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Research Paper

PLANAR, OUTERPLANAR AND RING GRAPH OF THE INTERSECTION GRAPH

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Abstract. Let R be a commutative ring and M be an R-module. The M-intersection graph of ideals of R, denoted by $G_M(R)$ is a graph with the vertex set $I(R)^*$, and two distinct vertices I and J are adjacent if and only if $IM \cap JM \neq 0$. In this paper, we study $G_{R/J}(R/I)$, where I and J are ideals of R and $I \subseteq J$. We characterize all ideals I and J for which $G_{R/J}(R/I)$ is planar, outerplanar or ring graph.

1. Introduction

Let R be a commutative ring, and $I(R)^*$ be the set of all non-zero proper ideals of R. Recently, there has been considerable research done on associating graphs with rings, for instance see [1], [2], [5], [6] and [14]. Also, the intersection graphs of some algebraic structures such as groups, rings and modules have been studied by several authors, see [3, 4, 8, 9]. In [8], the intersection graph of ideals of R, denoted by $G(R)$, was introduced as the graph with

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vertices $I(R)^*$ and for distinct $I, J \in I(R)^*$, the vertices I and J are adjacent if and only if $I \cap J \neq 0$. Also, in [4], the intersection graph of submodules of an R-module M, denoted by $G(M)$, is defined to be the graph whose vertices are the non-zero proper submodules of M and two distinct vertices are adjacent if and only if they have non-zero intersection. In [12], the M intersection graph of ideals of R denoted by $G_M(R)$, is defined to be the graph with the vertex set $I(R)^*$, and two distinct vertices I and J are adjacent if and only if $IM \cap JM \neq 0$. Clearly, if $I = J$, then $G_{R/J}(R/I)$ is exactly the same as the intersection graph of ideals of R/I. This implies that $G_{R/J}(R/I)$ is a generalization of $G(R/I)$.

Now, we recall some definitions and notations on graphs. Let G be a graph with the vertex set $V(G)$ and the edge set $E(G)$. If $\{a, b\} \in E(G)$, we say a is adjacent to b and write $a — b$. If $|V(G)| \geq 2$, then a path from a to b is a series of adjacent vertices $a — x_1 — x_2 — \cdots — x_n — b$. For $a, b \in V(G)$ with $a \neq b$, $d(a, b)$ denotes the length of a shortest path from a to b. If there is no such path, then we define $d(a, b) = \infty$. We say that G is *connected* if there is a path between any two distinct vertices of G. A *cycle* is a path that begins and ends at the same vertex in which no edge is repeated and all vertices other than the starting and ending vertex are distinct. We denote the complete graph of order n by K_n. A graph is *bipartite* if its vertices can be partitioned into two disjoint subsets X_1 and X_2 such that each edge connects a vertex from X_1 to one from X_2. A bipartite graph is a *complete bipartite* graph if every vertex in X_1 is adjacent to every vertex in X_2. We denote the complete bipartite graph, with part sizes m and n by $K_{m,n}$. The disjoint union of graphs G_1 and G_2, which is denoted by $G_1 \cup G_2$, where G_1 and G_2 are two vertex-disjoint graphs, is a graph with $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. A graph G may be expressed uniquely as a disjoint union of connected graphs. These graphs are called the connected components, or simply the components, of G. Recall that a graph is said to be *planar* if it can be drawn in the plane so that its edges intersect only at their ends. A *subdivision* of a graph is any graph that can be obtained from the original graph by replacing edges by paths.

As usual, $\mathbb{Z}_m$ denotes the integers modulo m. The $\mathbb{Z}_n$ intersection graph of $\mathbb{Z}_m$, $G_n(\mathbb{Z}_m)$ was studied in [13], where $n, m > 1$ are integers and $\mathbb{Z}_n$ is a $\mathbb{Z}_m$-module. For instance, the values of n and m for which $G_n(\mathbb{Z}_m)$ is connected, complete, Eulerian or has a cycle is determined. In this article, we study $G_{R/J}(R/I)$, where R is a Dedekind domain, I and J are ideals of R and $I \subseteq J$. We characterize all ideals I and J for which $G_{R/J}(R/I)$ is planar, outerplanar or ring graph. As a corollary, we determine all integer numbers n and m for which $G_n(\mathbb{Z}_m)$ is planar, outerplanar or ring graph.

2. Results

A remarkable simple characterization of the planar graphs was given by Kuratowski in 1930.

