot s$; (3) $r \cdot m \cdot (s_1 + s_2) = r \cdot m \cdot s_1 + r \cdot m \cdot s_2$; (4) $r_1 \cdot (r_2 \cdot m \cdot s_1) \cdot s_2 = (r_1 r_2) \cdot m \cdot (s_1 s_2)$. Moreover, the concepts around (R, S)-module have been studied in [17]. An (R, S)-module has an (R, S)-bimodule structure when both rings R and S have central idempotent elements.

According to [13], a proper (R, S)-submodule P of M is called a jointly prime (R, S)-submodule if for each left ideal I of R, right ideal J of S, and (R, S)-submodule K of M with $IKJ \subseteq P$ implies $IMJ \subseteq P$ or $K \subseteq P$. If R and S are commutative rings, then we have a proper (R, S)-submodule P of M is called a jointly prime (R, S)-submodule if for each ideal I of R, ideal J of S, and (R, S)-submodule K of M with $IKJ \subseteq P$ implies $IMJ \subseteq P$ or $K \subseteq P$. Furthermore, a non-zero (R, S)-module M is said to be jointly prime if its zero (R, S)-submodule is a jointly prime (R, S)-submodule of M.

Weakly prime submodules are generalizations of prime submodules. Weakly prime submodules have been introduced and studied over an associative ring with identity in [7, 6]. Assume that R is an associative ring with identity. According to [7], a proper submodule P of M is said to be weakly prime if for any $a, b \in R$ and submodule K of M with $aRbK \subseteq P$ implies either $aK \subseteq P$ or $bK \subseteq P$. If R is a commutative ring, then a proper submodule P of M is weakly prime if for each submodule K of M and elements a, b of R with $abK \subseteq P$, implies either $aK \subseteq P$ or $bK \subseteq P$. Moreover, weakly prime submodules over a commutative ring have been studied in [3, 5, 2, 1]. Next, we extend these facts to (R, S)-modules. Following to [16], a proper (R, S)-submodule P of M is said to be left weakly jointly prime if for each (R, S)-submodule N of M and element $a, b \in R$ such that $abNS \subseteq P$ implies either $aNS \subseteq P$ or $bNS \subseteq P$. According to [7], an R-module M is called a weakly prime module if its zero submodules is a weakly prime submodule of M. An R-module M is said to be weakly prime if the annihilator of any non-zero submodule of M is a prime ideal. Moreover, this work aims to define left weakly jointly prime (R, S)-modules and then investigate their properties.

In Section 2, we present the definition of left weakly jointly prime (R, S)-module and give some of their properties. First, we provide the necessary and sufficient condition for (R, S)-modules to be left weakly jointly prime. And then we present the set of $(0 :_R \langle m \rangle)$ for each $0 \neq m \in M$ is a chain of prime ideals if and only if M is a left weakly jointly prime (R, S)-module where $S^2 = S$ and $a \in RaS$ for all $a \in M$. At the end of this section, we present the sufficient condition for every non-zero summand of (R, S)-module M to be left weakly jointly prime.

According to [7], a ring R is called a fully prime ring if each proper ideal of R is prime. This ring type is fully investigated in [8, 15]. Based on [7], we have that an R-module M is called a fully weakly prime module if each proper submodule of M is weakly prime. When we extend these facts to (R, S)-module, we have an (R, S)-module of M is fully left weakly jointly prime if every proper (R, S)-submodule of M is a left weakly jointly prime (R, S)-submodule. Section 3 presents some properties of fully left weakly jointly prime (R, S)-modules. We develop some properties of fully weakly prime modules studied in [7]. Moreover, at the end of this section, we show our main results about the necessary and sufficient conditions for an arbitrary (R, S)-module to be fully left weakly jointly prime.

Throughout this paper, R and S are commutative rings unless stated otherwise, and M is an additive abelian group.

2. Left weakly jointly prime (R, S)-modules

In this section, we present the definition of left weakly jointly prime (R, S)-modules and give some of their properties. We begin by defining a left weakly jointly prime (R, S)-submodule as follows.

Definition 2.1. [16] Let M be an (R, S)-module. A proper (R, S)-submodule P of M is called a left weakly jointly prime (R, S)-submodule if for each (R, S)-submodule N of M and element $a, b \in R$ such that $abNS \subseteq P$ implies either $aNS \subseteq P$ or $bNS \subseteq P$.

When $a \in RaS$ for all $a \in M$, we have another definition of a left weakly jointly prime (R, S)-submodule as follows.

Definition 2.2. [16] Let M be an (R, S)-module satisfied $a \in RaS$ for all $a \in M$. A proper (R, S)-submodule P of M is called a left weakly jointly prime (R, S)-submodule if for each ideal I, J of R and (R, S)-submodule N of M with $IJNS \subseteq P$, implies either $INS \subseteq P$ or $JNS \subseteq P$.

