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Research Paper

MINIMAL PRIME FILTERS OF COMMUTATIVE BE-ALGEBRAS

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Abstract. In this paper we introduced the concept of minimal prime filters in commutative BE-algebras. A characterization theorem for minimal prime filters of BE-algebras is derived. Some properties of minimal prime filters of a commutative BE-algebras are derived with the help of congruences. A necessary and sufficient is derived for a pair of minimal prime filters to become co-maximal.

1. Introduction

The notion of BE-algebras was introduced and extensively studied by H.S. Kim and Y.H. Kim in [4]. These classes of BE-algebras were introduced as a generalization of the class of BCK-algebras of K. Iseki and S. Tanaka [3]. Some properties of filters of BE-algebras were studied by S.S. Ahn and Y.H. Kim in [1] and by B.L. Meng in [5]. In [12], A. Walendziak discussed some properties of commutative BE-algebras. He also investigated the relationship between BE-algebras, implicative algebras and J-algebras. In 2013, Borumand Saeid, Rezaei

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and Borzooei [2] extensively studied the properties of some types of filters in BE-algebras. In [8], it is observed by A. Rezaei and et. al. that a KU–algebra is equivalent to the commutative self distributive BE–algebra. In 2018, Rezaei and Borumand Saeid [9] proved that any self distributive commutative BE-algebras is a gi-algebra and any gi-algebra is strong and transitive if and only if it is a commutative BE-algebra. In [6], Meng introduced the notion of prime filters in BCK-algebras, and then gave a description of the filter generated by a set, and obtained some of fundamental properties of prime filters. He also studied in [7], some properties of prime ideals in BCK-algebras.

Motivated by the characterizations given by Rao in [10], the author introduced the notion of prime filters in BE-algebras. Some properties of prime filters and maximal filters are then studied. He characterized generalized prime filters of a commutative BE-algebra. In [11], some properties of dual annihilator filters of commutative BE-algebras are studied. It is proved that the class of all dual annihilator filters of a BE-algebra is a complete Boolean algebra. A set of equivalent conditions is derived for every prime filter of a commutative BE-algebra to become a maximal filter.

Filters are important substructures in a BE-algebra and play an important role. It is well understood that filters are the kernels of congruences. Filter theory is crucial in the study of any class of logical algebras. From a logical standpoint, different filters correspond to different sets of valid formulas in an appropriate logic. Designing various types of filters in some logical algebra, on the other hand, is also algebraically interesting. With this motivation, we investigate the concept of a minimal prime filter of a commutative BE-algebra in this paper. A characterization theorem is derived for minimal prime filters of commutative BE-algebras. A set of equivalent conditions is derived for every prime filter of a BE-algebra to become a minimal prime filter. Some properties of minimal prime filters of a commutative BE-algebra are studied with the help of congruences. An equivalency is obtained between the minimal prime filters of a commutative BE-algebra and the minimal prime filters of its quotient algebra with respect to this congruence.

2. Preliminaries

In this section, we present certain definitions and results which are taken mostly from the papers [1], [4], [10] and [12] for the ready reference of the reader.

Definition 2.1. [4] An algebra $(X, *, 1)$ of type $(2, 0)$ is called a BE-algebra if it satisfies the following properties:

(1) $x * x = 1$,

(2) $x * 1 = 1$,

(3) $1 * x = x$,

(4) $x * (y * z) = y * (x * z)$ for all $x, y, z \in X$.

A BE-algebra X is called self-distributive if $x * (y * z) = (x * y) * (x * z)$ for all $x, y, z \in X$. A BE-algebra X is called transitive if $y * z \leq (x * y) * (x * z)$ for all $x, y, z \in X$. Every self-distributive BE-algebra is transitive. A BE-algebra X is called commutative if $(x * y) * y = (y * x) * x$ for all $x, y \in X$. Every commutative BE-algebra is transitive. For any $x, y \in X$, define $x \vee y = (y * x) * x$. If X is commutative, then $(X, \vee)$ is a semilattice [12]. We introduce a relation $\leq$ on a BE-algebra X by $x \leq y$ if and only if $x * y = 1$ for all $x, y \in X$. Clearly $\leq$ is reflexive. If X is commutative, then $\leq$ is transitive, anti-symmetric and hence a partial order on X.

Definition 2.2. [1] A non-empty subset F of a BE-algebra X is called a filter of X if, for all $x, y \in X$, it satisfies the following properties:

(1) $1 \in F$,

(2) $x \in F$ and $x * y \in F$ imply that $y \in F$.

For any non-empty subset A of a transitive BE-algebra X, the set $\langle A \rangle = \{x \in X \mid a_1 * (a_2 * (\cdots * (a_n * x) \cdots)) = 1$ for some $a_1, a_2, \ldots a_n \in A\}$ is the smallest filter containing A. For any $a \in X, \langle a \rangle = \{x \in X \mid a^n * x = 1$ for some $n \in \mathbb{N}\}$, where $a^n * x = a * (a * (\cdots * (a * x) \cdots))$ with the repetition of a is n times, is called the principal filter generated a. If X is self-distributive, then $\langle a \rangle = \{x \in X \mid a * x = 1\}$. A proper filter P of a BE-algebra is called prime [10] if $\langle x \rangle \cap \langle y \rangle \subseteq P$ implies $x \in P$ or $y \in P$ for any $x, y \in X$. A proper filter M of a transitive BE-algebra X is called maximal [10] if there exists no proper filter Q such that $M \subset Q$. Every maximal filter of a commutative BE-algebra is prime.

Theorem 2.3. [10] *Let S be a $\vee$-closed subset of a commutative BE-algebra X. If F is a filter of X such that $F \cap S = \emptyset$, then there exists a prime filter P of X such that $F \subseteq P$ and $P \cap S = \emptyset$.*

Theorem 2.4. [10] *If X is a self-distributive and commutative BE-algebra, then*

(1) *$x \leq y$ implies $\langle y \rangle \subseteq \langle x \rangle$,*

(2) *$\langle x \rangle \cap \langle y \rangle = \langle x \vee y \rangle$ for all $x, y \in X$.*

Theorem 2.5. [10] *If X is a self-distributive and commutative BE-algebra, then the following are equivalent:*

(1) *P is prime;*

(2) *for any $x, y \in X$, $x \vee y \in P$ implies $x \in P$ or $y \in P$;*

(3) *for any two filters F and G of X, $F \cap G \subseteq P$ implies $F \subseteq P$ or $G \subseteq P$.*

For any non-empty subset A of a commutative BE-algebra X, the dual annihilator [11] of A is defined as $A^+ = \{x \in X \mid x \vee a = 1 \text{ for all } a \in A\}$. Clearly A^+ is a filter of X. Obviously $X^+ = \{1\}$ and $\{1\}^+ = X$. For $A = \{a\}$, we simply denote $\{a\}^+$ by $(a)^+$.

