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Research Paper

SOME RESULTS ON UNIFORM MV-ALGEBRAS

FARZANEH RAJABISOTUDEH, NADER KOUHESTANI* AND GHOLAM REZA REZAEI

Abstract. In this paper, we study Tychonoff spaces and uniformities on MV-algebras. In particular, we find some conditions under which an MV-algebra can be made into a Tychonoff space. Also, we find uniformities that make an MV-algebra into a uniform MV-algebra. Next, we discuss some algebraic and topological properties of uniform MV-algebras. More precisely, we study the existence of closed ideals and closed filters, and examine the way these are related to uniform MV-algebras. Furthermore, we obtain some results on the uniform continuity of the operations and its consequences.

1. Introduction

Some of the concepts which are related to the notion of measure, like uniform convergence and uniform continuity, can be easily defined in pseudo-metric spaces. But, such concepts cannot be defined in topological spaces. In 1938, Weil [16] introduced uniform spaces as spaces between pseudo-metric spaces and topological spaces. In 1948, Bourbaki presented a

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*Corresponding author

systematic theory for these spaces and determined their relationship to topological spaces. Uniformities defined on algebraic structures are important research topics in mathematics, of which three natural uniform structures on topological groups are the most important ones.

Algebraic structures related to logic are structures which have been introduced to the mathematics community around the second half of the last century, and their algebraic properties have been studied. These include BCK-algebras, BCC-algebras, BL-algebras, MV-algebras, etc. An important research topic of recent decades is the study of the aforementioned structures equipped with topology (see [11], [14] and [13]). Algebraic structures related to logic which are endowed with uniformity have also been discussed in recent years. For example, Khanegir et al. [12] introduced the notion of uniform BL-algebra and studied some of its properties. See [15], [7] and [6] for some other examples.

MV-algebras are among the most important algebraic structures related to logic, which were introduced by Chang [8] in 1958. These structures prove the completeness theorem for $\aleph_0$-valued Lukasiewicz logic. Barbieri and Weber introduced the notion of uniform MV-algebra in [2], and studied submeasures, real-valued measures, and the uniformity generated by a measure.

Our aim in this paper is to find some conditions under which a topology on an MV-algebra can be made into a Tychonoff space. Also, we study the relation between uniformities and algebraic properties of MV-algebras.

The paper is organized as follows. In Section 2, we recall some definitions and results from the theories of MV-algebras and uniform spaces. In Section 3, we recall the definition of Tychonoff spaces, and derive some conditions under which an MV-algebra can be made into a Tychonoff space. Also, we study the way T_0, T_1, T_2 and Urysohn spaces are related to each other. In Section 4, we first recall the definition of uniform MV-algebras and then, in Theorem 4.5, we find those uniformities that make an MV-algebra into a uniform MV-algebra. Moreover, we investigate closed ideals and closed filters of uniform MV-algebras at the end of this section. Finally, we prove the existence of a contravariant functor from the category of uniform MV-algebras to the category of semigroups.

2. **Preliminaries**

In this section, we recall some definitions and results from the theories of MV-algebras and uniform spaces.

MV-algebras

Recall from [10] that an *MV-algebra* is an algebra $(A, \oplus, *, 0)$ of type $(2, 1, 0)$ such that for every $x, y \in A$,

$(M1)$ $(A, \oplus, 0)$ is a commutative monoid,

$(M2)$ $x \oplus 0^* = 0^*$,

$(M3)$ $(x^*)^* = x$, and

$(M4)$ $(x^* \oplus y)^* \oplus y = (x \oplus y^*)^* \oplus x$.

In an MV-algebra A, for every $x, y \in A$ define

$(M5)$ $1 := 0^*$;

$(M6)$ $x \odot y := (x^* \oplus y^*)^*$;

$(M7)$ $x \ominus y := x \odot y^*$;

$(M8)$ $x \to y := (x \odot y^*)^*$;

$(M9)$ $x \rightsquigarrow y := (x \oplus y^*)^*$.

In an MV-algebra A, for every $x, y \in A$ we write $x \leq y$ if and only if $x^* \oplus y = 1$. It is well-known that $\leq$ is a partial order on A, which gives A the structure of a distributive lattice, where join and meet are defined by $x \wedge y = y \odot (y^* \oplus x)$ and $x \vee y = x \oplus (y \ominus x)$, respectively, 0 is the least element and 1 is the greatest element. By $(M6)$ and $(M7)$, $x \leq y \Longleftrightarrow x \ominus y = 0$ for every $x, y \in A$.