Below, we give an example of left weakly jointly prime (R, S)-submodules.

Example 2.3. Let $\mathbb{Z}$ be a $(4\mathbb{Z}, 3\mathbb{Z})$-module. A proper $(4\mathbb{Z}, 3\mathbb{Z})$-submodule $12\mathbb{Z}$ of $\mathbb{Z}$ is a left weakly jointly prime $(4\mathbb{Z}, 3\mathbb{Z})$-submodule of $\mathbb{Z}$. Let $a, b \in 4\mathbb{Z}$ with $a = 4k$ and $b = 4l$ and N

be a $(4\mathbb{Z}, 3\mathbb{Z})$-submodule of $\mathbb{Z}$ with $N = x\mathbb{Z}$ for some $k, l, x \in \mathbb{Z}$. We have

$$abN(3\mathbb{Z}) = (4k)(4l)(x\mathbb{Z})(3\mathbb{Z}) = 48klx\mathbb{Z}^2 \subseteq 48klx\mathbb{Z} \subseteq 12\mathbb{Z}.$$

In the other side, we obtain $aN(3\mathbb{Z}) = (4k)(x\mathbb{Z})(3\mathbb{Z}) = 12kx\mathbb{Z}^2 \subseteq 12kx\mathbb{Z} \subseteq 12\mathbb{Z}$ or $bN(3\mathbb{Z}) = (4l)(x\mathbb{Z})(3\mathbb{Z}) = 12lx\mathbb{Z}^2 \subseteq 12lx\mathbb{Z} \subseteq 12\mathbb{Z}$. Hence, $12\mathbb{Z}$ is a left weakly jointly prime $(4\mathbb{Z}, 3\mathbb{Z})$-submodule of $\mathbb{Z}$.

Now, we present the definition of left weakly jointly prime (R, S)-module.

Definition 2.4. An (R, S)-module M is called left weakly jointly prime if it's zero (R, S)-submodule is left weakly jointly prime.

Example 2.5. Let $\mathbb{Z}$ be an $(4\mathbb{Z}, 3\mathbb{Z})$-module. It is easy to show that the zero $(4\mathbb{Z}, 3\mathbb{Z})$-submodule of $\mathbb{Z}$ is a left weakly jointly prime $(4\mathbb{Z}, 3\mathbb{Z})$-submodule. Thus, $\mathbb{Z}$ is a left weakly jointly prime $(4\mathbb{Z}, 3\mathbb{Z})$-module.

Example 2.6. Let R and S are commutative rings with

$$R = \left\{ \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{0} & \bar{0} \end{pmatrix} \middle| \bar{a}, \bar{b} \in \mathbb{Z}_4 \right\} \text{ and } S = \left\{ \begin{pmatrix} \bar{a} & \bar{0} \\ \bar{b} & \bar{0} \end{pmatrix} \middle| \bar{a}, \bar{b} \in \mathbb{Z}_4 \right\}.$$

Let an (R, S)-module M with

$$M = \left\{ \begin{pmatrix} \bar{a} & \bar{0} \\ \bar{b} & \bar{c} \end{pmatrix} \middle| \bar{a}, \bar{b}, \bar{c} \in \mathbb{Z}_4 \right\}.$$

We can show that M is not a left weakly jointly prime (R, S)-module. Let (R, S)-submodule $N = \left\{ \begin{pmatrix} \bar{a} & \bar{0} \\ \bar{0} & \bar{b} \end{pmatrix} \middle| \bar{a}, \bar{b} \in \mathbb{Z}_4 \right\}$ and any element $a, b \in R$ with $a = \begin{pmatrix} \bar{0} & \bar{y} \\ \bar{0} & \bar{0} \end{pmatrix}$ and $b = \begin{pmatrix} \bar{0} & \bar{x} \\ \bar{0} & \bar{0} \end{pmatrix}$. Let any element $n \in N$ with $n = \begin{pmatrix} \bar{a} & \bar{0} \\ \bar{0} & \bar{b} \end{pmatrix}$. We obtain

$$abnS = \begin{pmatrix} \bar{0} & \bar{y} \\ \bar{0} & \bar{0} \end{pmatrix} \begin{pmatrix} \bar{0} & \bar{x} \\ \bar{0} & \bar{0} \end{pmatrix} \begin{pmatrix} \bar{a} & \bar{0} \\ \bar{0} & \bar{b} \end{pmatrix} S = \left\{ \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix} \right\}.$$