Proposition 2.6. [11] *Let X be a commutative BE-algebra and $x, y, z \in X$. Then $(x*y)\vee z \leq (x \vee z) * (y \vee z)$.*

Proposition 2.7. [11] *For any two filters F, G of a commutative BE-algebra X, we have*

(1) $F \cap F^+ = \emptyset$,
(2) $F \subseteq F^{++}$,
(3) $F^{+++} = F^+$,
(4) $F \subseteq G$ *implies* $G^+ \subseteq F^+$,
(5) $(F \vee G)^+ = F^+ \cap G^+$,
(6) $(F \cap G)^{++} = F^{++} \cap G^{++}$.

Corollary 2.8. [11] *For any two elements a, b of a commutative BE-algebra X, we have*

(1) $(\langle a\rangle)^+ = (a)^+$,
(2) $\langle a\rangle \subseteq (a)^{++}$,
(3) $a \leq b$ *implies* $(a)^+ \subseteq (b)^+$.

3. Minimal prime filters of BE-algebras

In this section, the notion of minimal prime filters is introduced in BE-algebras. An equivalent condition is derived for every prime filter of a BE-algebra to become a minimal prime filter.

Definition 3.1. Let F be a filter and P a prime filter of a commutative BE-algebra X such that $F \subseteq P$. Then P is called a *minimal prime filter* belonging to F if there exists no prime filter Q such that $F \subseteq Q \subset P$.

In a BE-algebra X, the minimal prime filters belonging to $\{1\}$ are simply called minimal prime filters of X. In the other version, a minimal prime filter of a BE-algebra is the minimal element of the partial order set of all prime filters. Thus a prime filter P of X is a minimal prime filter if for any prime filter F of X such that $F \subseteq P$, then $P = F$.

Example 3.2. Let $X = \{1, a, b, c\}$ be a set. Define a binary operation $*$ on X as follows:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	a	c
b	1	1	1	c
c	1	a	b	1

$\vee$	1	a	b	c
1	1	1	1	1
a	1	a	a	1
b	1	a	b	1
c	1	1	1	c

Then $(X, *, \vee, 1)$ is a commutative BE-algebra. Consider the subset $P = \{1, c\}$ of X. Clearly P is a prime filter of X. Observe that the improper filter $\{1\}$ is not prime because of $a \vee c = 1 \in \{1\}$ but neither $a \in \{1\}$ nor $c \in \{1\}$. Therefore P is a minimal prime filter of X. Similarly, we observe that $G = \{1, a, b\}$ is another minimal prime filter of X.

Proposition 3.3. *Let F be a proper filter of a commutative BE-algebra X. Then every prime filter of X, containing F, contains at least a minimal prime filter belonging to F.*

Proof. Let P be a prime filter of X such that $F \subseteq P$. Consider the collection

$$\mathfrak{T} = \{Q \mid Q \text{ is a prime filter of } X \text{ such that } F \subseteq Q \subseteq P\}.$$

Clearly $P \in \mathfrak{T}$ and hence $\mathfrak{T} \neq \emptyset$. Let $\{Q_\alpha\}_{\alpha\in\Delta}$ be a chain of elements in $\mathfrak{T}$. Since $\{Q_\alpha\}_{\alpha\in\Delta}$ is a chain, we get that $\bigcap\limits_{\alpha\in\Delta} Q_\alpha$ is a prime filter of X. Since $F \subseteq Q_\alpha \subseteq P$ for all $\alpha \in \Delta$, it is clear that $F \subseteq \bigcap\limits_{\alpha\in\Delta} Q_\alpha \subseteq P$. Hence $\bigcap\limits_{\alpha\in\Delta} Q_\alpha$ is a lower bound for $\{Q_\alpha\}_{\alpha\in\Delta}$. Therefore by Zorn's lemma, $\mathfrak{T}$ has a minimal element, say Q_0. Therefore Q_0 is a minimal prime filter such that $F \subseteq Q_0 \subseteq P$. □

By taking $F = \{1\}$, we get the following easy consequence:

Corollary 3.4. *Every prime filter of a commutative BE-algebra X contains at least a minimal prime filter.*

Proposition 3.5. *Let F be a proper filter of a commutative BE-algebra X. Then F is the intersection of all minimal prime filters of X, belonging to F.*

Proof. Since F is contained in every minimal prime filter of X, belonging to F and so contained in the intersection of all minimal prime filters belonging to F. To prove the converse, let $x \notin F$. Then by Corollary 3.4 (F is the intersection of all prime filters containing F), there exists a prime filter P of X such that $F \subseteq P$ and $x \notin P$. Then there exists a minimal prime filter M of X such that $F \subseteq M \subseteq P$. Since $x \notin P$, we get $x \notin M$. Hence M is a minimal prime filter of X, belonging to F, such that $x \notin M$. Thus x is not in the intersection of all minimal prime filters of X, belonging to F. □

If we take $F = \{1\}$ in the above proposition, the following is a direct consequence.

Corollary 3.6. *Let X be a commutative BE-algebra. Then the intersection of all minimal prime filters of X is equal to $\{1\}$.*

In Proposition 3.5, it is observed that every proper filter of a BE-algebra X can be decomposed as the intersection of all minimal prime filters of X, belonging to F.

Theorem 3.7. *(Unique decomposition theorem) Let F be a proper filter of a commutative BE-algebra X. If there exist positive integers m and n such that*

$$F = P_1 \cap P_2 \cap \cdots \cap P_m \quad \text{and} \quad F = Q_1 \cap Q_2 \cap \cdots \cap Q_n$$

are two representations of distinct minimal prime filters of X, belonging to F, then $m = n$, and for any P_i in the first expression there is Q_j in the second expression such that $P_i = Q_j$.

