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Research Paper

CONNECTIONS BETWEEN GE ALGEBRAS AND PRE-HILBERT ALGEBRAS

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Abstract. GE algebras (generalized exchange algebras) and pre-Hilbert algebras are a generalization of well-known Hilbert algebras. In the paper, connections between these algebras are studied. In particular, it is proven that pi-BE algebras are equivalent with GE algebras satisfying the exchange property. Some characterizations of transitive GE algebras and exchange pre-Hilbert algebras are given. It is shown that the intersection of classes of GE algebras and pre-Hilbert algebras is the class of transitive GE algebras. Moreover, GE, BE and pre-Hilbert algebras with the antisymmetry property are investigated. It is proven that transitive GE algebras satisfying the property of antisymmetry coincide with Hilbert algebras. Finally, positive implicative GE and pre-Hilbert algebras are considered, their connections with some algebras of logic are presented. In addition, the hiearchies existing between the classes of algebras studied here are shown.

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1. Introduction

L. Henkin [8] introduced the notion of "implicative model", as a model of positive implicative propositional calculus. In 1960, Monteiro [15] has given the name "Hilbert algebras" to the dual algebras of Henkin's implicative models. In 1966, Iséki [11] introduced a new notion called a BCK algebra. It is an algebraic formulation of the BCK-propositional calculus system of Meredith [14], and generalize the concept of implicative algebras (see [1]). In 2021, Cintula and Noguera [6] presented of the most important logics that one can find in the literature. In particular, they considered the *$\mathcal{BCK}$ logic* and its many extensions. In [13], as a generalization of BCK algebras, Kim and Kim introduced BE algebras. Rezaei et al. [16] investigated connections between Hilbert algebras and BE algebras. In 2008, Walendziak [20] defined commutative BE algebras and proved that they are BCK algebras. Later on, in 2010, Buşneag and Rudeanu [5] introduced the notion of pre-BCK algebra. A BCK algebra is just a pre-BCK algebra satisfying also the antisymmetry. In 2016, Iorgulescu [9] introduced new generalizations of BCK and Hilbert algebras (RML, aBE, pi-BE, pimpl-RML algebras and many others). Bandaru et al. [2] introduced the concepts of GE algebra (generalized exchange algebra) and transitive GE algebra (tGE algebra for short). They studied the various properties and filter theory of GE algebras. Some types of filters (imploring, prominent, belligerent, interior, voluntary GE-filters) have been widely investigated and many important results are obtained (see [18, 17, 3, 19, 4]). Recently, as a natural generalization of Hilbert algebras, Walendziak [22] introduced pre-Hilbert algebras (the definition of a pre-Hilbert algebra is inspired by Henkin's Positive Implicative Logic [8]) and studied their properties.

In the paper, we consider connections between GE algebras, BE algebras and pre-Hilbert algebras. In particular, we prove that pi-BE algebras are equivalent with GE algebras satisfying the exchange property. We give some characterizations of transitive GE algebras and show that the class of these algebras is the intersection of classes of GE algebras and pre-Hilbert algebras. We introduce and study exchange pre-Hilbert algebras. Moreover, we consider GE, BE and pre-Hilbert algebras with the antisymmetry property and give some examples. We also prove that tGE algebras verifying the property of antisymmetry are equivalent with Hilbert algebras. Finally, we present the interrelationships between the classes of algebras considered here.

The motivation of this study consists algebraic and logical arguments. Namely, generalized exchange algebras and pre-Hilbert algebras are related to Henkin's Positive Implicative Logic, they belong to a wide class of algebras of logic. An additional motivation is the fact that the present paper is a continuation of previous papers: [2, 21] on GE algebras and [22] on pre-Hilbert algebras. Moreover, the results of the paper may have applications for future studies of the relationships between some generalizations of Hilbert algebras.