However, we have

$$anS = \begin{pmatrix} \bar{0} & \bar{y} \\ \bar{0} & \bar{0} \end{pmatrix} \begin{pmatrix} \bar{a} & \bar{0} \\ \bar{0} & \bar{b} \end{pmatrix} S = \begin{pmatrix} \bar{0} & \overline{yb} \\ \bar{0} & \bar{0} \end{pmatrix} S \neq \left\{ \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix} \right\},$$

and

$$bnS = \begin{pmatrix} \bar{0} & \bar{x} \\ \bar{0} & \bar{0} \end{pmatrix} \begin{pmatrix} \bar{a} & \bar{0} \\ \bar{0} & \bar{b} \end{pmatrix} S = \begin{pmatrix} \bar{0} & \overline{xb} \\ \bar{0} & \bar{0} \end{pmatrix} \neq \left\{ \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix} \right\}.$$

Thus, M is not a left weakly jointly prime (R, S)-module.

According to [7], a proper submodule N of an R-module M is weakly prime if and only if the quotient R-module M/N is weakly prime. Proposition 2.7 extends this result to (R,S)-modules as follows.

Proposition 2.7. *Let M be an (R,S)-module. Then a proper (R,S)-submodule X of M is a left weakly jointly prime (R,S)-module if and only if M/X is a left weakly jointly prime (R,S)-module.*

Proof. Let X be a left weakly jointly prime (R,S)-submodule of M. We have M/X is a left weakly jointly prime (R,S)-module. Conversely, assume that M/X is a left weakly jointly prime (R,S)-module. Then X is a left weakly jointly prime (R,S)-submodule of M. □

According to [13], for each (R,S)-submodule N of M, let $(N :_R M) = \{r \in R \mid rMS \subseteq N\}$. Clearly that $(N :_R M)$ is only an additive subgroup of R. However, if we have the condition $S^2 = S$, clearly that $(N :_R M)$ is an ideal of R. We may also say that $(N :_R M)$ is the annihilator of the quotient (R,S)-module M/N over the ring R.

Before we present the next properties of left weakly jointly prime (R,S)-modules, we need the following properties.

Proposition 2.8. *Let M be an (R,S)-module with $S^2 = S$ and $a \in RaS$ for all $a \in M$ and N be a proper (R,S)-submodule of M. Then N is a left weakly jointly prime (R,S)-submodule of M if and only if $(N :_R K)$ is a prime ideal of R for each (R,S)-submodule K of M with $K \nsubseteq N$.*

Proof. Let K be an (R,S)-submodule of M with $K \nsubseteq N$. Since $S^2 = S$ and $a \in RaS$ for all $a \in M$, then $(N :_R K)$ is a proper ideal of R. Let any elements a and b of R such that $ab \in (N :_R K)$, so we have $abKS \subseteq N$. Since N is a left weakly jointly prime (R,S)-submodule, then $aKS \subseteq N$ or $bKS \subseteq N$. Thus, we obtain $a \in (N :_R K)$ or $b \in (N :_R K)$. Hence, $(N :_R K)$ is a prime ideal of R. Conversely, let a and b be elements of R and L be an (R,S)-submodule of M such that $abLS \subseteq N$ and $aLS \nsubseteq N$. Then $L \nsubseteq N$. So, we have $ab \in (N :_R L)$. Based on the hypothesis, $(N :_R L)$ is a prime ideal of R. Thus from $ab \in (N :_R L)$ and $aLS \nsubseteq N$ we obtain $a \notin (N :_R L)$ and $b \in (N :_R L)$. Thus, we have $bLS \subseteq N$. Thus, N is a left weakly jointly prime (R,S)-submodule of M. □

Proposition 2.9 presents the necessary and sufficient condition for an (R,S)-module to be left weakly jointly prime.

Proposition 2.9. *Let M be an (R,S)-module with $S^2 = S$ and $a \in RaS$ for all $a \in M$. Then M is a left weakly jointly prime if and only if $(0 :_R K)$ is a prime ideal of R, for each non-zero (R,S)-submodule K of M.*

Proof. Let K be a non-zero (R,S)-submodule of M. Clearly that $(0 :_R K) = \{r \in R \mid rKS = 0\}$ is a proper ideal of R. Since M is a left weakly jointly prime (R,S)-module, 0 is a left weakly jointly prime (R,S)-submodule of M. Thus, based on Proposition 2.8, we have $(0 :_R K)$ is a prime ideal of R. Conversely, it is known that for each non-zero (R,S)-submodule K of M satisfy $(0 :_R K)$ is a prime ideal of R. Using Proposition 2.8, 0 is a left weakly jointly prime (R,S)-submodule of M. Hence, M is a left weakly jointly prime (R,S)-module. □

Let M be an (R,S)-modules. Following to [13], for any non-empty subsets Y of M we define

$$\langle Y\rangle = \bigcap\{K \mid K \text{ is an } (R,S)-\text{submodule of } M \text{ containing } Y\}.$$

It is obvious that $\langle Y\rangle$ is an (R,S)-submodule of M containing Y. If $Y = \{a\}$, then we have

$$\langle\{a\}\rangle = \langle a\rangle = \bigcap\{K \mid K \text{ is an } (R,S)-\text{submodule of } M \text{ containing } a\}.$$

Clearly that $\langle a\rangle$ is an (R,S)-submodule of M for any element $a \in M$. Moreover, element a is contained in $\langle a\rangle$.