Proof. Let $P_i (i = 1, 2, \ldots, m)$ be a minimal prime filter in the first representation. Clearly $F \subseteq P_i$. By the second representation, we have $Q_1 \cap (Q_2 \cap \cdots \cap Q_n) \subseteq P_i$. Since P_i is prime, we get

$$Q_1 \subseteq P_i \quad \text{or} \quad Q_2 \cap \cdots \cap Q_n \subseteq P_i.$$

If $Q_1 \subseteq P_i$, then the minimality of P_i provides that $P_i = Q_1$. If $Q_1 \nsubseteq P_i$, then $Q_2 \cap \cdots \cap Q_n \subseteq P_i$. Repeating the same argument, we finally get that there exists $j \in \{2, 3, \ldots, m\}$ such that $P_i = Q_j$. It remains to show that $m = n$. Note that $P_i, P_2, \ldots, P_m$ are distinct, the preceding argument actually implies $m \leq n$. If we begin with the second representation, by the entirely similar argument, we will obtain $n \leq m$. Therefore $m = n$. □

Corollary 3.8. *If a proper filter F of a commutative BE-algebra X can be expressed as the intersection of a finite number of distinct minimal prime filters of X, belonging to F, then such representation is unique except their occurring order.*

In the following theorem, minimal prime filters are characterized.

Theorem 3.9. *Let F be a filter and P a prime filter of a self-distributive and commutative BE-algebra X such that $F \subseteq P$. Then P is a minimal prime filter belonging to F if and only if for each $x \in P$, there exists $y \notin P$ such that $x \vee y \in F$.*

Proof. Assume that P is a minimal prime filter belonging to a filter F of X. Let $a \in P$. Put $S_0 = \{a \vee x \mid x \in X - P\}$. Consider $S = S_0 \cup (X - P)$ and $a \in S$. We first observe that S is a $\vee$-closed subset of X. Let $x, y \in S$. Then we have the following cases:

Case I: Suppose $x, y \in S_0$. Then there exists $a_0, b_0 \in X - P$ such that $x = a \vee a_0$ and $y = a \vee b_0$. Hence $x \vee y = (a \vee a_0) \vee (a \vee b_0) = a \vee (a_0 \vee b_0)$. Since P is prime, we get $a_0 \vee b_0 \in X - P$. Hence $x \vee y \in S_0$. Therefore $x \vee y \in S$.

Case II: Suppose $x, y \in X - P$. Since P is prime, we get $x \vee y \in X - P \subseteq S$.

Case III: Suppose $x \in S_0$ and $y \in X - P$. Then $x = a \vee a_0$ for some $a_0 \in X - P$. Since P is prime, we get $a_0 \vee y \in X - P$. Hence $x \vee y = (a \vee a_0) \vee y = a \vee (a_0 \vee y) \in S_0$. Therefore $x \vee y \in S$.

Case IV: Suppose $x \in X - P$ and $y \in S_0$. Then $y = a \vee b_0$ for some $b_0 \in X - P$. Hence

$x \vee y = x \vee (a \vee b_0) = a \vee (x \vee b_0) \in S_0$ because of $x \vee b_0 \in X - P$.

From the above cases, we obtain that S is a $\vee$-closed subset of X. Therefore S is a $\vee$-closed subset with $X - P \subseteq S$ and $a \in S$. By Theorem 2.3, we get

$$\begin{aligned} F \cap S = \emptyset \quad &\Rightarrow \quad \text{there exists a prime filter } Q \text{ such that } F \subseteq Q \text{ and } Q \cap S = \emptyset \\ &\Rightarrow \quad Q \cap (X - P) = \emptyset \text{ and } a \notin Q \\ &\Rightarrow \quad Q \subseteq P \text{ and } a \notin Q \\ &\Rightarrow \quad Q \subsetneq P \end{aligned}$$

which is a contradiction to the minimality of P. Hence $F \cap S \neq \emptyset$. Choose $z_0 \in F \cap S$. Hence $z_0 \in S_0 \cup (X - P)$. Suppose $z_0 \in X - P$. Since $F \subseteq P$, we get $X - P \subseteq X - F$ and hence $z_0 \in X - F$. Hence $z_0 \notin F$, which is a contradiction. Hence $z_0 \notin X - P$. Thus $z_0 \in S_0$. Therefore $a \vee x = z_0 \in F$ for some $x \in X - P$.

Conversely, assume the condition. Let Q be a prime filter of X such that $F \subseteq Q \subset P$. Choose $a \in P - Q$. Then by the assumed condition, there exists $x \notin P$ such that $a \vee x \in F \subseteq Q$. Since Q is prime and $a \notin Q$, we get $x \in Q \subset P$, which is a contradiction. Therefore P is a minimal prime filter belonging to F. □

Corollary 3.10. *Let X be a self-distributive and commutative BE-algebra. A prime filter P of X is a minimal prime filter if and only if for each $x \in P$, there exists $y \notin P$ such that $x \vee y = 1$.*

4. Characterization of Minimal prime filters

In this section, some properties of minimal prime filters of BE-algebras are observed. Characterization theorems are derived for minimal prime filters of commutative BE-algebras.

Definition 4.1. For any filter P of a commutative BE-algebra X, define the set $O(P)$ as

$$O(P) = \{x \in X \mid x \vee a = 1 \text{ for some } a \notin P\} = \bigcup_{a \notin P} (a)^+.$$

In general, if P is a filter, then $O(P)$ need not be a filter.

Example 4.2. Let $X = \{1, a, b, c, d\}$ be a set. Define a binary operation $*$ on X as follows:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	1	d
b	1	c	1	c	d
c	1	b	b	1	d
d	1	a	b	c	1

$\vee$	1	a	b	c	d
1	1	1	1	1	1
a	1	a	b	c	1
b	1	b	b	1	1
c	1	c	1	c	1
d	1	1	1	1	d

Then clearly $(X, *, \vee, 1)$ is a commutative BE-algebra. Consider the filter $P = \{1, d\}$ of X. Now $O(P) = \{x \mid x \vee y = 1 \text{ for some } y \notin P\} = \{1, b, c, d\}$, which is not a filter of X.

In the following, we can observe that $O(P)$ is a filter for a prime filter P.