2. Preliminaries

Let $\mathcal{A} = (A, \rightarrow, 1)$ be an algebra of type $(2, 0)$. We consider the following list of properties ([9]) that can be satisfied by $\mathcal{A}$ (the properties in the list are the most important properties satisfied by a Hilbert algebra):

(An) (Antisymmetry) $x \rightarrow y = 1 = y \rightarrow x \Longrightarrow x = y$,

(B) $(y \rightarrow z) \rightarrow [(x \rightarrow y) \rightarrow (x \rightarrow z)] = 1$,

(C) $[x \rightarrow (y \rightarrow z)] \rightarrow [y \rightarrow (x \rightarrow z)] = 1$,

(Ex) (Exchange) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,

(GE) (Generalized exchange) $x \rightarrow (y \rightarrow z) = x \rightarrow [y \rightarrow (x \rightarrow z)]$,

(K) $x \rightarrow (y \rightarrow x) = 1$,

(L) (Last element) $x \rightarrow 1 = 1$,

(M) $1 \rightarrow x = x$,

(Re) (Reflexivity) $x \rightarrow x = 1$,

(Tr) (Transitivity) $x \rightarrow y = 1 = y \rightarrow z \Longrightarrow x \rightarrow z = 1$,

(pi) $x \rightarrow y = x \rightarrow (x \rightarrow y)$,

(p-1) $x \rightarrow (y \rightarrow z) \leq (x \rightarrow y) \rightarrow (x \rightarrow z)$,

(p-2) $(x \rightarrow y) \rightarrow (x \rightarrow z) \leq x \rightarrow (y \rightarrow z)$,

(pimpl) $x \rightarrow (y \rightarrow z) = (x \rightarrow y) \rightarrow (x \rightarrow z)$.

Lemma 2.1. *Let $\mathcal{A} = (A, \rightarrow, 1)$ be an algebra of type $(2, 0)$. Then the following hold:*

(i) *(M) + (B) imply (Re), (Tr);*

(ii) *(An) + (C) imply (Ex);*

(iii) *(Re) + (L) + (Ex) imply (K);*

(iv) *(M) + (L) + (B) imply (K);*

(v) *(Re) + (M) + (pimpl) imply (pi);*

(vi) *(Ex) + (pi) imply (GE);*

(vii) *(B) + (Ex) + (pi) imply (p-1).*

Proof. (i) – (v) follow from Propositions 2.1 and 6.4 of [9].

(vi) follows from Proposition 2.7 of [21].

(vii). Let $x, y, z \in A$. By (Ex), $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$. Applying (B) and (pi), we get $y \rightarrow (x \rightarrow z) \leq (x \rightarrow y) \rightarrow [x \rightarrow (x \rightarrow z)] = (x \rightarrow y) \rightarrow (x \rightarrow z)$. Then $x \rightarrow (y \rightarrow z) \leq (x \rightarrow y) \rightarrow (x \rightarrow z)$, that is, (p-1) holds. □

Following Iorgulescu [9], we say that $(A, \rightarrow, 1)$ is an *RML algebra* if it verifies the axioms (Re), (M), (L). We recall now the following definition.

Definition 2.2. ([9]) Let $\mathcal{A} = (A, \to, 1)$ be an RML algebra. The algebra $\mathcal{A}$ is said to be:

1. a *BE algebra* if it verifies (Ex),

2. an *aRML algebra* if it verifies (An),

3. an *aBE algebra* if it verifies (Ex) and (An), that is, it is a BE algebra with (An),

4. a *pre-BCK algebra* if it verifies (B) and (Ex), that is, it is a BE algebra with (B),

5. a *BCK algebra* if it is a pre-BCK algebra verifying (An).

Denote by **RML**, **BE**, **aRML**, **aBE**, **pre-BCK** and **BCK** the classes of RML, BE, aRML, aBE, pre-BCK and BCK algebras, respectively.

By definition, **BE** = **RML** + (Ex), **aRML** = **RML** + (An), **aBE** = **BE** + (An), **pre-BCK** = **BE** + (B), **BCK** = **pre-BCK** + (An).

Let $\mathcal{A} = (A, \to, 1)$ be an algebra of type $(2, 0)$. We define the binary relation $\leq$ by: for all $x, y \in A$,

$$x \leq y \Longleftrightarrow x \to y = 1.$$

It is known that $\leq$ is an order relation in BCK algebras. By definition, in RML and BE algebras, $\leq$ is a reflexive relation; in aRML and aBE algebras, $\leq$ is reflexive and antisymmetric. Since (M) + (B) imply (Tr), see Lemma 2.1 (i), in pre-BCK algebras, $\leq$ is reflexive and transitive (i.e., it is a pre-order relation).

Definition 2.3. ([2]) Let $\mathcal{A} = (A, \to, 1)$ be an algebra of type *(2,0)*. We say that $\mathcal{A}$ is:

6. a GE algebra (generalized exchange algebra) if it verifies (Re), (M), (GE);

7. a transitive GE algebra (tGE algebra, for short) if it is a GE algebra verifying (B);

8. an antisymmetric GE algebra (aGE algebra, for short) if it is a GE algebra verifying (An).