If I is a subset of A and $x \in A$, then we denote the set $\{x \diamond z : z \in I\}$ by $x \diamond I$, and the set $\{z \diamond x : z \in I\}$ by $I \diamond x$, where $\diamond \in \{\oplus, \odot, \ominus, \to, \rightsquigarrow\}$.

Proposition 2.1. [10] *In an MV-algebra A, the following statements are true.*

$(M10)$ $x \oplus x^* = 1, x \odot x^* = 0$.

$(M11)$ $(A, \odot, 1)$ *is a commutative monoid.*

$(M12)$ $x \odot 0 = x \odot x^* = 0$.

$(M13)$ $x \oplus y = 0 \Longrightarrow x = y = 0$.

$(M14)$ $x \odot y = 1 \Longrightarrow x = y = 1$.

$(M15)$ $(x \wedge y)^* = x^* \vee y^*, (x \vee y)^* = x^* \wedge y^*$.

$(M16)$ $x \leq y \Longleftrightarrow y^* \leq x^*$.

$(M17)$ $x \leq y \Longrightarrow x \oplus z \leq y \oplus z, x \odot z \leq y \odot z$.

$(M18)$ $x \odot y \leq x \wedge y \leq x \leq x \vee y \leq x \oplus y$.

$(M19)$ $x \ominus y \leq x \leq x \oplus y$.

$(M20)$ $y \odot (x \oplus z) \leq x \oplus (y \odot z)$.

$(M21)$ $z \odot x^* \leq (z \odot y^*) \oplus (y \odot x^*)$.

$(M22)$ $(z \oplus y) \odot y^* \leq z$.

$(M23)$ $y \odot (z \oplus y)^* = 0$.

$(M24)$ $(x^* \odot y)^* \odot y = (y^* \odot x)^* \odot x$.

$(M25)$ $x \odot (y \odot z) = (x \odot y) \odot z$.

$(M26)$ $(x \oplus y) \ominus y \leq y^*$.

$(M27)$ $x \odot z \leq y \Longleftrightarrow x \leq z^* \oplus y$.

$(M28)$ $x \odot (y \to z) \leq (x \odot y) \to (x \odot z)$.

$(M29)$ $(x_1 \to y_1) \odot (x_2 \to y_2) \leq (x_1 \odot x_2) \to (y_1 \odot y_2)$.

$(M30)$ $(x \ominus y) \leq y^*$.

$(M31)$ $(x \oplus y) \ominus (a \oplus b) \leq (x \ominus a) \oplus (y \ominus b)$.

$(M32)$ $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$.

$(M33)$ $(x \ominus y) \oplus y = (y \ominus x) \oplus x$.

$(M34)$ $x \odot (x^* \oplus y) = y \odot (y^* \oplus x)$.

$(M35)$ $x \ominus 0 = x, x \ominus x = 0 \ominus x = x \ominus 1 = 0, 1 \ominus x = x^*$.

$(M36)$ $x \oplus x = x \Longleftrightarrow x \odot x = x$.

$(M37)$ $x \oplus (\wedge_{i \in I} x_i) = \wedge(x \oplus x_i), x \odot (\vee_{i \in I} x_i) = \vee(x \odot x_i)$.

$(M38)$ $x \leq y \Longleftrightarrow x \wedge y = x \Longleftrightarrow x \vee y = y$.

$(M39)$ $(x \wedge y) \wedge z = x \wedge (y \wedge z), x \wedge (y \wedge z) = (x \wedge y) \wedge (x \wedge z)$.

$(M40)$ $(x \to a) \odot (b \to y) \leq (a \to b) \to (x \to y)$.

Definition 2.2. Let A be an MV-algebra.

(1) [8] A non-empty subset I of A is called an *ideal* if it satisfies the following conditions.

(I1) For every $x, y \in I, x \oplus y \in I$.

(I2) If $x \in I$ and $y \leq x$, then $y \in I$.

(2) [10] A non-empty subset F of A is called a *filter* if it satisfies the following conditions.

(F1) For every $x, y \in F, x \odot y \in F$.