Theorem 2.1. [7, Theorem 10.30] *(Kuratowski's Theorem) A graph is planar if and only if it contains no subdivision of either K_5 or $K_{3,3}$.*

Let R be a commutative ring, I, J be ideals of R and let $I \subseteq J$. Then R/J is an R/I-module. First, the planarity of $G_{R/J}(R/I)$ is investigated. We begin with the following remark.

Remark 2.2. We note that $V(G_{R/J}(R/I)) = \{K/I : K \trianglelefteq R \text{ and } I \subsetneqq K \subsetneqq R\}$. Hence the vertices of $G_{R/J}(R/I)$ can be identified by the set $\{K \trianglelefteq R : I \subsetneqq K \subsetneqq R\}$ and two distinct vertices K, L are adjacent if and only if $(K + J) \cap (L + J) \neq J$.

Remark 2.3. (1) $G_{R/J}(R/I)$ contains the intersection graph of R/J.

(2) The vertices $\{K \trianglelefteq R : I \subsetneqq K \subseteq J\}$ are isolated.

(3) If K and L are two vertices and $K \cap L \nsubseteq J$, then K and L are adjacent.

(4) Let $\mathfrak{m}_1, \cdots, \mathfrak{m}_n, \mathfrak{m}_{n+1}$ be maximal ideals which contains properly I and $J \subseteq \mathfrak{m}_{n+1}$, for some positive integer n. For any $\emptyset \neq C \subseteq \{1, \ldots, n\}$, set $\mathfrak{m}_C = \bigcap_{i \in C} \mathfrak{m}_i$. Then $\{\mathfrak{m}_C : \emptyset \neq C \subseteq \{1, \ldots, n\}\}$ is a clique of size $2^n - 1$ by part (3).

(5) If $G_{R/J}(R/I)$ is planar, then R/I has at most three maximal ideals by part (4).

(6) Every chain of ideals which contain J is a clique.

Remark 2.4. Throughout the paper, we consider R is a Dedekind domain, I and J are two ideals and R/J is an R/I-module. Then every non-zero proper ideal of R is a finite product of maximal ideals in a unique way [10, Proposition 2.1.3]. Without loss of generality, we assume that $I = \mathfrak{m}_1^{\alpha_1} \cdots \mathfrak{m}_s^{\alpha_s}$ and $J = \mathfrak{m}_1^{\beta_1} \cdots \mathfrak{m}_{s'}^{\beta_{s'}}$, where $1 \leq s' \leq s$, $\mathfrak{m}_i$'s are distinct maximal ideals, α_i's and β_i's are positive integers, and $0 < \beta_i \leq \alpha_i$ for $i = 1, \ldots, s'$. Let $K = \bigcap_{i \in A} \mathfrak{m}_i^{t_i}$ and $L = \bigcap_{i \in B} \mathfrak{m}_i^{t'_i}$ be two ideals of R, such that $\{\mathfrak{m}_i\}_{i \in A \cup B} \subseteq Max(R)$, A, B are finite subsets and t_i, t'_i are positive integers. By the maximality of $\mathfrak{m}_i$'s, we find that $K \cap L = \bigcap_{i \in A \cup B} \mathfrak{m}_i^{max\{t_i, t'_i\}}$ and

$$K + L = \begin{cases} \bigcap_{i \in A \cap B} \mathfrak{m}_i^{min\{t_i, t'_i\}}, & \text{if } A \cap B \neq \emptyset; \\ R, & \text{if } A \cap B = \emptyset. \end{cases}$$

Also, if $A \cap B = \emptyset$, then $K \cap L = KL$. We use these facts in the proof of results.

Theorem 2.5. *Let R be a Dedekind domain, I and J be ideals and let R/J be an R/I-module. Then $G_{R/J}(R/I)$ is planar if and only if one of the following holds:*

(1) $I = \mathfrak{m}_1^{\alpha_1}, J = \mathfrak{m}_1^{\beta_1}$ *and* $\beta_1 \leq 5$.

(2) $I = \mathfrak{m}_1^{\alpha_1}\mathfrak{m}_2^{\alpha_2}, J = \mathfrak{m}_1$ *and* $\alpha_2 \leq 4$.

(3) $I = \mathfrak{m}_1^{\alpha_1}\mathfrak{m}_2, J = \mathfrak{m}_1^2$.

(4) $I = \mathfrak{m}_1^{\alpha_1}\mathfrak{m}_2\mathfrak{m}_3, J = \mathfrak{m}_1$.

(5) $I = \mathfrak{m}_1\mathfrak{m}_2^{\alpha_2}, J = \mathfrak{m}_1\mathfrak{m}_2^2$ *and* $\alpha_2 = 2, 3, 4$.