Next, we give a proposition explaining the elements' form in (R,S)-submodule $\langle Y\rangle$.

Theorem 2.10. *Let M be an (R,S)-module and the set $Y \subseteq M$. If $Y = \emptyset$, then $\langle Y\rangle = \{0\}$. If $Y \neq \emptyset$, then we get*

$$\langle Y\rangle = \Big\{\sum_{i=1}^{t} r_i y_i s_i + \sum_{j=1}^{k} n_j y'_j \Big| r_i \in R, y_i, y'_j \in Y, s_i \in S, n_j \in \mathbb{Z}, \forall i = 1,2,...,t, \forall j = 1,2,...,k\Big\}.$$

Proof. We assume $Y \neq \emptyset$ and

$$A = \Big\{\sum_{i=1}^{t} r_i y_i s_i + \sum_{j=1}^{k} n_j y'_j \Big| r_i \in R, y_i, y'_j \in Y, s_i \in S, n_j \in \mathbb{Z}, \forall i = 1,2,...,t, \forall j = 1,2,...,k\Big\}.$$

We will prove that $\langle Y\rangle = A$. Since $\langle Y\rangle$ is the intersection of all (R,S)-submodules of M that contain Y, it is clear that $Y \subseteq \langle Y\rangle$. Since $\langle Y\rangle$ is closed to the scalar addition and multiplication operations, then $A \subseteq Y \subseteq \langle Y\rangle$. Next, we will prove $\langle Y\rangle \subseteq A$. It is equivalent to show that A is an (R,S)-submodule of M containing Y. Let any $y \in Y$, we have $y = 0y0 + 1y \in A$, so $Y \subseteq A$. Let any $\Big(\sum_{i=1}^{t} r_i y_i s_i + \sum_{j=1}^{k} n_j y'_j\Big), \Big(\sum_{i=1}^{q} r'_i y'_i s'_i + \sum_{j=1}^{l} n'_j y"_j\Big) \in A$, we have

$$\Big(\sum_{i=1}^{t} r_iy_is_i+\sum_{j=1}^{k} n_jy'_j\Big)-\Big(\sum_{i=1}^{q} r'_iy'_is'_i+\sum_{j=1}^{l} n'_jy"_j\Big) = \Big(\sum_{i=1}^{t} r_iy_is_i-\sum_{i=1}^{q} r'_iy'_is'_i\Big) +\Big(\sum_{j=1}^{k} n_jy'_j-\sum_{j=1}^{l} n'_jy"_j\Big)\in A.$$

Next, let any element $r \in R$ and $s \in S$, we obtain

$$r\Big(\sum_{i=1}^{t} r_iy_is_i+\sum_{j=1}^{k} n_jy'_j\Big)s=\sum_{i=1}^{t}(rr_i)y_i(s_is)+\sum_{j=1}^{k}(rn_j)y'_js\in A.$$

Thus, A is an (R,S)-submodule of M containing Y. So, we obtain $\langle Y\rangle \subseteq A$. Hence, it has proved that $\langle Y\rangle = A$. □

Following to Theorem 2.10, If $Y = \{a\}$, then we have

$$\langle a\rangle=\Big\{\sum_{i=1}^{t} r_ias_i+\sum_{j=1}^{k} n_jx\Big| r_i\in R, s_i\in S, n_j\in\mathbb{Z}, \forall i=1,2,...,t, \forall j=1,2,...,k\Big\}.$$

If $a \in RaS$ for all $a \in M$, then we have

$$\langle a\rangle=\Big\{\sum_{i=1}^{t} r_ias_i\Big| r_i\in R, s_i\in S, \forall i=1,2,...,t\Big\}.$$

Next, we define cyclic (R,S)-submodules as follows.

Definition 2.11. Let M be an (R,S)-module. An (R,S)-submodule N of M is called a cyclic (R,S)-submodule of M if N is generated by an element $x \in M$, i.e. $N = \langle x\rangle$.