(F2) If $x \in F$ and $x \leq y$, then $y \in F$.

Proposition 2.3. [10] *Let I and F be subsets of an MV-algebra A. Then, I is an ideal if and only if the following conditions are satisfied.*

(I3) $0 \in I$.

(I4) $y \in I$ *and* $x \ominus y \in I$ *imply* $x \in I$.

Also, F is a filter if and only if the following conditions are satisfied.

(F3) $1 \in F$.

(F4) $x \in F$ *and* $x \to y \in F$ *imply* $y \in F$.

Proposition 2.4. [10] *Let F be a filter and I be an ideal of an MV-algebra A. Then, the following relations are congruence relations on A.*

$$x \overset{F}{\equiv} y \iff x \to y \in F \text{ and } y \to x \in F.$$

$$x \overset{I}{\equiv} y \iff x \ominus y \in I \text{ and } y \ominus x \in I.$$

Moreover, if $x/F = \{y \in A : x \overset{F}{\equiv} y\}$, $A/F = \{x/F : x \in A\}$, $x/I = \{y \in A : x \overset{I}{\equiv} y\}$ and $A/I = \{x/I : x \in A\}$, then both A/F and A/I are quotient MV-algebras with the operations

$$x/F \odot y/F = (x \odot y)/F,\ x/I \oplus y/I = (x \oplus y)/I,\ (x/F)^* = x^*/F \text{ and } (x/I)^* = x^*/I.$$

Uniform Spaces

Let X be a non-empty set. A *uniformity* on X is a non-empty family $\mathcal{U}$ of subsets of $X \times X$ with the following properties.

(U_1) $\triangle = \{(x,x) : x \in X\} \subseteq U$, for each $U \in \mathcal{U}$.

(U_2) If $U \in \mathcal{U}$, then $U^{-1} = \{(x,y) \in X \times X : (y,x) \in U\}$ belongs to $\mathcal{U}$.

(U_3) If $U \in \mathcal{U}$, $V \circ V \subseteq U$ for some $V \in \mathcal{U}$, where $V \circ V = \{(x,y) : \exists z \in X \text{ s.t. } (x,z),(z,y) \in V\}$.

(U_4) If $U, V \in \mathcal{U}$, then $U \cap V \in \mathcal{U}$.

(U_5) If $U \in \mathcal{U}$ and $U \subseteq V$, then $V \in \mathcal{U}$.

The pair $(X,\mathcal{U})$ is called a *uniform space*.

Let $(X,\mathcal{U})$ be a uniform space. Each element of $\mathcal{U}$ is called an *entourage*, and $U \in \mathcal{U}$ is *symmetric* if $U = U^{-1}$. A subfamily $\mathcal{B}$ of $\mathcal{U}$ is called a *base* for $\mathcal{U}$ if each member of $\mathcal{U}$ contains a member of $\mathcal{B}$. A subfamily $\mathcal{S}$ of $\mathcal{U}$ is called a *subbase* for $\mathcal{U}$ if the collection of all finite intersections of members of $\mathcal{S}$ is a base for $\mathcal{U}$. The set

$$\tau = \{G \subseteq X : \forall x \in G \ \ \exists U \in \mathcal{U} \text{ s.t. } U[x] \subseteq G\},$$

where $U[x] = \{y \in X : (x,y) \in U\}$, is a topology on X which is called the *uniform topology* on X induced by $\mathcal{U}$. [9]

Lemma 2.5. [9] *A non-empty family $\mathcal{B}$ of subsets of $X \times X$ is a base for the uniformity $\mathcal{U} = \{U \subseteq X \times X : \exists B \in \mathcal{B}, B \subseteq U\}$ if and only if the following statements are true.*

$(B1)$ $\triangle = \{(x,x) : x \in X\} \subseteq U$, *for each* $U \in \mathcal{B}$.