Denote by **GE**, **tGE** and **aGE** the classes of all GE algebras, transitive GE algebras and aGE algebras, respectively.

Proposition 2.4. *(Corollary 3.2 of* [21]*) Any GE algebra satisfies (Re), (M), (L), (C), (K), (GE), (pi).*

Remark 2.5. Since GE algebras satisfy (L), we get **GE** = **RML** + (GE). By definition, **tGE** = **GE** + (B) and **aGE** = **GE** + (An).

Remark 2.6. GE algebras do not have to satisfy (An), (B), (Tr), (Ex), (p-1), (pimpl); see example below.

Example 2.7. ([21]) Consider the set $A = \{a, b, c, d, e, 1\}$ and the operation $\to$ given by the following table:

$\to$	a	b	c	d	e	1
a	1	1	c	c	1	1
b	a	1	d	d	1	1
c	a	1	1	1	1	1
d	a	1	1	1	1	1
e	a	1	1	1	1	1
1	a	b	c	d	e	1

We can observe that the properties (Re), (M), (L), (GE) (hence (pi)) are satisfied. Therefore, $(A, \to, 1)$ is a GE algebra. It does not satisfy (An) for $(x, y) = (c, d)$; (Ex) for $(x, y, z) = (a, b, c)$; (Tr), (B), (p-1), (pimpl) for $(x, y, z) = (a, e, c)$.

Proposition 2.8. *(Corollary 3.12 of* [21]*) Any transitive GE algebra verifies (Re), (M), (L), (B), (Tr), (GE), (pi), (p-1), (p-2).*

Remark 2.9. Transitive GE algebras do not have to verify (An), (Ex), (pimpl); see example below.

Example 2.10. Let $A = \{a, b, c, 1\}$ and $\to$ be defined as follows:

$\to$	a	b	c	d	1
a	1	1	c	c	1
b	a	1	d	d	1
c	a	1	1	1	1
d	a	1	1	1	1
1	a	b	c	d	1

It is easy to see that $\mathcal{A} = (A, \to, 1)$ is a transitive GE algebras. It does not verify (An) for $x = c$, $y = d$; (Ex) and (pimpl) for $x = b$, $y = a$, $z = c$.

In [22], we introduced the following notion:

Definition 2.11. A pre-Hilbert algebra is an algebra $(A, \to, 1)$ of type $(2, 0)$ satisfying (M), (K) and (p-1).

Recall that an algebra $(A, \to, 1)$ is a *Hilbert algebra* ([7]) if it verifies the axioms (An), (K), (p-1).

Remark 2.12. In [7], A. Diego proved that Hilbert algebras satisfy (Re), (M), (L), (B), (Ex), (pi), (p-2), (pimpl). Moreover, he showed that the class of all Hilbert algebras is a variety.

Let us denote by **pre-H** and **H** the classes of pre-Hilbert and Hilbert algebras, respectively.

Remark 2.13. Since (An) + (K) + (p-1) imply (M) (see [7]), a Hilbert algebra is in fact a pre-Hilbert algebra verifying (An), that is, $\mathbf{H} = \mathbf{pre} - \mathbf{H} + (An)$.

Proposition 2.14. *(Theorem 3.9 of* [22]*) Pre-Hilbert algebras satisfy (Re), (M), (L), (K), (Tr), (B), (C), (p-1), (p-2).*

Proposition 2.15. *Let* $\mathcal{A} = (A, \to, 1)$ *be an algebra of type* $(2, 0)$. *The following are equivalent:*

(i) $\mathcal{A}$ *is a pre-Hilbert algebra;*
(ii) $\mathcal{A}$ *satisfies (M), (L), (B) and (p-1).*

Proof. (i) $\Longrightarrow$ (ii). Follows from Proposition 2.14.
(ii) $\Longrightarrow$ (i). By Lemma 2.2 (iv), (M) + (L) + (B) imply (K). Then $\mathcal{A}$ satisfies (M), (K) and (p-1). Thus $\mathcal{A}$ is a pre-Hilbert algebra. $\square$

Remark 2.16. From Proposition 2.15 we see that $\mathbf{pre} - \mathbf{H} = \mathbf{RML} + (B) + (p-1)$.