Proposition 4.3. *Let P be a filter of a self-distributive and commutative BE-algebra X. If P is a prime filter of X, then $O(P)$ is a filter containing P.*

Proof. Clearly $1 \in O(P)$. Let $x, x * y \in O(P)$. Then $x \vee a = 1$ and $(x * y) \vee b = 1$ for some $a, b \in X - P$. Put $c = a \vee b$. Then $1 = x \vee a \leq x \vee c$ and $1 = (x * y) \vee b \leq (x * y) \vee c$. Hence $x \in (c)^+$ and $x * y \in (c)^+$. Since $(c)^+$ is a filter, we get $y \in (c)^+$. Thus $y \vee c = 1$. Since $a \notin P, b \notin P$ and P is prime, we get $c = a \vee b \notin P$. Thus $y \in O(P)$. Therefore $O(P)$ is a filter of X. Now, let $x \in O(P)$. Then there exists some $t \notin P$ such that $x \vee t = 1$. Since $x \vee t \in P$ and $t \notin P$, it imply that $x \in P$. Therefore $O(P) \subseteq P$. □

Corollary 4.4. *Let P be a prime filter of a self-distributive and commutative BE-algebra X. Then $x \in O(P)$ if and only if there exists $a \in X - P$ such that $x \vee a = 1$.*

Proposition 4.5. *Let P be a prime filter of a self-distributive and commutative BE-algebra X. Then every minimal prime filter belonging to $O(P)$ is contained in P.*

Proof. Let Q be a minimal prime filter belonging to $O(P)$. Suppose $Q \nsubseteq P$. Choose $x \in Q - P$. Since Q is a minimal prime filter belonging to $O(P)$, by Theorem 3.9, there exists $y \notin Q$ such that $x \vee y \in O(P)$. Hence $y \vee (x \vee z) = (x \vee y) \vee z = 1$ for some $z \notin P$. Since P is prime, we get that $x \vee z \notin P$. Hence $y \in O(P) \subseteq Q$, which is a contradiction. Therefore $Q \subseteq P$. □

Proposition 4.6. *Let P be a prime filter of a self-distributive and commutative BE-algebra X. Then $O(P)$ is the intersection of all the minimal prime filters contained in P.*

Proof. Let P be a prime filter of X. By Zorn's lemma, we can observe that P contains a minimal prime filter. Let $\{S_\alpha\}_{\alpha \in \Delta}$ be a family of all minimal prime filters contained in P. Let $x \in O(P)$. Then there exists $a \notin P$ such that $x \vee a = 1$. Since each $S_\alpha \subseteq P$ and $a \notin P$, we get $a \notin S_\alpha$ for all $\alpha \in \Delta$. Since $x \vee a \in S_\alpha$ and $a \notin S_\alpha$ for all $\alpha \in \Delta$, we get $x \in S_\alpha$ for all $\alpha \in \Delta$. Hence $x \in \bigcap_{\alpha \in \Delta} S_\alpha$. Therefore $O(P) \subseteq \bigcap_{\alpha \in \Delta} S_\alpha$. Conversely, let $x \notin O(P)$. Clearly $x \neq 1$. Consider $S = \{\bigvee_{i=1}^{n} x_i \mid x_i \in (X - P) \cup \{x\}, n \in N\}$. Then clearly S is closed under $\vee$, $X - P \subseteq S$ and $x \in S$. Suppose $1 \in S$. Then

$$1 = \bigvee_{i=1}^{n} x_i \text{ where } x_i \in (X - P) \cup \{x\}$$

Suppose $x_i \in X - P$ for all $i = 1, 2, ..., n$. Then $\bigvee_{i=1}^{n} x_i = 1 \in P$. Since P is prime, it yields that $x_i \in P$ for some $i = 1, 2, ..., n$, which is a contradiction. Hence at least one $x_i = x$. Suppose that $x_i = x$ for all $i = 1, 2, ..., n$. Then $x = \bigvee_{i=1}^{n} x_i = 1$, which is a contradiction. Hence there exists at least one $x_i \neq x$. Therefore $1 = x_1 \vee x \vee x_2$ or $1 = x_1 \vee x$ or $1 = x \vee x_2$ where $x_1, x_2 \in X - P$.

Case I: Suppose $x_1 \vee x \vee x_2 = 1$. Then $x \in (x_1 \vee x_2)^+$ and $x_1 \vee x_2 \notin P$. Therefore $x \in O(P)$, which is a contradiction.

Case II: Suppose $x_1 \vee x = 1$. Then $x \in (x_1)^+$ and $x_1 \notin P$. Hence $x \in O(P)$, which is a contradiction.

Case III: Suppose $x \vee x_2 = 1$. Then $x \in (x_2)^+$ and $x_2 \notin P$. Hence $x \in O(P)$, which is a contradiction.

Hence $1 \notin S$, which gives $\langle 1 \rangle \cap S = \emptyset$. Then there exists a prime filter Q such that $\langle 1 \rangle \subseteq Q$ and $S \cap Q = \emptyset$. Since $x \in S$, we get $x \notin Q$. Hence

$$\begin{aligned} Q \cap S = \emptyset &\Rightarrow Q \cap (X - P) = \emptyset \text{ and } x \notin Q \\ &\Rightarrow Q \subseteq P \text{ and } x \notin Q \end{aligned}$$

Consider $\mathfrak{K} = \{Q_i \mid Q_i \text{ is a prime filter }, Q_i \subseteq P \text{ and } x \notin Q_i\}$. Clearly $Q \in \mathfrak{K}$. Let $\{Q_\alpha\}_{\alpha \in \Delta}$ be a chain in $\mathfrak{K}$. Then clearly $\bigcap_{\alpha \in \Delta} Q_\alpha \in \mathfrak{K}$. Hence by Zorn's Lemma, $\mathfrak{K}$ contains a minimal element, say M. Clearly M is a minimal prime filter of X. Therefore M is a minimal prime filter such that $M \subseteq P$ and $x \notin M$. Hence $x \notin \bigcap_{\alpha \in \Delta} S_\alpha$. Therefore $\bigcap_{\alpha \in \Delta} S_\alpha \subseteq O(P)$. □

The following corollary is a direct consequence of the above theorem.