$(B2)$ *If U belongs to $\mathcal{B}$, then U^{-1} contains a member of $\mathcal{B}$.*

$(B3)$ *If U belongs to $\mathcal{B}$, then there exists V in $\mathcal{B}$ such that $V \circ V \subseteq U$.*

$(B4)$ *If U and V are in $\mathcal{B}$, then there exists $W \in \mathcal{B}$ such that $W \subseteq U \cap V$.*

Suppose that $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ are uniform spaces. *The product* of $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ is a uniform space $(Z,\mathcal{W})$ with the underlying set $Z = X \times Y$ and the uniformity $\mathcal{W}$ on Z whose base consists of the sets

$$W_{U,V} = \{((x,y),(x',y')) \in Z \times Z : \ (x,x') \in U, (y,y') \in V\},$$

where $U \in \mathcal{U}$ and $V \in \mathcal{V}$. The uniformity $\mathcal{W}$ is written as $\mathcal{W} = \mathcal{U} \times \mathcal{V}$.[9]

Definition 2.6. [9] Let $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ be a map between uniform spaces. The map f is *uniformly continuous* if for each $V \in \mathcal{V}$, there exists $U \in \mathcal{U}$ such that $(f(x), f(y)) \in V$ for all $(x,y) \in U$, that is, $(f \times f)(U) \subseteq V$. We denote $f \times f$ by $f^{(2)}$.

In Definition 2.6, if f is bijective and the maps f and f^{-1} are uniformly continuous, then the map f is called a *unimorphism*, and X and Y are said to be *uniformly equivalent*. [9].

3. Tychonoff spaces and MV-algebras

It is well-known that every Tychonoff space induces a uniformity. Hence, in this section, we first recall the definition of Tychonoff spaces, and then find some conditions under which a topology on an MV-algebra can make it into a Tychonoff space. For example in Propositions 3.3,3.4 and 3.6 we use locally compact sets to do this. In Propositions 3.8 and 3.9, we use open neighborhoods of 0 and 1 to convert a topological MV-algebra to a Tychonff space.

Recall from [9] that a topological space (X, τ) is said to be
(i) a *regular* space if for each $x \in U \in \tau$, there exists an open set H such that $x \in H \subseteq \overline{H} \subseteq U$. A regular T_1-space is called a T_3-space;
(ii) a *normal* space if for each closed set S and each open set U containing S, there is an open set H such that $S \subseteq H \subseteq \overline{H} \subseteq U$;
(iii) a *completely regular* space if for every closed set F in X and for each $x \in X$ not belonging to F, there exists a continuous function $f : X \to [0;1]$ such that $f(x) = 0$ and $f(F) = 1$;
(iv) a *Tychonoff* space if it is a T_1, completely regular space.

Lemma 3.1. *Let I be an ideal in an MV-algebra A. Then, the following statements are true.*
(i) y is the maximum of $\frac{x}{I}$ if and only if $\frac{x}{I} = y \ominus I$.
(ii) y is the minimum of $\frac{x}{I}$ if and only if $\frac{x}{I} = y \oplus I$.

Proof. (i) Let y be the maximum of $\frac{x}{I}$ and $z \in \frac{x}{I} = \frac{y}{I}$. Then, $(y \to z)^* = y \ominus z \in I$ and $z = y \wedge z = y \odot (y \to z) \in y \odot I^* = y \ominus I$. So, $\frac{x}{I} \subseteq y \ominus I$. If $z \in y \ominus I$, then $z = y \ominus a$ for some $a \in I$. Now,

$$z \ominus y = (y \ominus a) \ominus y = y \odot a^* \odot y^* = 0 \in I.$$

On the other hand, by $(M24)$ and $(M19)$, $y \ominus z = y \ominus (y \ominus a) = a \ominus (a \ominus y) \leq a \in I$. Hence, $z \in \frac{y}{I} = \frac{x}{I}$.

Conversely, if $\frac{x}{I} = y \ominus I$, then $y = y \ominus 0 \in y \ominus I = \frac{x}{I}$. If $z \in \frac{x}{I}$, then $z = y \ominus a$ for some $a \in I$. The inequality $z = y \ominus a \leq y$ implies $y = max\frac{x}{I}$.
(ii) Let y be the minimum of $\frac{x}{I}$ and $z \in y \oplus I$. Then, there exists $b \in I$ such that $z = y \oplus b$. By $(M22)$, $(y \oplus b) \ominus y \leq b \in I$ which implies that $(y \oplus b) \ominus y \in I$. On the other hand,

$$y \ominus (y \oplus b) = y \odot (y \oplus b)^* = y \odot y^* \odot b^* = 0 \in I.$$