Remark 2.17. Pre-Hilbert algebras do not have to satisfy (An), (Ex), (GE), (pi), (pimpl); see example below.

Example 2.18. Consider the set $A = \{a, b, c, d, 1\}$ and the operation $\to$ given by the following table:

$\to$	a	b	c	d	1
a	1	c	b	d	1
b	a	1	1	d	1
c	a	1	1	d	1
d	a	c	c	1	1
1	a	b	c	d	1

We can observe that the properties (M), (K), (p-1) are verified. Then, $(A, \to, 1)$ is a pre-Hilbert algebra. It does not verify (An) for $(x, y) = (b, c)$; (Ex) and (pimpl) for $(x, y, z) = (a, d, b)$; (pi) for $(x, y) = (a, b)$; (GE) for $(x, y, z) = (a, 1, b)$.

Remark 2.19. It is easy to see that in GE algebras, $\leq$ is a reflexive relation; in tGE and pre-Hilbert algebras, $\leq$ is reflexive and transitive (i.e., it is a pre-order relation), in aGE algebras, $\leq$ is reflexive and antisymmetric. In Hilbert algebras, $\leq$ is an order relation.

3. Connections between GE, BE algebras and pre-Hilbert algebras

We recall the following definitions:

Definition 3.1. ([9]) **1.** A pi-RML algebra is an RML algebra verifying property (pi).
2. A positive implicative RML algebra, or a pimpl-RML algebra for short, is an RML algebra verifying property (pimpl).

Denote by **pi-RML** and **pimpl-RML** the classes of pi-RML and pimpl-RML algebras, respectively; similarly for the subclasses of **RML**.

By definition, **pi-RML** = **RML** + (pi), **pi-BE** = **BE** + (pi), **pi-pre-BCK** = **pre-BCK** + (pi) = **pi-BE** + (B), **pimpl-RML** = **RML** + (pimpl).

Remark 3.2. From Lemma 2.1 (v) it follows that in RML algebras, (pimpl) implies (pi). For BCK algebras, (pimpl) and (pi) are equivalent (cf. Theorem 8 of [12]).

By definition and Proposition 2.4, we have

Proposition 3.3. *Any GE algebra is a pi-RML algebra.*

From Lemma 2.1 (vi) we get

Proposition 3.4. *Any pi-BE algebra is a GE algebra.*

Proposition 3.5. *Let* $\mathcal{A} = (A, \to, 1)$ *be an algebra of type* $(2, 0)$. *The following statements are equivalent:*

(i) $\mathcal{A}$ *is a pi-BE algebra,*
(ii) $\mathcal{A}$ *is a GE algebra satisfying (Ex).*

Proof. (i) $\Longrightarrow$ (ii). Let $\mathcal{A}$ be a pi-BE algebra. It is sufficient to show that $\mathcal{A}$ satisfies (GE). Let $x, y, z \in A$. We obtain

$$x \to (y \to z) \overset{\text{(pi)}}{=} x \to [x \to (y \to z)] \overset{\text{(Ex)}}{=} x \to [y \to (x \to z)].$$

Thus (GE) holds.
(ii) $\Longrightarrow$ (i). Since GE algebras satisfy (pi), we see that (ii) implies (i). $\square$

Now we give a characterization of transitive GE algebras.

Theorem 3.6. *Let* $\mathcal{A} = (A, \to, 1)$ *be an algebra of type* $(2, 0)$. *The following are equivalent:*

(i) $\mathcal{A}$ *is a transitive GE algebra;*
(ii) $\mathcal{A}$ *satisfies (M), (GE), (B);*
(iii) $\mathcal{A}$ *satisfies (Re), (M), (GE) and (p-1);*
(iv) $\mathcal{A}$ *is a pre-Hilbert algebra with (GE).*

Proof. (i) $\Longrightarrow$ (ii). Obviously, by definition.
(ii) $\Longrightarrow$ (iii). By Lemma 2.1 (i), (M) + (B) imply (Re). Consequently, $\mathcal{A}$ is a tGE algebra. From Proposition 2.8 it follows that $\mathcal{A}$ satisfies (p-1).
(iii) $\Longrightarrow$ (iv). Let $\mathcal{A}$ satisfy (Re), (M), (GE) and (p-1). Then $\mathcal{A}$ is a GE algebra. By Proposition 2.4, $\mathcal{A}$ satisfies (K). Therefore, (M), (K), (p-1) and (GE) hold in $\mathcal{A}$. Thus $\mathcal{A}$ is a pre-Hilbert algebra with (GE).
(iv) $\Longrightarrow$ (i). Since pre-Hilbert algebras satisfy (B), we conclude that (iv) implies (i). $\square$

From Theorem 3.6 we have

Corollary 3.7. $\mathbf{tGE} = \mathbf{GE} \cap \mathbf{pre-H}$.