(6) $I = \mathfrak{m}_1\mathfrak{m}_2\mathfrak{m}_3, J = \mathfrak{m}_1\mathfrak{m}_2$.

(7) $I = \mathfrak{m}_1^{\alpha_1}\mathfrak{m}_2^{\alpha_2}, J = \mathfrak{m}_1\mathfrak{m}_2$ *and* $\alpha_1, \alpha_2 \leq 4$.

(8) $I = J = \mathfrak{m}_1\mathfrak{m}_2\mathfrak{m}_3$.

Proof. One side is obvious. For the other side assume that $G_{R/J}(R/I)$ is planar. We note that by Remark 2.3 part (5), $s \leq 3$. Consider three following cases:

Case 1. $s' = 1$. If $\beta_1 \geq 6$, then $\mathfrak{m}_1, \mathfrak{m}_1^2, \mathfrak{m}_1^3, \mathfrak{m}_1^4, \mathfrak{m}_1^5$ forms a K_5, a contradiction. Hence $\beta_1 \leq 5$. There are three following subcases:

Subcase 1. $s = 1$. Clearly, $G_{R/\mathfrak{m}_1^{\beta_1}}(R/\mathfrak{m}_1^{\alpha_1})$ is planar, where $\beta_1 \leq 5$ and (1) holds.

Subcase 2. $s = 2$. We note that $\alpha_2 \leq 4$. Otherwise, $\mathfrak{m}_2, \mathfrak{m}_2^2, \mathfrak{m}_2^3, \mathfrak{m}_2^4, \mathfrak{m}_2^5$ forms a K_5, a contradiction. If $\beta_1 \geq 3$, then $\mathfrak{m}_1, \mathfrak{m}_1^2, \mathfrak{m}_2, \mathfrak{m}_1\mathfrak{m}_2, \mathfrak{m}_1^2\mathfrak{m}_2$ forms a K_5, a contradiction. This implies that $\beta_1 = 1, 2$. If $\beta_1 = 1$, then it is easy to check that $G_{R/\mathfrak{m}_1}(R/\mathfrak{m}_1^{\alpha_1}\mathfrak{m}_2^{\alpha_2})$ is planar, where $\alpha_2 \leq 4$. Therefore (2) holds. Now, let $\beta_1 = 2$. If $\alpha_2 \geq 2$, then $\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_1\mathfrak{m}_2, \mathfrak{m}_2^2, \mathfrak{m}_1\mathfrak{m}_2^2$ forms a K_5, a contradiction. Therefore $\alpha_2 = 1$. Obviously, $G_{R/\mathfrak{m}_1^2}(R/\mathfrak{m}_1^{\alpha_1}\mathfrak{m}_2)$ is planar and (3) holds.

Subcase 3. $s = 3$. If $\alpha_2 \geq 2$, then $\mathfrak{m}_2, \mathfrak{m}_3, \mathfrak{m}_2^2, \mathfrak{m}_2\mathfrak{m}_3, \mathfrak{m}_2^2\mathfrak{m}_3$ forms a K_5, a contradiction. Hence $\alpha_2 = 1$. Similarly, we find that $\alpha_3 = 1$. If $\beta_1 \geq 2$, then $\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3, \mathfrak{m}_1\mathfrak{m}_2, \mathfrak{m}_2\mathfrak{m}_3$ forms a K_5, a contradiction. It is easy to see that $G_{R/\mathfrak{m}_1}(R/\mathfrak{m}_1^{\alpha_1}\mathfrak{m}_2\mathfrak{m}_3)$ is planar and (4) holds.

Case 2. $s' = 2$. If $\beta_1, \beta_2 \geq 2$, then $\mathfrak{m}_1, \mathfrak{m}_1^2, \mathfrak{m}_2, \mathfrak{m}_1\mathfrak{m}_2, \mathfrak{m}_1^2\mathfrak{m}_2$ forms a K_5, a contradiction. Hence with no loss of generality we may assume that $\beta_1 = 1$.