Hence, $z = y \oplus b \in \frac{y}{I} = \frac{x}{I}$. Thus, $y \oplus I \subseteq \frac{x}{I}$. If $z \in \frac{x}{I} = \frac{y}{I}$, then $y = min\frac{x}{I}$ implies $z = y \vee z = y \oplus (y \ominus z) \in y \oplus I$. So, $\frac{x}{I} = \frac{y}{I} \subseteq y \oplus I$. Conversely, let $\frac{x}{I} = y \oplus I$. It is obvious that $y \in y \oplus I = \frac{x}{I}$. If $z \in \frac{x}{I}$, then $z = y \oplus b \geq y$ for some $b \in I$. Hence, y is the minimum of $\frac{x}{I}$. □

Example 3.2. If X is a non-empty set, then $(P(X), \oplus, *, \varphi)$ is an MV-algebra, where $P(X)$ is the power set of X, $* : P(X) \to P(X)$ is a map defined by $B^* = X \setminus B$ and $B \oplus C = B \cup C$, for every $B, C \in P(X)$. Let a be an element of X. It is easy to see that the set $I = \{\varphi, \{a\}\}$ is an ideal of $P(X)$. If $B \in P(X)$, then $\frac{B}{I} = \{B\}$ if $a \in B$ and otherwise, $\frac{B}{I} = \{B, B \cup \{a\}\}$. Now, by Lemma 3.1, $\frac{B}{I} = B \ominus I = B \oplus I$ if $a \in B$ and otherwise, $\frac{B}{I} = (B \cup \{a\}) \ominus I = B \oplus I$.

Proposition 3.3. *Let τ be a Hausdorff topology on an MV-algebra A, and I be an open locally compact ideal of A. Assume that for every $a \in A$, the set $\frac{a}{I}$ has a maximum and $a \ominus I$, is an open set. Then, (A, τ) is a Thychonoff space.*

Proof. Let $x \in A$. By Lemma 3.1, there exists $y \in A$ such that $\frac{x}{I} = y \ominus I$. By the hypothesis, the map $l_y(z) = y \ominus z$ is an open map from I onto $\frac{x}{I}$. By [9, Theorem 3.3.15]], $\frac{x}{I}$ is locally compact. The identity $A = \bigcup_{x \in A} \frac{x}{I}$ allows us to deduce that A is a union of open, locally compact subspaces of A. By [9, Exercise 3.3.B], A is also locally compact. By [9, Theorem 3.3.1], (A, τ) is a Tychonoff space. □

Proposition 3.4. *Let τ be a topology on an MV-algebra A, and U be an open locally compact neighborhood of 0 such that for any $a \in A$, $U \oplus a$ is an open neighborhood of a. Then, (A, τ) is a Tychonoff space.*

Proof. By [9, Theorem 3.3.15], the set $U \oplus a$ is an open, locally compact subset of A. Since $A = \bigcup_{a \in A} U \oplus a$, by [9, Exercise 3.3.B], A is a locally compact space. Let $a \neq b$. Then by $(M19)$, it is easy to prove that $U \oplus a \cap U \oplus b = \varphi$. This shows that (A, τ) is a Hausdorff space. By [9, Theorem 3.3.1], (A, τ) is a Tychonoff space. □

Example 3.5. Define $\oplus : [0,1] \times [0,1] \longrightarrow [0,1]$ by $x \oplus y = \min\{x+y, 1\}$, and $* : [0,1] \longrightarrow [0,1]$ by $x^* = 1 - x$. Then, $([0,1], \oplus, *, 0)$ is an MV-algebra which is called the *standard MV-algebra* [8]. It is easy to prove that $A = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$ is a subalgebra of $[0,1]$. Let $\tau_{[0,1]}$ be the discrete topology on $[0,1]$ and $B = \{\{0, 1/3\}, \{1/3, 2/3\}, \{2/3, 1\}\}$ be a subbase for a topology τ_A on A. If τ is the Tychonoff topology on the MV-algebra $A \times [0,1]$, then $U = \{0, 1/3\} \times \{0\}$ is an open, locally compact neighborhood of $(0,0)$ in τ and for any $(a,b) \in A \times [0,1]$, the set $U \oplus (a,b)$ is an open neighborhood of (a,b). By Proposition 3.4, $(A \times [0,1], \tau)$ is a Tychonoff space.