Clearly that (R,S)-module M is cyclic if M is generated by an element $a \in M$, i.e. $M = \langle a\rangle$. We have the following properties according to Proposition 2.9.

Corollary 2.12. *Let M be an (R,S)-module with $S^2 = S$ and $a \in RaS$ for all $a \in M$. Then M is a left weakly jointly prime (R,S)-module if and only if for each $0 \neq m \in M$ satisfy $(0 :_R \langle m\rangle)$ is a prime ideal of R.*

Proof. Let $m \in M\backslash\{0\}$ and we form the set $(0 :_R \langle m\rangle)$. Let $x, y \in R$ such that $xy \in (0 :_R \langle m\rangle)$, so we have $xy\langle m\rangle S = 0$. Since M is a left weakly jointly prime (R,S)-modules, we have $x\langle m\rangle S = 0$ or $y\langle m\rangle S = 0$. Thus, we obtain $x \in (0 :_R \langle m\rangle)$ or $y \in (0 :_R \langle m\rangle)$. Hence, $(0 :_R \langle m\rangle)$ is a prime ideal of R. Conversely, let any (R,S)-submodule N of M and elements $a, b \in R$ such that $abNS = 0$ and $aNS \neq 0$. Let element $n \in N$. If element $n = 0$, clearly that from $abnS \subseteq abNS = 0$ we have $bnS = 0$. If element $n \neq 0$, then we get a cyclic (R,S)-submodule $\langle n\rangle$. Clearly that $\langle n\rangle \subseteq N$. So, we have $ab\langle n\rangle S \subseteq abNS = 0$. Based on the hypothesis, we obtain $b\langle n\rangle S = 0$. Since $n \in \langle n\rangle$, then we have $bnS = 0$. Thus, we get $bnS = 0$ for all element $n \in N$, so $bNS = 0$. Hence, M is left weakly jointly prime (R, S)-modules. □

Based on Corollary 2.12, we have the following properties.

Proposition 2.13. *Let M be an (R, S)-module with $S^2 = S$ and $a \in RaS$ for all $a \in M$. Then M is a left weakly jointly prime (R, S)-module if and only if the set $\mathfrak{J} = \{(0 :_R \langle m\rangle) \mid 0 \neq m \in M\}$ is a chain of prime ideals of R.*

Proof. Let M be a left weakly jointly prime (R, S)-module. Then for each $0 \neq m \in M$ satisfy $(0 :_R \langle m\rangle)$ is a prime ideal of R. We have to show that $\mathfrak{J}$ is a chain of prime ideals of R. Let $m, n \in M \setminus \{0\}$. Clearly, $(0 :_R \langle m\rangle) \cap (0 :_R \langle n\rangle) \subseteq (0 :_R \langle m\rangle + \langle n\rangle)$. Since M is a left weakly jointly prime (R, S)-module, then $(0 :_R \langle m\rangle + \langle n\rangle)$ is a prime ideal of R. Since $(0 :_R \langle m\rangle)(0 :_R \langle n\rangle) \subseteq (0 :_R \langle m\rangle) \cap (0 :_R \langle n\rangle) \subseteq (0 :_R \langle m\rangle + \langle n\rangle)$ then $(0 :_R \langle m\rangle) \subseteq (0 :_R \langle m\rangle + \langle n\rangle)$ or $(0 :_R \langle n\rangle) \subseteq (0 :_R \langle m\rangle + \langle n\rangle)$. So, we have $(0 :_R \langle m\rangle) = (0 :_R \langle m\rangle) \cap (0 :_R \langle m\rangle + \langle n\rangle) \subseteq (0 :_R \langle n\rangle)$ or $(0 :_R \langle n\rangle) = (0 :_R \langle n\rangle) \cap (0 :_R \langle m\rangle + \langle n\rangle) \subseteq (0 :_R \langle m\rangle)$. Thus, $\mathfrak{J}$ is a chain of prime ideals of R. Conversely, assume that $\mathfrak{J}$ is a chain prime ideal of R. It means that for each $0 \neq m \in M$, $(0 :_R \langle m\rangle)$ is a prime ideal of R. Thus, using Corollary 2.12, we have M a left weakly jointly prime (R, S)-module. □

Now, we recall from [7] that each summand of a weakly prime R-module is a weakly prime R-module. Next, we present the generalization of these properties to (R, S)-modules.