We now introduce a new algebra. We say that an algebra $\mathcal{A} = (A, \to, 1)$ is an *exchange pre-Hilbert algebra* if it is a pre-Hilbert algebra verifying (Ex).

Denote by **Ex-pre-H** the class of exchange pre-Hilbert algebras. By definition, **Ex-pre-H** = **pre-H** + (Ex).

Example 3.8. (Example 8.20 of [10]) Consider the set $A = \{a, b, c, 1\}$ with the following table of $\to$:

$\to$	a	b	c	1
a	1	a	a	1
b	1	1	1	1
c	1	1	1	1
1	a	b	c	1

The algebra $\mathcal{A} = (A, \to, 1)$ verifies (Re), (M), (L), (Ex), (B), (p-1). It does not verify (An) for $x = b, y = c$; (pi) for $x = a, y = b$. Hence, $\mathcal{A}$ is an exchange pre-Hilbert algebra, without (pi).

Proposition 3.9. *Let $\mathcal{A} = (A, \to, 1)$ be an algebra of type $(2, 0)$. The following statements are equivalent:*

(i) *$\mathcal{A}$ is an exchange pre-Hilbert algebra,*
(ii) *$\mathcal{A}$ is a pre-BCK algebra with (p-1),*
(iii) *$\mathcal{A}$ is a BE algebra satisfying (p-1).*

Proof. (i) $\Longrightarrow$ (ii) and (ii) $\Longrightarrow$ (iii) are obvious.
(iii) $\Longrightarrow$ (i). By Lemma 2.1 (iii), (Re) + (L) + (Ex) imply (K). Then $\mathcal{A}$ satisfies (M), (K), (p-1), (Ex). Thus $\mathcal{A}$ is an exchange pre-Hilbert algebra. $\square$

Example 3.10. (Example 9.35 of [10]) Let $A = \{a, b, c, 1\}$ and $\to$ be defined as follows:

$\rightarrow$	a	b	c	1
a	1	a	b	1
b	1	1	1	1
c	1	1	1	1
1	a	b	c	1

Properties (Re), (M), (L), (Ex) hold in $\mathcal{A} = (A, \rightarrow, 1)$. Hence, $\mathcal{A}$ is a BE algebra. It does not verify (An) for $x = b, y = c$; (B) and (p-1) for $x = a, y = b, z = c$; (pi) for $x = a, y = b$. Therefore $\mathbf{Ex - pre - H} \subset \mathbf{BE}$ and $\mathbf{pi - BE} \subset \mathbf{BE}$.

Remark 3.11. (1) By Propositions 3.5, 3.6 and 3.9, $\mathbf{pi - BE} = \mathbf{GE} + (Ex)$, $\mathbf{tGE} = \mathbf{pre - H} + (GE)$, $\mathbf{Ex - pre - H} = \mathbf{pre - BCK} + (p-1) = \mathbf{BE} + (p-1)$.
(2) Hence $\mathbf{Ex - pre - H} + (pi) = \mathbf{pre - BCK} + (p-1) + (pi) = \mathbf{pi - pre - BCK}$, since (B) + (Ex) + (pi) imply (p-1), see Lemma 2.1 (vii).
(3) We have $\mathbf{tGE} + (Ex) = \mathbf{GE} + (B) + (Ex) = \mathbf{pi - BE} + (B) = \mathbf{pi - pre - BCK}$.

The interrelationships between the classes of algebras mentioned before are visualized in Figure 1 (see Remarks 2.5, 2.16, 3.11).