Proposition 3.6. *Let I be a locally compact ideal in a Hausdorff MV-algebra (A, τ). Then, (A, τ) is a Tychonoff space provided that for every $a \in A \setminus \{0\}$, the maps $r_a(x) = x \ominus a$ and $l_a(x) = a \ominus x$ are continuous, and the compactness of some $S \subseteq A$ implies that of the sets $r_a^{-1}(S)$ and $l_a^{-1}(S)$.*

Proof. First we prove that for every $a \in A \setminus \{0\}$, $r_a^{-1}(I)$ and $l_a^{-1}(I)$ are locally compact. To do so, let $a \in A$ and $x \in r_a^{-1}(I)$. Since I is locally compact, there is an open set V in A such that

$r_a(x) \in V \subseteq \overline{V} \subseteq I$ and $\overline{V}$ is compact in A. From the continuity of r_a we infer that $r_a^{-1}(V)$ is open. Thus, by the hypothesis, $r_a^{-1}(\overline{V})$ is compact. Hence, $x \in r_a^{-1}(V) \subseteq \overline{r_a^{-1}(V)} \subseteq r_a^{-1}(\overline{V}) \subseteq r_a^{-1}(I)$ and $\overline{r_a^{-1}(V)}$ is compact. Therefore, $r_a^{-1}(I)$ is locally compact. Similarly, $l_a^{-1}(I)$ is also locally compact. The identity $\frac{a}{I} = r_a^{-1}(I) \bigcap l_a^{-1}(I)$ implies that $\frac{a}{I}$ is locally compact. Thus, $A = \bigcup_{a\in A} \frac{a}{I}$ is a union of open, locally compact subspaces of A. By [9, Exercise 3.3.B], (A, τ) is locally compact. Now [9, Theorem 3.3.1] allows us to deduce that (A, τ) is a Tychonoff space. $\square$

Proposition 3.7. *Let $\mathcal{I}$ be a family of ideals in an MV-algebra A which is closed under intersection. Also, assume that for any $x \neq y$,*
(i) there exists $I \in \mathcal{I}$ such that $y \notin I \oplus x$, and
(ii) for every $I, J \in \mathcal{I}$, if $I \oplus x \bigcap J \oplus y \neq \varphi$, then either $x \in J \oplus y$ or $y \in I \oplus x$.
Then, there exists a topology τ on A such that (A, τ) is a Tychonoff space and $\oplus$ is continuous.

Proof. Since $\mathcal{I}$ is a family of ideals, it is easy to show that for any $I \in \mathcal{I}$ and $x \in I$, $I \oplus x$ is in the set $\tau = \{G \subseteq A : \forall x \in G\ \exists I \in \mathcal{I}$ such that $I \oplus x \subseteq G\}$. Since τ is closed under arbitrary unions and intersections, it is a topology on A. If $x \neq y$, then by (i), $x \notin J \oplus y$ and $y \notin I \oplus x$ for some $I, J \in \mathcal{I}$. Hence $x \notin K \oplus y$ and $y \notin K \oplus x$, where $K = I \bigcap J \in \mathcal{I}$. Therefore, (A, τ) is a T_1-space. By the definition of τ, the continuity of $\oplus$ is clear. Now, we show that for every $x \in A$ and $I \in \mathcal{I}$, $I \oplus x$ is closed in (A, τ). Let $x \in A$, $I \in \mathcal{I}$ and $y \in \overline{I \oplus x}$. If $y = x$, then it is clear that $y \in I \oplus x$. Let $y \neq x$. By (i), $x \notin J \oplus y$ for some $J \in \mathcal{I}$. Since $J \oplus y$ is an open neighborhood of y, $J \oplus y \bigcap I \oplus x \neq \varphi$. So, $y \in I \oplus x$. Now, the set $\{I \oplus x : x \in A, I \in \mathcal{I}\}$ is a base for τ whose all elements are open and closed sets in (A, τ). Therefore, (A, τ) is a Tychonoff space. $\square$

Proposition 3.8. *Let (A, τ) be a topological MV-algebra and let for any $a \in A$, $r_a(x) = x \ominus a$ be an open map from A into A. Then, (A, τ) is a Tychonoff space if for each $x \neq 0$ and any open neighborhood U of 0, there exists an open set V such that $0 \in V \subseteq \overline{V} \subseteq U$, $\overline{V}$ is compact and $x \notin \overline{V}$.*