First assume that $\beta_2 \geq 2$. If $\alpha_1 \geq 2$, then $\mathfrak{m}_1, \mathfrak{m}_1^2, \mathfrak{m}_2, \mathfrak{m}_1\mathfrak{m}_2, \mathfrak{m}_1^2\mathfrak{m}_2$ forms a K_5, a contradiction. Therefore $\alpha_1 = 1$. If $s = 3$, then $\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_1\mathfrak{m}_2, \mathfrak{m}_3, \mathfrak{m}_1\mathfrak{m}_3$ forms a K_5, a contradiction. Thus $s = 2$. If $\beta_2 \geq 3$, then $\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_2^2, \mathfrak{m}_1\mathfrak{m}_2, \mathfrak{m}_1\mathfrak{m}_2^2$ forms a K_5, a contradiction. Therefore $\beta_2 = 2$. If $\alpha_2 \geq 5$, then $\mathfrak{m}_2, \mathfrak{m}_2^2, \mathfrak{m}_2^3, \mathfrak{m}_2^4, \mathfrak{m}_2^5$ forms a K_5, a contradiction. Therefore $\alpha_2 \leq 4$. It is easy to see that $G_{R/\mathfrak{m}_1\mathfrak{m}_2^2}(R/\mathfrak{m}_1\mathfrak{m}_2^{\alpha_2})$ is planar, where $\alpha_2 = 2, 3, 4$ and (5) holds.

Now, assume that $\beta_2 = 1$. Let $s = 3$. If $\alpha_3 \geq 2$, then $\mathfrak{m}_1, \mathfrak{m}_3, \mathfrak{m}_3^2, \mathfrak{m}_1\mathfrak{m}_3, \mathfrak{m}_1\mathfrak{m}_3^2$ forms a K_5, a contradiction. Hence $\alpha_3 = 1$. If $\alpha_1 \geq 2$, then $\mathfrak{m}_1, \mathfrak{m}_1^2, \mathfrak{m}_3, \mathfrak{m}_1\mathfrak{m}_3, \mathfrak{m}_1^2\mathfrak{m}_3$ forms a K_5, a contradiction. Therefore $\alpha_1 = 1$ and similarly, $\alpha_2 = 1$. It is clear that $G_{R/\mathfrak{m}_1\mathfrak{m}_2}(R/\mathfrak{m}_1\mathfrak{m}_2\mathfrak{m}_3)$ is planar and (6) holds. Now, assume that $s = 2$. Then we find that $\alpha_1, \alpha_2 \leq 4$. Also, one can easily check that $G_{R/\mathfrak{m}_1\mathfrak{m}_2}(R/\mathfrak{m}_1^{\alpha_1}\mathfrak{m}_2^{\alpha_2})$ is planar, where $\alpha_1, \alpha_2 \leq 4$. Therefore (7) holds.

Case 3. $s' = 3$. Then $s = 3$. If $\alpha_1 \geq 3$, then $\mathfrak{m}_1, \mathfrak{m}_1^2, \mathfrak{m}_1^3, \mathfrak{m}_2, \mathfrak{m}_3$ forms a K_5, a contradiction. Therefore $\alpha_1 \leq 2$ and similarly $\alpha_2, \alpha_3 \leq 2$. If $\beta_1 \geq 2$, then $\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3, \mathfrak{m}_1\mathfrak{m}_2, \mathfrak{m}_1\mathfrak{m}_3$ forms a K_5, a contradiction. Therefore $\beta_1 = 1$ and similarly we have $\beta_2 = \beta_3 = 1$. If $\alpha_1 = \alpha_2 = 2$, then $\mathfrak{m}_1, \mathfrak{m}_1^2, \mathfrak{m}_2, \mathfrak{m}_2^2, \mathfrak{m}_1\mathfrak{m}_2$ forms a K_5, a contradiction. Therefore $\alpha_1 = 1$ or $\alpha_2 = 1$. With no loss of generality, we may assume that $\alpha_1 = 1$. By The same argument as we saw, we find that $\alpha_2 = 1$ or $\alpha_3 = 1$. Thus we have $\alpha_1 = \alpha_2 = \alpha_3 = 1$ or $\alpha_1 = \alpha_2 = 1$ and $\alpha_3 = 2$. It is clear that if $\alpha_1 = \alpha_2 = \alpha_3 = 1$, then $G_{R/J}(R/I)$ is planar. Therefore (8) holds. Also,

$G_{R/\mathfrak{m}_1\mathfrak{m}_2\mathfrak{m}_3}(R/\mathfrak{m}_1\mathfrak{m}_2\mathfrak{m}_3^2)$ is not planar because $\mathfrak{m}_3, \mathfrak{m}_3^2, \mathfrak{m}_2\mathfrak{m}_3, \mathfrak{m}_2\mathfrak{m}_3^2, \mathfrak{m}_2$ forms a K_5 which is impossible. □

As an immediate consequence of the previous theorem, we have the next result.