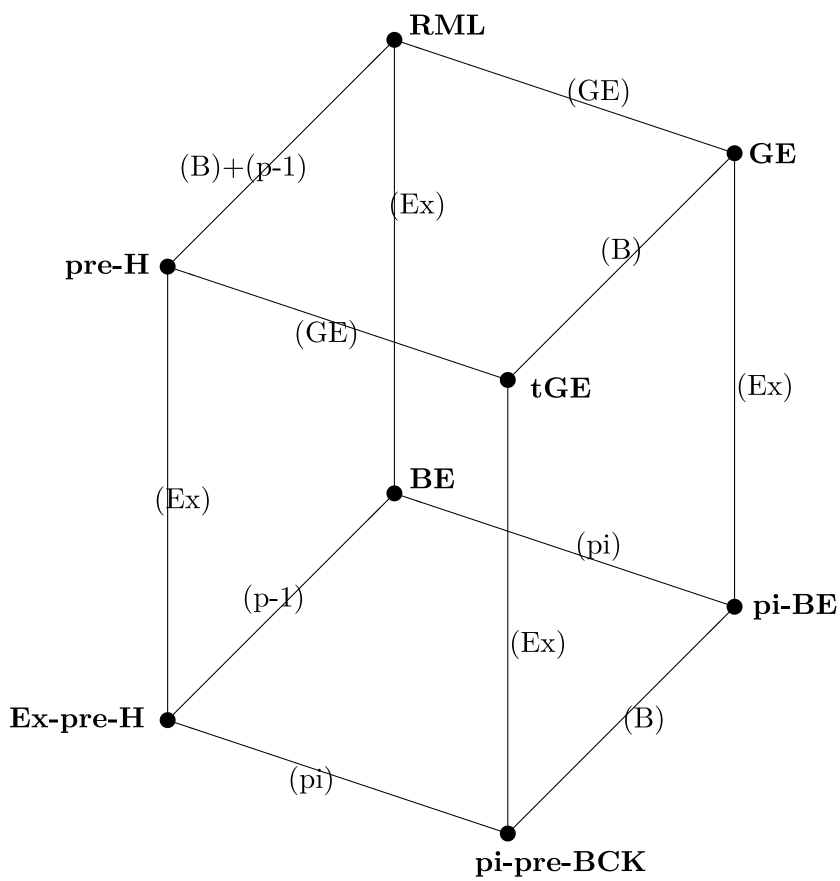

FIGURE 1.

We now consider GE, BE and pre-Hilbert algebras with the antisymmetry. First we give a characterization of antisymmetric GE algebras.

Proposition 3.12. *Let $\mathcal{A} = (A, \to, 1)$ be an algebra of type $(2, 0)$. The following statements are equivalent:*

(i) *$\mathcal{A}$ is an aGE algebra,*

(ii) *$\mathcal{A}$ is a pi-aBE algebra,*

(iii) *$\mathcal{A}$ is an aRML algebra with (GE).*

Proof. (i) $\Longrightarrow$ (ii). Let $\mathcal{A}$ be a GE algebra with (An). From Proposition 2.4 it follows that $\mathcal{A}$ satisfies (C) and (pi). By Lemma 2.1 (ii), (An) + (C) imply (Ex). Therefore $\mathcal{A}$ is a pi-aBE algebra.

(ii) $\Longrightarrow$ (iii). To prove (GE), let $x, y, z \in A$. We obtain

$$x \to (y \to z) \overset{\text{(pi)}}{=} x \to [x \to (y \to z)] \overset{\text{(Ex)}}{=} x \to [y \to (x \to z)].$$

Thus (GE) holds. Hence $\mathcal{A}$ is an aRML algebra with (GE).

(iii) $\Longrightarrow$ (i). It is obvious. □

Remark 3.13. By Proposition 3.12, $\mathbf{aGE} = \mathbf{pi} - \mathbf{aBE}$.

Proposition 3.14. *Let $\mathcal{A} = (A, \to, 1)$ be an algebra of type $(2, 0)$. The following statements are equivalent:*

(i) *$\mathcal{A}$ is a pre-Hilbert algebra with (An),*

(ii) *$\mathcal{A}$ is a Hilbert algebra,*

(iii) *$\mathcal{A}$ is a pi-aBE algebra with (B),*

(iv) *$\mathcal{A}$ is an aGE algebra with (B),*

(v) *$\mathcal{A}$ is a tGE algebra with (An).*

Proof. (i) $\Longleftrightarrow$ (ii), by Remark 2.13.

(ii) $\Longrightarrow$ (iii). Let $\mathcal{A}$ be a Hilbert algebra. Then $\mathcal{A}$ satisfies (An), (Re), (M), (L), (Ex), (B), (pi). Hence $\mathcal{A}$ is a pi-aBE algebra with (B).

(iii) $\Longrightarrow$ (ii). By Lemma 2.1 (iii) and (vii), (Re) +(L) + (Ex) imply (K) and (B) + (Ex) + (pi) imply (p-1). Hence (iii) forces (ii).