Proof. First, we prove that (A, τ) is a Hausdorff space. Let $x \neq y$. Without loss of generality, assume that $x \ominus y \neq 0$. Let U be an open neighborhood of 0. By the hypothesis, there exists an open set V such that $0 \in V \subseteq \overline{V} \subseteq U$, $\overline{V}$ is compact and $x \ominus y \notin \overline{V}$. Since $\ominus$ is continuous, there exist open sets U_1 and U_2 such that $x \in U_1$, $y \in U_2$ and $U_1 \ominus U_2 \subseteq A \setminus \overline{V}$. Now, it is easy to show that $U_1 \bigcap U_2 = \varphi$. Thus, (A, τ) is a Hausdorff space. Now, we prove that (A, τ) is locally compact. To see this, let $x \in U \in \tau$. If $x = 0$, then by the hypothesis, there exists $V \in \tau$ such that $0 \in V \subseteq \overline{V} \subseteq U$ and $\overline{V}$ is compact. If $x \neq 0$, then from $x \ominus 0 = x \in U$ we obtain an open neighborhood W of 0 such that $x \ominus W \subseteq U$. Let V be an open set such that $0 \in V \subseteq \overline{V} \subseteq W$ and $\overline{V}$ is compact. Since r_x is an open and continuous map, the set $x \ominus V$ is an open neighborhood of x and $x \ominus \overline{V}$ is a compact set. Since (A, τ) is Hausdorff, the set $x \ominus \overline{V}$ is closed. Moreover, the set $\overline{x \ominus V}$ is also compact because $\overline{x \ominus V} \subseteq \overline{X \ominus \overline{V}} = x \ominus \overline{V}$. The relation

$$x \ominus V \subseteq \overline{x \ominus V} \subseteq x \ominus \overline{V} \subseteq x \ominus W \subseteq U$$

implies that (A, τ) is locally compact. By [9, Theorem 3.3.1], (A, τ) is a Tychonoff space. □

Proposition 3.9. *Let τ be a topology on an MV-algebra A such that $\to$ is continuous. Moreover, assume that for every $a \in A$, $\lambda_a(x) = x \to a$ is an open map from A into A. Then, (A, τ) is a Tychonoff space if for each $x \neq 1$ and any open neighborhood U of 1, there exists an open set V such that $1 \in V \subseteq \overline{V} \subseteq U$, $\overline{V}$ is compact and $x \notin \overline{V}$.*

Proof. Let $x^* \in U \in \tau$. Since $x \to 0 \in U$ and $\to$ is continuous, there exists $V \in \tau$ such that $x \in V$, and $V^* = V \to 0 \subseteq U$. Hence $*$ is continuous and clearly, it is a homeomorphism. By $(M7)$ and $(M8)$, $\ominus = *\circ \to$ and $r_a = * \circ \lambda_a$, for every $a \in A$. So, $\ominus$ is continuous and the map $r_a(x) = x \ominus a$ is open. Let $0 \neq x \in U \in \tau$. Then, $1 \neq x^* \in U^*$. By the hypothesis, there exists $V \in \tau$ such that $1 \in V \subseteq \overline{V} \subseteq U^*$, $\overline{V}$ is compact and $x^* \notin \overline{V}$. Thus, $0 \in V^* \subseteq \overline{V}^* = \overline{V^*} \subseteq U$, $\overline{V^*}$ is compact and $x \notin \overline{V^*}$. Therefore, by Proposition 3.8, (A, τ) is a Tychonoff space. □

4. Uniform MV-algebras

In this section, we recall the notion of uniform MV-algebra. Moreover, we find some uniformities under which the operations of MV-algebras are uniformly continuous. Proposition 4.10 reveals a connection between Tychonoff spaces and uniform MV-algebras. Also, we discuss the existence of closed ideals, closed filters, and a contravariant functor from the category of uniform MV-algebras to the category of semigroups.