Proposition 2.14. *An (R, S)-module M is left weakly jointly prime if and only if every direct summand of M, including the zero summands, is a left weakly jointly prime (R, S)-submodule.*

Proof. Assume that $M = N \oplus K$. Let $a, b \in R$ and $x \in M \setminus N$ such that $ab\langle x\rangle S = 0$. Since M is a left weakly jointly prime (R, S)-module, then we have $a\langle x\rangle S = 0$ or $b\langle x\rangle S = 0$. Since $0 \subseteq N$, then for each $a, b \in R$ that satisfy $ab\langle x\rangle S \subseteq N$ implies either $a\langle x\rangle S \subseteq N$ or $b\langle x\rangle S \subseteq N$. Thus, N is a left weakly jointly prime (R, S)-submodule of M. Hence, every direct summand of M is a left weakly jointly prime (R, S)-submodule. Conversely, let $M = M \oplus \{0\}$. By our hypothesis, $\{0\}$ is a left weakly jointly prime (R, S)-submodule, i.e., M is a left weakly jointly prime (R, S)-module. □

From Proposition 2.14, we have that each direct summand of left weakly jointly prime (R, S)-modules is a left weakly jointly prime (R, S)-module. Therefore it is natural to consider (R, S)-modules which are not indecomposable and not left weakly jointly prime, but their non-zero summands are left weakly jointly prime (R, S)-modules.

Corollary 2.15. *Let M be an (R,S)-module with $S^2 = S$ and $a \in RaS$ for all $a \in M$, M is not a left weakly jointly prime (R,S)-module and not indecomposable. If every decomposition of M is of the form $M = N \oplus K$, where N and K are non-zero indecomposable left weakly jointly prime (R,S)-module, then every non-zero summand of M is left weakly jointly prime (R,S)-modules.*

Proof. Assume that $M = N \oplus K$ where N, K are non-zero indecomposable left weakly jointly prime (R,S)-submodule. Following Proposition 2.14, the non-zero summand of M is a left weakly jointly prime (R,S)-module. □

3. Fully left weakly jointly prime (R,S)-modules

In this section, we present the definition of fully left weakly jointly prime (R,S)-modules and give some of their properties. We recall the definition of the fully prime ring and fully weakly prime modules as follows.

Definition 3.1. [8] Let R be an associative ring with identity. A ring R is called a fully prime ring if each proper ideal of R is prime.

Definition 3.2. [7] Let R be an associative ring with identity. An R-module M is a fully weakly prime module if every proper submodule of M is a weakly prime submodule.

We extend the definition of fully weakly prime modules to (R,S)-modules as follows.

Definition 3.3. An (R,S)-module M is called fully left weakly jointly prime if every proper (R,S)-submodule of M is left weakly jointly prime.

Example 3.4. Let $\mathbb{Z}_6$ be a $(2\mathbb{Z},\mathbb{Z})$-module. The proper $(2\mathbb{Z},\mathbb{Z})$-submodules of $\mathbb{Z}_6$ are $\{\bar{0}\}$, $\{\bar{0},\bar{2},\bar{4}\}$, and $\{\bar{0},\bar{3}\}$. Those proper submodules are left weakly jointly prime $(2\mathbb{Z},\mathbb{Z})$-submodules. Thus, $\mathbb{Z}_6$ is a fully left weakly jointly prime $(2\mathbb{Z},\mathbb{Z})$-module.

Example 3.5. Let $4\mathbb{Z}$ be a $(2\mathbb{Z},3\mathbb{Z})$-module. The proper $(2\mathbb{Z},3\mathbb{Z})$-submodules of $4\mathbb{Z}$ are $\{\bar{0}\}$ and $(4n)\mathbb{Z}$ for all $n > 1$. Both of them are left weakly jointly prime $(2\mathbb{Z},3\mathbb{Z})$-submodules. Thus, $4\mathbb{Z}$ is a fully left weakly jointly prime $(2\mathbb{Z},3\mathbb{Z})$-module.

Example 3.6. Following to Example 2.6, M is not a fully left weakly jointly prime (R,S)-module since $\left\{\begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix}\right\}$ is not a left weakly jointly prime (R,S)-submodule of M.

Next, this proposition below shows that fully prime rings give us a big set of fully left weakly jointly prime (R,S)-modules.

Proposition 3.7. *Let R be a fully prime ring. Then each (R, S)-module M with $S^2 = S$ and $a \in RaS$ for all $a \in M$ is fully left weakly jointly prime (R, S)-module.*

Proof. Let R be a fully prime ring. We will show that M is a fully left weakly jointly prime (R, S)-module. Let N be a proper (R, S)-submodule of M. We will prove that N is a left weakly jointly prime (R, S)-submodule of M. Let K be an (R, S)-submodule of M that is not contained in N, so we have $(N :_R K)$ is an ideal of R. Since $a \in RaS$ for all $a \in M$, then $(N :_R K)$ is a proper ideal of R. Since R is a fully prime ring, $(N :_R K)$ is a prime ideal of R. Using Proposition 2.8, N is a left weakly jointly prime (R, S)-submodule of M. □

Proposition 3.8. *Let M be an (R, S)-module with $S^2 = S$ and $a \in RaS$ for all $a \in M$. Then the following statements are equivalent. (i) Each (R, S)-module M is fully left weakly jointly prime.*

(ii) If $R^2 = R$ and $a \in RaR$ for all $a \in R$, then the (R, R)-module R is fully left weakly jointly prime.