Corollary 2.6. *Let R be a Dedekind domain and let I be an ideal of R. Then $G(R/I)$ is planar if and only if $I \in \{\mathfrak{m}_1, \mathfrak{m}_1^2, \mathfrak{m}_1^3, \mathfrak{m}_1^4, \mathfrak{m}_1^5, \mathfrak{m}_1\mathfrak{m}_2, \mathfrak{m}_1\mathfrak{m}_2^2, \mathfrak{m}_1\mathfrak{m}_2\mathfrak{m}_3\}$.*

Let $n, m > 1$ be integers and $\mathbb{Z}_n$ be a $\mathbb{Z}_m$-module. Clearly, $\mathbb{Z}_n$ is a $\mathbb{Z}_m$-module if and only if n divides m. By Remark 2.4, we may assume that $m = p_1^{\alpha_1} \cdots p_s^{\alpha_s}$ and $n = p_1^{\beta_1} \cdots p_{s'}^{\beta'_s}$, where p_i's are distinct primes, α_i's and β_i's are positive integers, and $0 < \beta_i \le \alpha_i$ for $i = 1, \ldots, s'$. Since $\mathbb{Z}$ is a Dedekind domain, we conclude the following corollary.

Corollary 2.7. *Let $\mathbb{Z}_n$ be a $\mathbb{Z}_m$-module. Then $G_n(\mathbb{Z}_m)$ is planar if and only if one of the following holds:*

(1) $m = p_1^{\alpha_1}, n = p_1^{\beta_1}$ *and* $\beta_1 \le 5$.

(2) $m = p_1^{\alpha_1}p_2^{\alpha_2}, n = p_1$ *and* $\alpha_2 \le 4$.

(3) $m = p_1^{\alpha_1}p_2, n = p_1^2$.

(4) $m = p_1^{\alpha_1}p_2p_3, n = p_1$.

(5) $m = p_1p_2^{\alpha_2}, n = p_1p_2^2$ *and* $\alpha_2 = 2, 3, 4$.

(6) $m = p_1p_2p_3, n = p_1p_2$.

(7) $m = p_1^{\alpha_1}p_2^{\alpha_2}, n = p_1p_2$ *and* $\alpha_1, \alpha_2 \le 4$.

(8) $m = n = p_1p_2p_3$.

Also, we have the next result.

Corollary 2.8. *Let m be a positive integer number. Then $G(\mathbb{Z}_m)$ is planar if and only if $m \in \{p_1, p_1^2, p_1^3, p_1^4, p_1^5, p_1p_2, p_1p_2^2, p_1p_2p_3\}$.*

Let G be a graph. We recall that a *chord* is any edge of G joining two nonadjacent vertices in a cycle of G. Let C be a cycle of G. We say C is a *primitive cycle* if it has no chords. Also, a graph G has the *primitive* cycle property (PCP) if any two primitive cycles intersect in at most one edge. The number $frank(G)$ is called the *free rank* of G and it is the number of primitive cycles of G. Also, the number $rank(G) = |E(G)| - |V(G)| + r$ is called the *cycle rank* of G, where r is the number of connected components of G. The cycle rank of G can be expressed as the dimension of the cycle space of G. By [11, Proposition 2.2], we have $rank(G) \le frank(G)$. A graph G is called a *ring graph* if it satisfies in one of the following equivalent conditions (see [11, Theorem 2.13]).

(1) $rank(G) = frank(G)$,

(2) G satisfies the PCP and G does not contain a subdivision of K_4 as a subgraph. Also, an undirected graph is an *outerplanar* graph if it can be drawn in the plane without crossings in such a way that all of the vertices belong to the unbounded face of the drawing. There is a characterization for outerplanar graphs that says a graph is outerplanar if and only if it does not contain a subdivision of K_4 or $K_{2,3}$. By [11, Proposition 2.17], we find that every outerplanar graph is a ring graph and every ring graph is a planar graph.

Example 1. In Fig.1, let $v_1 = p_2\mathbb{Z}_{p_1p_2^2}, v_2 = p_1p_2\mathbb{Z}_{p_1p_2^2}, v_3 = p_2^2\mathbb{Z}_{p_1p_2^2}, v_4 = p_1\mathbb{Z}_{p_1p_2^2}$. Also, in Fig.2, assume that $v_1 = p_1p_2\mathbb{Z}_{p_1p_2^3}, v_2 = p_1p_2^2\mathbb{Z}_{p_1p_2^3}, v_3 = p_2^2\mathbb{Z}_{p_1p_2^3}, v_4 = p_1\mathbb{Z}_{p_1p_2^3}, v_5 = p_2\mathbb{Z}_{p_1p_2^3}, v_6 = p_2^3\mathbb{Z}_{p_1p_2^3}$