Corollary 4.7. *Let X be a self-distributive and commutative BE-algebra. A prime filter P of X is minimal if and only if $O(P) = P$.*

Proof. Assume that P is minimal. By the main theorem, it is clear that $O(P) = P$. Conversely, assume that $O(P) = P$. Let $x \in P = O(P)$. Then there exists $y \notin P$ such that $x \vee y = 1$. Therefore P is minimal. □

Theorem 4.8. *Let F and G be two filters of a commutative BE-algebra X. Then $F \vee G = \langle F \cup G \rangle = \{x \in X \mid a * (b * x) = 1$ for some $a \in F$ and $b \in G\}$ is the smallest filter containing containing both F and G.*

Proof. Clearly $1 \in F \vee G$. Let $x, x * y \in F \vee G$. Then there exists $a, c \in F$ and $b, d \in G$ such that $a * (b * x) = 1$ and $c * (d * (x * y)) = 1$. Hence $x * (c * (d * y)) = 1$. Thus $x \leq c * (d * y)$.

Therefore $1 = a * (b * x) \leq a * (b * (c * (d * y))) = a * (c * (b * (d * y)))$. Since $a, c \in F$, we get $b * (d * y) \in F$. Put $f = b * (d * y)$.

$$\begin{aligned} b * (d * (f * y)) &= f * (b * (d * y)) \\ &= (b * (d * y)) * (b * (d * y)) \\ &= 1 \in G \end{aligned}$$

Since $b, d \in G$, it infers $f * y \in G$. Put $g = f * y$. Hence $f * (g * y) = g * (f * y) = (f * y) * (f * y) = 1$. Therefore $y \in F \vee G$. Therefore $F \vee G$ is a filter of X. Let $x \in F$. Clearly $x * (1 * x) = 1$. Hence $x \in F \vee G$. Therefore $F \subseteq F \vee G$. Similarly, we get $G \subseteq F \vee G$. Let H be a filter of X such that $F \subseteq H$ and $G \subseteq H$. Let $x \in F \vee G$. Then there exists $a \in F \subseteq H$ and $b \in G \subseteq H$ such that $a * (b * x) = 1 \in H$. Since $a, b \in H$, we get $x \in H$. Hence $F \vee G \subseteq H$. Therefore $F \vee G$ is the smallest filter of X such that $F \subseteq H$ and $G \subseteq H$. □

In view of the above theorem, it can be observed that the class $\mathcal{F}(X)$ of all filters of a commutative BE-algebra X forms a semi-lattice with respect to the operation $\vee$. In the following theorem, a sufficient condition is derived for the class of filter of the form $(x)^+, x \in X$ to become a sub semi-lattice of $(\mathcal{F}(X), \vee)$.

Theorem 4.9. *Let X be a self-distributive and commutative BE-algebra. If every prime filter contains a unique minimal prime filter, then*

(1) *for any prime filter P, $O(P)$ is a prime filter,*

(2) *for any $a, b \in X$, $a \vee b = 1$ implies $(a)^+ \vee (b)^+ = X$.*

Proof. (1). Let P be a prime filter of X. Then P contains a unique minimal prime filter, say Q. Then by Proposition 4.3, we get $O(P) = Q$. Therefore $O(P)$ is a minimal prime filter of X.

(2). Let $a, b \in X$ be such that $a \vee b = 1$. Suppose $(a)^+ \vee (b)^+ \neq X$. Then there exists a prime filter P such that $(a)^+ \vee (b)^+ \subseteq P$. Hence $a \notin O(P)$ and $b \notin O(P)$. Since $O(P)$ is prime, we get that $1 = a \vee b \notin O(P)$, which is a contradiction. Therefore $(a)^+ \vee (b)^+ = X$. □

Let X be a self-distributive and commutative BE-algebra and $\mathcal{P}_F(X)$ denote the set of all prime filters of X. For any $A \subseteq X$, let $K(A) = \{P \in \mathcal{P}_F(X) \mid A \nsubseteq P\}$ and for any $x \in X, K(x) = K(\{x\})$.

Lemma 4.10. *Let X be a self-distributive and commutative BE-algebra and $x, y \in X$. Then*

(1) $K(x) \cap K(y) = K(x \vee y)$,

(2) $K(x) = \emptyset \Leftrightarrow x = 1$,

(3) $\bigcup_{x \in X} K(x) = \mathcal{P}_F(X)$.

Proof. (1). Let $P \in \mathcal{P}_F(X)$ be such that $P \in K(x) \cap K(y)$. Then $x \notin P$ and $y \notin P$. Since P is prime, we get $x \vee y \notin P$. Hence $P \in K(x \vee y)$. Therefore $K(x) \cap K(y) \subseteq K(x \vee y)$. Conversely, assume that $P \in \mathcal{P}_F(X)$ and $P \in K(x \vee y)$. Hence $x \vee y \notin P$. If $x \in P$, then $x \vee y \in P$. Thus $x \notin P$. Therefore $P \in K(x)$. Similarly, we get $P \in K(y)$. Hence $P \in K(x) \cap K(y)$.
(2). Since $\{1\} \subseteq P$ for all $P \in \mathcal{P}_F(X)$, it is obvious.
(3). Let $P \in \mathcal{P}_F(X)$. Since P is a proper filter, there exists $a \in X$ such that $a \notin P$. Hence $P \in K(a) \subseteq \bigcup_{x \in X} K(x)$. Therefore $\mathcal{P}_F(X) \subseteq \bigcup_{x \in X} K(x)$. Clearly $\bigcup_{x \in X} K(x) \subseteq \mathcal{P}_F(X)$. Therefore $\bigcup_{x \in X} K(x) = \mathcal{P}_F(X)$. □

In the following theorem, a set of equivalent conditions is derived for every prime filter of a commutative BE-algebra to become a minimal prime filter.

Theorem 4.11. *Let X be a self-distributive and commutative BE-algebra X. Then the following are equivalent:*

(1) *For any $P, Q \in \mathcal{P}_F(X)$ with $P \neq Q$, there exist $x, y \in X$ such that* $P \in K(x); Q \in K(y)$ *and* $K(x) \cap K(y) = \emptyset$,

(2) *for each $P \in \mathcal{P}_F(X)$, P is the unique member of $\mathcal{P}_F(X)$ such that* $O(P) \subseteq P$,