(iii) $\Longleftrightarrow$ (iv), by Remark 3.13.

(iv) $\Longleftrightarrow$ (v), by definition. □

Remark 3.15. From Proposition 3.14 it follows that $\mathbf{H} = \mathbf{pre}-\mathbf{H}+(An) = \mathbf{pi}-\mathbf{aBE}+(B) = \mathbf{aGE} + (B) = \mathbf{tGE} + (An)$.

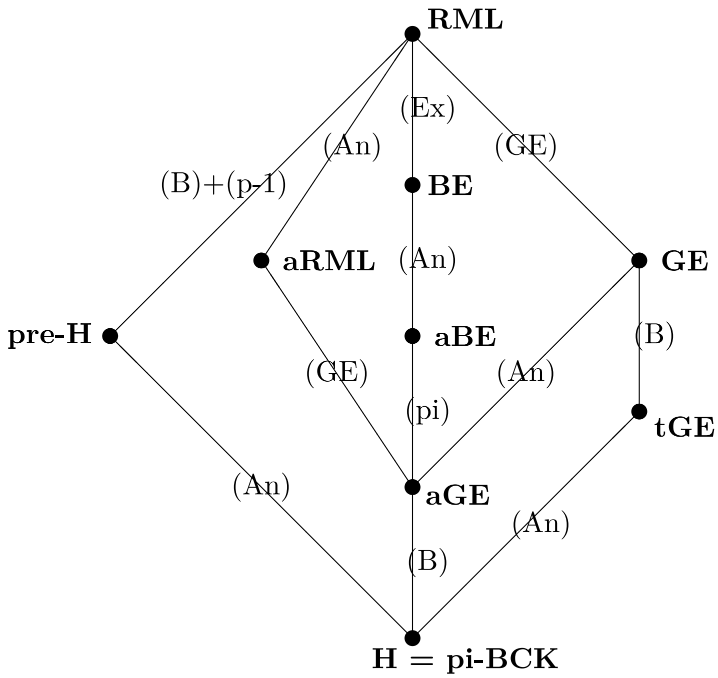

Figure 2.

By Definitions and Remarks 3.13 and 3.15, we can draw Figure 2.

Remark 3.16. (1) Positive implicative GE algebras were studied in [21]. By Remark 4.9 of [21], $\mathbf{pimpl-RML} = \mathbf{pimpl-GE} = \mathbf{pimpl-tGE}$.
(2) Positive implicative pre-Hilbert algebras were studied in [22]. By Proposition 4.7 of [22], $\mathbf{pimpl{-}pre{-}H} = \mathbf{pimpl{-}RML}$. Therefore $\mathbf{pimpl{-}pre{-}H} = \mathbf{pimpl{-}GE} = \mathbf{pimpl{-}tGE}$.
(3) From Corollary 6.19 of [9] we have $\mathbf{pimpl-BE} = \mathbf{pimpl-pre-BCK} = \mathbf{pimpl-RML} + (Ex)$ and $\mathbf{H} = \mathbf{pimpl-BCK} = \mathbf{pimpl-aRML} = \mathbf{pimpl-aBE}$.
(4) From the above we see that $\mathbf{H} \subset \mathbf{pimpl-BE} \subset \mathbf{pimpl-GE} = \mathbf{pimpl-pre-H}$.

4. Summary and future work

In this paper, we studied connections between BE algebras, GE algebras and pre-Hilbert algebras. We proved that GE algebras satisfying the exchange property are equivalent with pi-BE algebras and showed that the class of transitive GE algebras is the intersection of classes of GE algebras and pre-Hilbert algebras. We introduced exchange pre-Hilbert algebras and gave some examples and characterizations of these algebras. Moreover, we considered GE, BE and pre-Hilbert algebras verifying the antisymmetry property. In particular, we proved that tGE algebras verifying (An) are equivalent with Hilbert algebras. Finally, we presented the interrelationships between the classes of algebras considered here.

The results obtained in the paper can be a starting point for future research. We suggest the following topics:

(1) Studying GE and pre-Hilbert algebras with the implicative property, that is, verifying the identity $(x \to y) \to x = x$.

(2) Describing the deductive systems, the congruences, the quotient algebras, etc. of pre-Hilbert algebras and GE algebras.

(3) Investigating more deeply the exchange pre-Hilbert algebras.

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