(iii) R is a fully prime ring.

Proof. $(i) \Rightarrow (ii)$. The proof is trivial.

$(ii) \Rightarrow (iii)$. Let N be an ideal of R. Then N is an (R, R)-submodule of R. Let $a, b \in R$ with $ab \in N$. Thus, we get $abR \subseteq NR \subseteq N$ so $abRR \subseteq NR \subseteq N$. Since N is an (R, R)-submodule of R and R is fully left weakly jointly prime, we have N is a left weakly jointly prime (R, R)-submodule. Consequently, we obtain $aRR \subseteq N$ or $bRR \subseteq N$. Since R is commutative and $a, b \in R$ then we have $RaR \subseteq N$ or $RbR \subseteq N$. Since $a \in RaR$ for all $a \in R$, then we obtain $a \in N$ or $b \in N$. Thus, N is a prime ideal of R.

$(iii) \Rightarrow (i)$. The proof is equal to the proof of Proposition 3.7. □

The following result extends a fact of fully prime rings to fully left weakly jointly prime (R, S)-modules. This result is based on papers [8, 15].

Proposition 3.9. *Let M be an (R, S)-module with $a \in RaS$ for all $a \in M$. Then M is a fully left weakly jointly prime (R, S)-module if and only if for each (R, S)-submodule K of M and each ideal I of R, $IKS = I^2KS$ and also for any two ideals A and B of R satisfy AKS and BKS are comparable.*

Proof. Let M be a fully left weakly jointly prime (R, S)-module. Let K be an (R, S)-submodule of M and I be an ideal of R. If $I^2KS = M$, then clearly $M = I^2KS \subseteq IKS \subseteq M$ so that $M = I^2KS = IKS$. Thus, we may assume that $I^2KS \neq M$, then I^2KS is a left weakly jointly prime (R, S)-submodule. Consequently, from $IIKS \subseteq I^2KS$ implies that $IKS = I^2KS$.

Now, if A and B are two ideals of R, we may assume that $AKS \neq M \neq BKS$. However, $AKS \cap BKS$ is a left weakly jointly prime (R, S)-submodule of M. Consequently, from $ABKS \subseteq AKS \cap BKS$ implies that $AKS \subseteq BKS$ or $BKS \subseteq AKS$. Thus, it has proved that AKS and BKS are comparable. Conversely, we must show that each proper (R, S)-submodule N of M is a left weakly jointly prime (R, S)-submodule. To see this, let A and B be ideals of R and K be an (R, S)-submodule of M such that $ABKS \subseteq N$. By our hypothesis, we may assume that $AKS \subseteq BKS$. Then we obtain $AKS = A^2KS \subseteq ABKS \subseteq N$. Assuming that $BKS \subseteq AKS$ we get $BKS = B^2KS \subseteq BAKS = ABKS \subseteq N$. Hence, N is a left weakly jointly prime (R, S)-submodule. Thus, M is a fully left weakly jointly prime (R, S)-module. □

Next, we present the necessary and sufficient condition of fully left weakly jointly prime (R, S)-modules related to their cyclic (R, S)-submodules.

Proposition 3.10. *Let M be an (R, S)-module. Then M is a fully left weakly jointly prime if and only if each proper cyclic (R, S)-submodule of M is a left weakly jointly prime (R, S)-submodule.*

Proof. Let M be a fully left weakly jointly prime (R, S)-module. Thus, every proper (R, S)-submodule of M is a left weakly jointly prime (R, S)-submodule, and each proper cyclic (R, S)-submodule of M is a left weakly jointly prime. Conversely, assume that each proper cyclic (R, S)-submodule of M is a left weakly jointly prime. Let N be a proper (R, S)-submodule of M that is not cyclic. Let element $n \in N$, we can construct a proper cyclic (R, S)-submodule $\langle n \rangle$. Let $a, b \in R$ and (R, S)-submodule K of M with $abKS \subseteq \langle n \rangle$. Based on our hypothesis, $\langle n \rangle$ is a left weakly jointly prime (R, S)-submodule. Then we have $aKS \subseteq \langle n \rangle$ or $bKS \subseteq \langle n \rangle$. Since $\langle n \rangle$ is contained in N, then from $abKS \subseteq \langle n \rangle \subseteq N$ we have $aKS \subseteq N$ or $bKS \subseteq N$. Thus, it is proved that N a left weakly jointly prime (R, S)-submodule of M. Hence, M is a fully left weakly jointly prime (R, S)-module. □

We recall the following properties, a left weakly jointly prime (R, S)-submodule N of M is said to be minimal if it is minimal in the class of left weakly jointly prime (R, S)-submodules of M. Moreover, an (R, S)-module M satisfies the minimum condition if every non-empty family of (R, S)-submodules of M contains a minimal number.