(3) *every prime filter is minimal,*

(4) *every prime filter is maximal.*

Proof. (1) ⇒ (2): Assume that $\mathcal{P}_F(X)$ satisfies the condition (1). Let $P \in \mathcal{P}_F(X)$. Clearly $O(P) \subseteq P$. Suppose $Q \in \mathcal{P}_F(X)$ such that $Q \neq P$ and $O(P) \subseteq Q$. By the condition (1), there exists $x, y \in X$ such that $P \in K(x)$, $Q \in K(y)$ and $K(x \vee y) = K(x) \cap K(y) = \emptyset$. Hence $x \notin P, y \notin Q$. Since $K(x \vee y) = \emptyset$, by Lemma 4.10(2), we get $x \vee y = 1$. Therefore $y \in O(P) \subseteq Q$, which is a contradiction to that $y \notin Q$. Hence $P = Q$. Therefore P is the unique member of $\mathcal{P}_F(X)$ such that $O(P) \subseteq P$.
(2) ⇒ (3): Assume the condition (2). Let P be a prime filter of X. Suppose P is not minimal. Let Q be a prime filter in X such that $Q \subseteq P$. Hence $O(Q) \subseteq Q \subseteq P$, which is a contradiction to the assumption. Therefore P is a minimal prime filter in X.
(3) ⇒ (4): Since every maximal filter is prime, it is clear.
(4) ⇒ (1): Assume that every prime filter is maximal. Let P and Q be two distinct elements of $\mathcal{P}_F(X)$. Hence $O(Q) \nsubseteq P$. Choose $x \in O(Q)$ such that $x \notin P$. Since $x \in O(Q)$, there exists $y \notin Q$ such that $x \in (y)^+$. Hence $x \vee y = 1$. Thus it yields, $P \in K(x)$, $Q \in K(y)$. Since $x \vee y = 1$, we get that $K(x) \cap K(y) = K(x \vee y) = \emptyset$. □

5. Congruences and minimal prime filters

In this section, some properties of minimal prime filters of a self-distributive and commutative BE-algebras are studied with the help of congruences. An equivalency is obtained between the minimal prime filters of a BE-algebra and its quotient algebra with respect to this congruence.

Definition 5.1. Let S be any subset of a commutative BE-algebra X and $x, y \in X$. Define a binary relation ψ_S on X by $(x, y) \in \psi_S$ if and only if $x \vee a = y \vee a$ for some $a \in S$.

Obviously, the above relation ψ_S is reflexive and symmetric on the BE-algebra X. In general, the relation ψ_S is not transitive and hence it is not an equivalence relation on X.

Example 5.2. In Example 4.2, consider the subset $S = \{c, d\}$ of X. It is easy to check that for $x, y \in X$, $(x, y) \in \psi_S$ if and only if $x \vee t = y \vee t$ for some $t \in S$. Also, observe that ψ_S is not transitive because of $(a, b) \in \psi_S$ and $(b, d) \in \psi_S$ but $(a, d) \notin \psi_S$.

In the following result, it is observed that the above binary relation ψ_S is an equivalence relation on X.

Proposition 5.3. *Let S be a $\vee$-closed subset of a commutative BE-algebra X. Then ψ_S is an equivalence relation on X.*

Proof. Clearly ψ_S is reflexive and symmetric. Let $(x, y), (y, z) \in \psi_S$. Then $x \vee a = y \vee a$ and $y \vee b = z \vee b$ for some $a, b \in S$. Now $x \vee a \vee b = y \vee a \vee b = a \vee y \vee b = a \vee z \vee b = z \vee a \vee b$. Since $a \vee b \in S$, we get $(x, z) \in \psi_S$. Hence ψ_S is transitive and so ψ_S is an equivalence relation. □

In a commutative BE-algebra X, it is observed in Proposition 2.6 that $(x * y) \vee z \leq (x \vee z) * (y \vee z)$ for all $x, y, z \in X$. Here after by a commutative BE-algebra, we mean a commutative BE-algebra with the equality *i.e.* $\vee$ is right distributive over the operation $*$.

Example 5.4. Let $X = \{1, a, b, c\}$ be a set. Define a binary operation $*$ on X as follows:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	a	1	c
c	1	a	b	1

$\vee$	1	a	b	c
1	1	1	1	1
a	1	a	1	1
b	1	1	b	1
c	1	1	1	c

Then $(X, *, \vee, 1)$ is a commutative BE-algebra. It can be easily verified that $(x * y) \vee z = (x \vee z) * (y \vee z)$ for all $x, y, z \in X$. Hence $\vee$ is right distributive over $*$.

Under the presence of the above property, it is observed in the following theorem that the relation ψ_S is a congruence on X.

Proposition 5.5. *Let S be a $\vee$-closed subset of a commutative BE-algebra X. If $\vee$ is right distributive over the operation $*$, then ψ_S is a congruence on X.*

Proof. Let $(x,y),(z,w) \in \psi_S$. Then $x \vee a = y \vee a$ and $z \vee b = w \vee b$ for some $a, b \in S$. Now

$$\begin{aligned} (x * z) \vee (a \vee b) &= (x \vee (a \vee b)) * (z \vee (a \vee b)) \\ &= ((x \vee a) \vee b) * (a \vee (z \vee b)) \\ &= ((y \vee a) \vee b) * (a \vee (w \vee b)) \\ &= (y \vee (a \vee b)) * (w \vee (a \vee b)) \\ &= (y * w) \vee (a \vee b). \end{aligned}$$

Hence $(x * z, y * w) \in \psi_S$. Therefore ψ_S is a congruence on X. □

For any commutative BE-algebra X where $\vee$ is right distributive over $*$, it is clear that the quotient algebra $X_{/\psi_S}$ is also a commutative BE-algebra and a BE-semilattice with respect to the following operations.

$$[x]_{\psi_S} * [y]_{\psi_S} = [x * y]_{\psi_S} \text{ and } [x]_{\psi_S} \vee [y]_{\psi_S} = [x \vee y]_{\psi_S}$$

where $[x]_{\psi_S}$ is the congruence class of x modulo ψ_S. It can be routinely verified that the mapping $\Psi : X \longrightarrow X_{/\psi_S}$ defined by $\Psi(x) = [x]_{\psi_S}$ is an epimorphism.