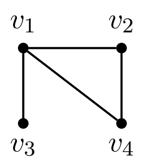

Fig.1 $G(\mathbb{Z}_{p_1p_2^2}) \cong G(R/\mathfrak{m}_1\mathfrak{m}_2^2)$

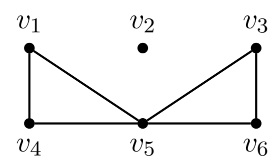

Fig.2 $G_{p_1p_2^2}(\mathbb{Z}_{p_1p_2^3}) \cong G_{R/\mathfrak{m}_1\mathfrak{m}_2^2}(R/\mathfrak{m}_1\mathfrak{m}_2^3)$

Example 2. In Fig.3, let $v_1 = p_1\mathbb{Z}_{p_1p_2p_3}, v_2 = p_1p_2\mathbb{Z}_{p_1p_2p_3}, v_3 = p_2\mathbb{Z}_{p_1p_2p_3}, v_4 = p_1p_3\mathbb{Z}_{p_1p_2p_3}, v_5 = p_3\mathbb{Z}_{p_1p_2p_3}, v_6 = p_2p_3\mathbb{Z}_{p_1p_2p_3}$. In Fig.4, let $v_1 = p_1p_2\mathbb{Z}_{p_1p_2p_3}, v_2 = p_1\mathbb{Z}_{p_1p_2p_3}, v_3 = p_1p_3\mathbb{Z}_{p_1p_2p_3}, v_4 = p_2\mathbb{Z}_{p_1p_2p_3}, v_5 = p_3\mathbb{Z}_{p_1p_2p_3}, v_6 = p_2p_3\mathbb{Z}_{p_1p_2p_3}$.

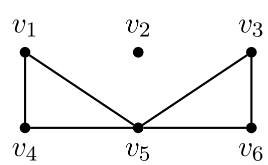

Fig.3 $G_{p_1p_2}(\mathbb{Z}_{p_1p_2p_3}) \cong G_{R/\mathfrak{m}_1\mathfrak{m}_2}(R/\mathfrak{m}_1\mathfrak{m}_2\mathfrak{m}_3)$

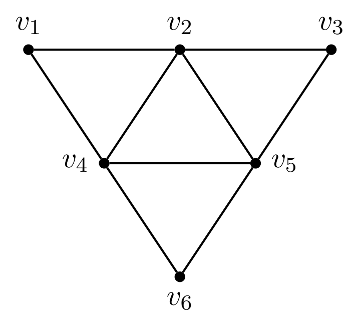

Fig.4 $G(\mathbb{Z}_{p_1p_2p_3}) \cong G(R/\mathfrak{m}_1\mathfrak{m}_2\mathfrak{m}_3)$

Example 3. In Fig.5, let $v_1 = p_2\mathbb{Z}_{p_1p_2^3}, v_2 = p_2^2\mathbb{Z}_{p_1p_2^3}, v_3 = p_2^3\mathbb{Z}_{p_1p_2^3}, v_4 = p_1p_2\mathbb{Z}_{p_1p_2^3}, v_5 = p_1\mathbb{Z}_{p_1p_2^3}, v_6 = p_1p_2^2\mathbb{Z}_{p_1p_2^3}$.

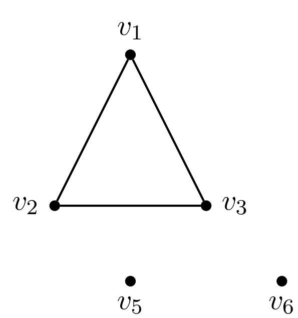

Fig.5 $G_{p_1p_2}(\mathbb{Z}_{p_1p_2^3}) \cong G_{R/\mathfrak{m}_1\mathfrak{m}_2}(R/\mathfrak{m}_1\mathfrak{m}_2^3)$

Theorem 2.9. *Let R be a Dedekind domain, I and J be ideals and let R/J be an R/I-module. Then $G_{R/J}(R/I)$ is a ring graph if and only if one of the following holds:*