According to [11], every prime submodule contains a minimal prime submodule. Based on this property, it is easy to show that every left weakly jointly prime (R, S)-submodule of M contains a minimal left weakly jointly prime (R, S)-submodule. Using this property, we present the last property of fully left weakly jointly prime (R, S)-modules as follows.

Proposition 3.11. *Let M be an (R,S)-module. If M is a fully left weakly jointly prime (R,S)-module, then M is a left weakly jointly prime (R,S)-module, and the set of proper cyclic (R,S)-submodules satisfies the minimum condition.*

Proof. Since M is a fully left weakly jointly prime (R,S)-module then (R,S)-submodule 0 is left weakly jointly prime. Therefore, M is a left weakly jointly prime (R,S)-module. Moreover, based on Proposition 3.10, we have that every proper cyclic (R,S)-submodule is left weakly jointly prime. Based on [Proposition 3.10, [16]], we have every cyclic (R,S)-submodule of M contains a minimal left weakly jointly prime (R,S)-submodule. Therefore, the set of proper cyclic (R,S)-submodules of M contains a minimal number. Hence, the proper cyclic (R,S)-submodules set satisfies the minimum condition. □

4. Conclusion

Further work on the properties of left weakly jointly prime (R,S)-submodules can be carried out. For example, research on the radical structure of the left weakly jointly prime (R,S)-modules and the dualization of the left weakly jointly prime (R,S)-modules.

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References

[1] H. Ansari-Toroghy and F. Farshadifar, *The dual notions of some generalizations of prime submodules*, Commun. Algebra, **39** No. 7 (2011) 2396-2416.

[2] S. E. Atani and F. Farzalipour, *On weakly prime submodules*, Tamkang J. Math., **38** No. 3 (2007) 247-252.

[3] A. Azizi, *Weakly prime submodules and prime modules*, Glasg. Math. J., **48** No. 2 (2006) 343-346.

[4] A. Azizi, *Radical formula and weakly prime submodules*, Glasg. Math. J., **51** No. 2 (2007) 405-412.

[5] A. Azizi, *On prime and weakly prime submodules*, Vietnam J. Math., **36** No. 3 (2008) 315-325.

[6] M. Behboodi, *On weakly prime radical of modules and semi-compatible modules*, Acta Math. Hung., **113** No. 3 (2006) 243-254.

[7] M. Behboodi and H. Koohy, *Weakly prime modules*, Vietnam J. Math., **32** No. 2 (2004) 185-195.

[8] W. D. Blair and H. Tsutsui, *Fully prime rings*, Comm. Algebra, **22** No. 13 (1994) 5389-5400.

[9] J. Dauns, *Prime modules*, J. fur Reine Angew. Math., **298** (1978) 156-181.

[10] T. Duraivel, S. Mangayarcarassy and K. Premkumar, *Prime extension dimension of a module*, J. Algebra Relat. Top., **6** No. 2 (2018) 97-106.

[11] K. R. Goodearl and R. B. Warfiled, *An Introduction to Noncommutative Notherian Rings*, Cambridge University Press, New York, 2004.

[12] A. Hajikarimi, *Modules whose nonzero finitely generated submodules are dense*, ASTA, **8** No. 1 (2021) 89-97.

[13] T. Khumprapussorn, S. Pianskool, and M. Hall, (R, S)*-modules and their fully and jointly prime submodules*, Int. Math. Forum, **7** No. 33 (2012) 163-1643.

[14] N. V. Sanh, A. V. Nguyen, K. F. U. Ahmed, S. Asawasamrit and L. P. Thao, *Primeness in module category*, Asian-Eur. J. Math., **3** No. 1 (2010) 145-154.

[15] T. Tsutsui, *Fully prime rings II*, Comm. Algebra, **24** No. 9 (1996) 2981-2989.

[16] D. A. Yuwaningsih, *Some properties of left weakly jointly prime* (R, S)*-submodules*, J. Indones. Math. Soc., **26** No. 2 (2020) 234-241.

[17] D. A. Yuwaningsih, I. E. Wijayanti and P. W. Prasetyo, *On* (R, S)*-module homomorphisms*, J. Phys. Conf. Ser., **1188** No. 1 (2019) 012114.

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