Lemma 5.6. *Let S be a $\vee$-closed subset of a self-distributive and commutative BE-algebra X. Then the following conditions hold:*

(1) *$[1]_{\psi_S}$ is the unit element of $X_{/\psi_S}$,*
(2) *If G is a filter of $X_{/\psi_S}$, then $\Psi^{-1}(G)$ is a filter of X,*
(3) *If P is a prime filter of $X_{/\psi_S}$, then $\Psi^{-1}(P)$ is a prime filter of X.*

Proof. (1). It is clear.
(2). Let G be a filter of $X_{/\psi_S}$. From condition (1), we get that $[1]_{\psi_S}$ is a unit element of $X_{/\psi_S}$. Let $x, x * y \in \Psi^{-1}(G)$. Then $[x]_{\psi_S} = \Psi(x) \in G$ and $[x]_{\psi_S} * [y]_{\psi_S} = \Psi(x) * \Psi(y) = \Psi(x * y) \in G$. Since G is a filter of $X_{/\psi_S}$, we get $[y]_{\psi_S} \in G$. Hence $y \in \Psi^{-1}(G)$. Therefore $\Psi^{-1}(G)$ is a filter of X.
(3). Let $x, y \in X$ and $x \vee y \in \Psi^{-1}(P)$. Then $[x]_{\psi_S} \vee [x]_{\psi_S} = [x \vee y]_{\psi_S} = \Psi(x \vee y) \in P$. Since P is prime in $X_{/\psi_S}$, we get $\Psi(x) = [x]_{\psi_S} \in P$ or $\Psi(y) = [y]_{\psi_S} \in P$. Hence $x \in \Psi^{-1}(P)$ or $y \in \Psi^{-1}(P)$. Therefore $\Psi^{-1}(P)$ is prime. □

Definition 5.7. Let S be a $\vee$-closed subset of a commutative BE-algebra X. For any filter F of X, define $\overline{F} = \{[x]_{\psi_S} \mid x \in F\}$.

By the nature of congruences of BE-algebras, it can be easily observed that $\overline{F}$ is a filter in $X_{/\psi_S}$ whenever F is a filter in X. In general, this fact may not hold for prime filters of BE-algebras. Unless, X is self-distributive and commutative BE-algebra.

Proposition 5.8. *Let P be a prime filter and S a $\vee$-closed subset of a self-distributive and commutative BE-algebra X such that $P \cap S = \emptyset$. Then*

(1) *$x \in P$ if and only if $[x]_{\psi_S} \in \overline{P}$,*
(2) *$\overline{P} \cap \overline{S} = \emptyset$,*
(3) *If P is a prime filter of X, then $\overline{P}$ is a prime filter of $X_{/\psi_S}$.*

Proof. (1). Let $x \in P$. Then clearly $[x]_{\psi_S} \in \overline{P}$. Conversely, let $[x]_{\psi_S} \in \overline{P}$. Then we get $[x]_{\psi_S} = [t]_{\psi_S}$ for some $t \in P$. Hence $(x,t) \in \psi_S$. Thus, it yields $x \vee a = t \vee a \in P$ for some $a \in S$. Since $P \cap S = \emptyset$, we get $a \notin P$. Hence $x \in P$. Therefore $x \in P$ if and only if $[x]_{\psi_S} \in \overline{P}$.
(2). Suppose $\overline{P} \cap \overline{S} \neq \emptyset$. Then choose $[x]_{\psi_S} \in \overline{P} \cap \overline{S}$ where $x \in X$. Then by condition (1), we get that $x \in P$ and $[x]_{\psi_S} \in \overline{S}$. Hence we get

$$\begin{aligned} [x]_{\psi_S} \in \overline{S} &\Rightarrow [x]_{\psi_S} = [y]_{\psi_S} \quad \text{for some } y \in S \\ &\Rightarrow (x,y) \in \psi_S \\ &\Rightarrow x \vee a = y \vee a \quad \text{for some } a \in S \\ &\Rightarrow x \vee a \in S \quad \text{since } y \vee a \in S \\ &\Rightarrow x \vee a \in P \cap S \quad \text{since } x \in P \end{aligned}$$

which is a contradiction to $P \cap S = \emptyset$. Therefore, it concludes that $\overline{P} \cap \overline{S} = \emptyset$.
(3). Since P is a filter of X, it is clear that $\overline{P}$ is a filter in $X_{/\psi_S}$. Since P is a proper filter in X, by (1), we get that $\overline{P}$ is a proper filter in $X_{/\psi_S}$. Let $[x]_{\psi_S}, [y]_{\psi_S} \in X_{/\psi_S}$. Then we have

$$\begin{aligned} [x]_{\psi_S} \vee [y]_{\psi_S} \in \overline{P} &\Rightarrow [x \vee y]_{\psi_S} \in \overline{P} \\ &\Rightarrow x \vee y \in P \quad \text{from (1)} \\ &\Rightarrow x \in P \text{ or } y \in P \\ &\Rightarrow [x]_{\psi_S} \in \overline{P} \text{ or } [y]_{\psi_S} \in \overline{P} \end{aligned}$$

Therefore $\overline{P}$ is a prime filter in $X_{/\psi_S}$. □

Proposition 5.9. *Let S be a $\vee$-closed subset of a self-distributive and commutative BE-algebra X and P a prime filter of X such that $P \cap S = \emptyset$. Then the mapping $P \mapsto \overline{P}$ is an order* *isomorphism of the set of all prime filters of X disjoint from S onto the set of all prime filters of $X_{/\psi_S}$.*

Proof. Let P, Q be two prime filters of X such that $P \cap S = \emptyset$ and $Q \cap S = \emptyset$. Then by Proposition 5.8(1), we get that $P \subseteq Q \Leftrightarrow \overline{P} \subseteq \overline{Q}$. Let P be a prime filter of X such that $P \cap S = \emptyset$. Then by Proposition 5.8(3), we get that $\overline{P}$ is a prime filter of $X_{/\psi_S}$. Let R be a prime filter of $X_{/\psi_S}$. Consider $P = \{x \in X \mid [x]_{\psi_S} \in R\}$. Since R is a filter of $X_{/\psi_S}$, we get that P is a filter of X. Let $x, y \in X$ such that $x \vee y \in P$. Then $[x]_{\psi_S} \vee [y]_{\psi_S} = [x \vee y]_{\psi_S} \in R$. Since R is prime, we get either $[x]_{\psi_S} \in R$ or $[y]_{\psi_S} \in R$. Hence either $x \in P$ or $y \in P$. Therefore P is a prime filter of X. Clearly $\overline{P} = R$. Suppose $P \cap S \neq \emptyset$. Choose $a \in P \cap S$. Then $[a]_{\psi_S} \in R$ and $a \in S$. Let $[y]_{\psi_S} \in X_{/\psi_S}$ be an arbitrary element. Now for any $a \in S$ and $y \in X$, we have the following:

$$\begin{aligned} a \vee y = a \vee y \vee a \quad &\Rightarrow \quad (y, y \vee a) \in \psi_S \\ &\Rightarrow \quad [y]_{\psi_S} = [y \vee a]_{\psi_S} \\ &\Rightarrow \quad [y]_{\psi_S} = [y]_{\psi_S} \vee [a]_{\psi_S} \in R \qquad \text{since } R \text{ is a filter} \\ &\Rightarrow \quad [y]_{\psi_S} \in R \end{aligned}$$

Hence we get $X_{/\psi_S} \subseteq R$, which is a contradiction. Thus, it infers that $P \cap S = \emptyset$. Therefore $P \mapsto \overline{P}$ is an order isomorphism from the set of all prime filters of X which are disjoint from S onto the set of all prime filters of $X_{/\psi_S}$. □

The following corollary is a direct consequence of the above theorem.