(1) $I = \mathfrak{m}_1^{\alpha_1}, J = \mathfrak{m}_1^{\beta_1}$ *and* $\beta_1 \leq 4$.
(2) $I = \mathfrak{m}_1^{\alpha_1}\mathfrak{m}_2^{\alpha_2}, J = \mathfrak{m}_1$ *and* $\alpha_2 \leq 3$.
(3) $I = \mathfrak{m}_1^{\alpha_1}\mathfrak{m}_2, J = \mathfrak{m}_1^2$.
(4) $I = \mathfrak{m}_1^{\alpha_1}\mathfrak{m}_2\mathfrak{m}_3, J = \mathfrak{m}_1$.
(5) $I = \mathfrak{m}_1\mathfrak{m}_2^{\alpha_2}, J = \mathfrak{m}_1\mathfrak{m}_2^2$ *and* $\alpha_2 = 2, 3$.
(6) $I = \mathfrak{m}_1\mathfrak{m}_2\mathfrak{m}_3, J = \mathfrak{m}_1\mathfrak{m}_2$.
(7) $I = \mathfrak{m}_1^{\alpha_1}\mathfrak{m}_2^{\alpha_2}, J = \mathfrak{m}_1\mathfrak{m}_2$ *and* $\alpha_1, \alpha_2 \leq 3$.
(8) $I = J = \mathfrak{m}_1\mathfrak{m}_2\mathfrak{m}_3$.

Proof. One side is obvious. For the other side assume that $G_{R/J}(R/I)$ is a ring graph. Then it is planar and by Theorem 2.5, we have eight following cases:

Case 1. $I = \mathfrak{m}_1^{\alpha_1}, J = \mathfrak{m}_1^{\beta_1}$ and $\beta_1 \leq 5$. If $\beta_1 = 5$, then $\mathfrak{m}_1, \mathfrak{m}_1^2, \mathfrak{m}_1^3, \mathfrak{m}_1^4$ forms a K_4, a contradiction. Therefore $\beta_1 \leq 4$. It is easy to check that $G_{R/J}(R/I)$ is a ring graph and (1) holds.

Case 2. $I = \mathfrak{m}_1^{\alpha_1}\mathfrak{m}_2^{\alpha_2}, J = \mathfrak{m}_1$ and $\alpha_2 \leq 4$. Clearly, if $\alpha_2 = 4$, then $\mathfrak{m}_2, \mathfrak{m}_2^2, \mathfrak{m}_2^3, \mathfrak{m}_2^4$ forms a K_4, a contradiction. Hence $\alpha_2 \leq 3$. Now, $G_{R/J}(R/I)$ has at most three non-isolated vertices and so it is a ring graph and (2) holds.

Case 3. $I = \mathfrak{m}_1^{\alpha_1}\mathfrak{m}_2, J = \mathfrak{m}_1^2$. In this case, $\{\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_1\mathfrak{m}_2\}$ is the set of all non-isolated vertices of $G_{R/J}(R/I)$. This yields that $G_{R/J}(R/I)$ is a ring graph and (3) holds.

Case 4. $I = \mathfrak{m}_1^{\alpha_1}\mathfrak{m}_2\mathfrak{m}_3, J = \mathfrak{m}_1$. It is easy to see that $G_{R/\mathfrak{m}_1}(R/\mathfrak{m}_1^{\alpha_1}\mathfrak{m}_2\mathfrak{m}_3)$ is a ring graph and (4) holds.

Case 5. $I = \mathfrak{m}_1\mathfrak{m}_2^{\alpha_2}, J = \mathfrak{m}_1\mathfrak{m}_2^2$ and $\alpha_2 = 2, 3, 4$. If $\alpha_2 = 4$, then $\mathfrak{m}_2, \mathfrak{m}_2^2, \mathfrak{m}_2^3, \mathfrak{m}_2^4$ forms a K_4, a contradiction. Therefore $\alpha_2 = 2, 3$. By Fig.1 and Fig.2, $G_{R/\mathfrak{m}_1\mathfrak{m}_2^2}(R/\mathfrak{m}_1\mathfrak{m}_2^2)$ and $G_{R/\mathfrak{m}_1\mathfrak{m}_2^2}(R/\mathfrak{m}_1\mathfrak{m}_2^3)$ are ring graphs and (5) holds.

Case 6. $I = \mathfrak{m}_1\mathfrak{m}_2\mathfrak{m}_3, J = \mathfrak{m}_1\mathfrak{m}_2$. In this case, by Fig.3, we conclude that $G_{R/J}(R/I)$ is a ring graph and (6) holds.