Corollary 5.10. *For any $\vee$-closed subset S and P a prime filter of a self-distributive and commutative BE-algebra X such that $P \cap S = \emptyset$, the above map $P \mapsto \overline{P}$ induces a one-to-one correspondence between the set of all minimal prime filters of X which are disjoint from S and the set of all minimal prime filters of $X_{/\psi_S}$.*

Two filters F and G of a BE-algebra X are called co-maximal if $F \vee G = X$. Clearly any two distinct maximal filters of a commutative BE-algebra are co-maximal but not any two minimal prime filters.

Example 5.11. Let $X = \{1, a, b, c\}$ be a set. Define a binary operation $*$ on X as follows:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	b	b
b	1	a	1	a
c	1	1	1	1

$\vee$	1	a	b	c
1	1	1	1	1
a	1	a	1	a
b	1	1	b	b
c	1	a	b	c

Then $(X, *, \vee, 1)$ is a commutative BE-algebra. Clearly $F = \{1, a\}$ and $G = \{1, b\}$ are two distinct maximal filters of X. Observe that $F \vee G = \{1, a, b, c\} = X$. Therefore F and G are co-maximal.

Example 5.12. Let $X = \{1, a, b, c, d, e\}$ be a set. Define a binary operation $*$ on X as follows:

$*$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	1	1	b	d	b	c
b	1	a	1	d	a	d
c	1	e	1	1	1	a
d	1	1	1	b	1	b
e	1	b	1	1	1	1

$\vee$	1	a	b	c	d	e
1	1	1	1	1	1	1
a	1	a	1	b	a	a
b	1	1	b	b	b	b
c	1	b	b	c	d	c
d	1	a	b	d	d	d
e	1	a	b	c	d	e

Then $(X, *, \vee, 1)$ is a commutative BE-algebra. Clearly $F = \{1, a\}$ and $G = \{1, b\}$ are two distinct minimal prime filters of X. Observe that $F \vee G = \{1, a, b, d\} \neq X$. Therefore F and G are not co-maximal.

In the following, we obtain an equivalent condition for two minimal prime filters to become co-maximal.

Theorem 5.13. *Let S be a $\vee$-closed subset of a self-distributive and commutative BE-algebra X. Then any two distinct minimal prime filters of X are co-maximal if and only if any two distinct minimal prime filters of $X_{/\psi_S}$ are co-maximal.*

Proof. Assume that any two distinct minimal prime filters of X are co-maximal. Let P_1, P_2 be two distinct minimal prime filters of $X_{/\psi_S}$. Then by above corollary, there exist two minimal prime filters Q_1 and Q_2 of X such that $Q_1 \cap S = \emptyset$ and $Q_2 \cap S = \emptyset$. Also $\overline{Q_1} = P_1$ and $\overline{Q_2} = P_2$. Since P_1 and P_2 are distinct, we get that Q_1 and Q_2 are distinct. By the assumption, we get that $Q_1 \vee Q_2 = X$. Hence for any $x \in X$, we can have

$$a * (b * x) = 1 \text{ where } a \in Q_1 \text{ and } b \in Q_2$$

Since $a \in Q_1$, we get $[a]_{\psi_S} \in \overline{Q_1} = P_1$. Similarly, we get $[b]_{\psi_S} \in \overline{Q_2} = P_2$. Hence we get

$$[a]_{\psi_S} * ([b]_{\psi_S} * [x]_{\psi_S}) = [a * (b * x)]_{\psi_S} = [1]_{\psi_S}$$

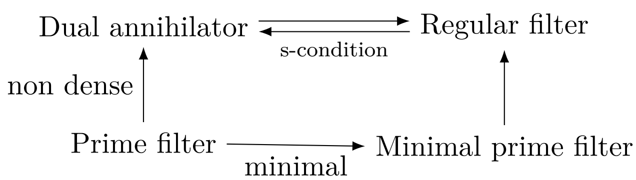
Thus, for any $x \in X$, we obtained $[x]_{\psi_S} \in P_1 \vee P_2$. Hence $P_1 \vee P_2 = X_{/\psi_S}$.

Conversely, assume that any two distinct minimal prime filters of $X_{/\psi_S}$ are co-maximal. Let P be a prime filter of X. Suppose P contains two distinct minimal prime filters, say P_1 and P_2. Consider $S = X - P$. Then clearly S is a $\vee$-closed subset of X such that $P_1 \cap S = \emptyset = P_2 \cap S$. Then by Corollary 5.10, we get that $\overline{P_1}$ and $\overline{P_2}$ are distinct minimal prime filters in $X_{/\psi_S}$ such that $\overline{P_1}, \overline{P_2} \subseteq \overline{P}$. Thus $\overline{P}$ is containing two distinct minimal prime filters of $X_{/\psi_S}$, which is a contradiction. Hence P contains a unique minimal prime filter. Therefore any two distinct minimal prime filters of X are co-maximal. □

6. Conclusion

In this paper, the notion of minimal prime filters is introduced in commutative BE-algebras. A characterization theorem is derived for minimal prime filters of commutative BE-algebras. A set of equivalent conditions is derived for every prime filter of a BE-algebra to become a minimal prime filter. Some properties of minimal prime filters of a commutative BE-algebra are studied with the help of congruences. An equivalency is obtained between the minimal prime filters of a commutative BE-algebra and the minimal prime filters of its quotient algebra with respect to this congruence. We think such results are very useful for the further characterization of minimal prime filters in terms of congruences of this structure.

In the following diagram we show the relationships between some filters. The notion "$A \to B$", means "A should be B".



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