Case 7. $I = \mathfrak{m}_1^{\alpha_1}\mathfrak{m}_2^{\alpha_2}, J = \mathfrak{m}_1\mathfrak{m}_2$ and $\alpha_1, \alpha_2 \leq 4$. If $\alpha_1 = 4$, then $\mathfrak{m}_1, \mathfrak{m}_1^2, \mathfrak{m}_1^3, \mathfrak{m}_1^4$ forms a K_4, a contradiction. Hence $\alpha_1 \leq 3$ and similarly, $\alpha_2 \leq 3$. It is not hard to see that $G_{R/J}(R/I) \setminus \mathfrak{A} \cong K_{\alpha_1} \cup K_{\alpha_2}$, where $\mathfrak{A}$ is the set of all isolated vertices of $G_{R/J}(R/I)$. Therefore (7) holds.(see Fig.5)

Case 8. $I = J = \mathfrak{m}_1\mathfrak{m}_2\mathfrak{m}_3$. By Fig.4, we find that $G_{R/J}(R/I)$ is a ring graph and (8) holds.

□

Corollary 2.10. *Let R be a Dedekind domain and let I be an ideal of R. Then $G(R/I)$ is a ring graph if and only if* $I \in \{\mathfrak{m}_1, \mathfrak{m}_1^2, \mathfrak{m}_1^3, \mathfrak{m}_1^4, \mathfrak{m}_1\mathfrak{m}_2, \mathfrak{m}_1\mathfrak{m}_2^2, \mathfrak{m}_1\mathfrak{m}_2\mathfrak{m}_3\}$.

Corollary 2.11. *Let $\mathbb{Z}_n$ be a $\mathbb{Z}_m$-module. Then $G_n(\mathbb{Z}_m)$ is a ring graph if and only if one of the following holds:*

(1) $m = p_1^{\alpha_1}, n = p_1^{\beta_1}$ *and* $\beta_1 \leq 4$.
(2) $m = p_1^{\alpha_1}p_2^{\alpha_2}, n = p_1$ *and* $\alpha_2 \leq 3$.
(3) $m = p_1^{\alpha_1}p_2, n = p_1^2$.
(4) $m = p_1^{\alpha_1}p_2p_3, n = p_1$.
(5) $m = p_1p_2^{\alpha_2}, n = p_1p_2^2$ *and* $\alpha_2 = 2, 3$.
(6) $m = p_1p_2p_3, n = p_1p_2$.
(7) $m = p_1^{\alpha_1}p_2^{\alpha_2}, n = p_1p_2$ *and* $\alpha_1, \alpha_2 \leq 3$.
(8) $m = n = p_1p_2p_3$.

Corollary 2.12. *Let m be a positive integer number. Then $G(\mathbb{Z}_m)$ is a ring graph if and only if $m \in \{p_1, p_1^2, p_1^3, p_1^4, p_1p_2, p_1p_2^2, p_1p_2p_3\}$.*

Using theorems in this section and the proof of pervious theorem, we have the following result.

Theorem 2.13. *Let R be a Dedekind domain, I and J be ideals and let R/J be an R/I-module. Then $G_{R/J}(R/I)$ is outerplanar if and only if one of the following holds:*

(1) $I = \mathfrak{m}_1^{\alpha_1}, J = \mathfrak{m}_1^{\beta_1}$ *and* $\beta_1 \leq 4$.
(2) $I = \mathfrak{m}_1^{\alpha_1}\mathfrak{m}_2^{\alpha_2}, J = \mathfrak{m}_1$ *and* $\alpha_2 \leq 3$.
(3) $I = \mathfrak{m}_1^{\alpha_1}\mathfrak{m}_2, J = \mathfrak{m}_1^2$.
(4) $I = \mathfrak{m}_1^{\alpha_1}\mathfrak{m}_2\mathfrak{m}_3, J = \mathfrak{m}_1$.
(5) $I = \mathfrak{m}_1\mathfrak{m}_2^{\alpha_2}, J = \mathfrak{m}_1\mathfrak{m}_2^2$ *and* $\alpha_2 = 2, 3$.
(6) $I = \mathfrak{m}_1\mathfrak{m}_2\mathfrak{m}_3, J = \mathfrak{m}_1\mathfrak{m}_2$.
(7) $I = \mathfrak{m}_1^{\alpha_1}\mathfrak{m}_2^{\alpha_2}, J = \mathfrak{m}_1\mathfrak{m}_2$ *and* $\alpha_1, \alpha_2 \leq 3$.
(8) $I = J = \mathfrak{m}_1\mathfrak{m}_2\mathfrak{m}_3$.

We close this article by the following corollary.

Corollary 2.14. *Let R be a Dedekind domain, I and J be ideals and let R/J be an R/I-module. Then $G_{R/J}(R/I)$ is outerplanar if and only if it is a ring graph.*

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