Let A be an MV-algebra and $\mathcal{U}$ be a uniformity on A. By Definition 2.6,
(i) the operation $\oplus : (A \times A, \mathcal{U} \times \mathcal{U}) \to (A, \mathcal{U})$ is *uniformly continuous* if for every $W \in \mathcal{U}$, there exist $U, V \in \mathcal{U}$ such that $U \oplus V \subseteq W$, or equivalently, $(x \oplus y, x' \oplus y') \in W$ for every $(x, x') \in U$ and $(y, y') \in V$;
(ii) the map $* : (A, \mathcal{U}) \to (A, \mathcal{U})$ is *uniformly continuous* if for every $W \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $(x, y) \in V$ implies $(*(x), *(y)) \in W$.
The pair $(A, \mathcal{U})$ is called a *uniform MV-algebra* if $\oplus$ and $*$ are uniformly continuous. [2]

Example 4.1. (i) Let $([0, 1], \oplus, *, 0)$ be the standard MV-algebra. The family $\{U_\varepsilon\}_{\varepsilon>0}$ is a base for a uniformity $\mathcal{U}$ on A, where $U_\varepsilon = \{(x, y) \in [0, 1] \times [0, 1] : |x - y| < \varepsilon\}$. If $W \in \mathcal{U}$, then

$U_\varepsilon \subseteq W$ for some $\varepsilon > 0$, and $|x^* - y^*| = |1 - x - 1 + y| = |y - x| < \varepsilon$ for every $(x, y) \in U_\varepsilon$. So, $(x^*, y^*) \in W$. Thus, $*$ is uniformly continuous. Now, we show that $\oplus$ is uniformly continuous. To see this, assume that $U_\varepsilon \in \mathcal{U}$ and $(x, y), (x', y') \in U_{\varepsilon/2}$. Then, the following steps show that $U_{\varepsilon/2} \oplus U_{\varepsilon/2} \subseteq U_\varepsilon$ and so, $\oplus$ is uniformly continuous.
Step 1. If $x + x' < 1$ and $y + y' < 1$, then

$$|x \oplus x' - y \oplus y'| = |x + x' - y - y'| \leq |x - y| + |x' - y'| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Step 2. If $x + x' < 1$ and $y + y' \geq 1$, then

$$|x \oplus x' - y \oplus y'| = |x + x' - 1| \leq |y + y' - x - x'| \leq |y - x| + |y' - x'| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Step 3. If $x + x' \geq 1$ and $y + y' < 1$, the proof is similar to that of Step 2.
Step 4. If $x + x' \geq 1$ and $y + y' \geq 1$, then $|x \oplus x' - y \oplus y'| = |1 - 1| = 0 < \varepsilon$.
(ii) Let $A = \{0, 1/4, 2/4, 3/4, 1\}$. Define $\oplus : A \times A \longrightarrow A$ by $x \oplus y = \min\{x + y, 1\}$, and $* : A \longrightarrow A$ by $x^* = 1 - x$. Then, $(A, \oplus, *, 0)$ is an MV-algebra [8]. Put $U_1 = \triangle$,

$$U_2 = \{(x, y) : |x - y| < 1/4\}, \qquad U_3 = \{(x, y) : |x - y| < 1/2\},$$
$$U_4 = \{(x, y) : |x - y| < 3/4\} \text{ and } \quad U_5 = \{(x, y) : |x - y| < 1\} = A \times A.$$

It is easy to see that $\mathcal{U} = \{U \subseteq X \times X : \text{There exists } i \text{ such that } U_i \subseteq U\}$ is a uniformity on A such that $U_{i-1} \oplus U_{i-1} \subseteq U_i, \forall i \geq 2$. Also, $U_i^* \subseteq U_i$ for every $i \geq 1$. These relations prove that $\oplus$ and $*$ are uniformly continuous. Therefore, $(A, \oplus, *, \mathcal{U})$ is a uniform MV-algebra.

Proposition 4.2. *Let A be an MV-algebra and $\mathcal{U}$ be a uniformity on it. If $*$ is uniformly continuous, then the following conditions are equivalent.*
(i) The operation $\oplus$ is uniformly continuous.
(ii) The operation $\odot$ is uniformly continuous.
(iii) The operation $\ominus$ is uniformly continuous.
(iv) The operation $\rightarrow$ is uniformly continuous.

Proof. By $(M6), (M7), (M8)$ and $(M9)$, $\odot = * \circ \oplus \circ (* \times *)$, $\ominus = \odot \circ (I \times *)$ and $\rightarrow = * \circ \odot \circ (I \times *)$, where $\circ$ is the composition operator and I is the identity map. Now the proof is clear because the composition of uniformly continuous functions is uniformly continuous. □

Theorem 4.3. *Let $\mathcal{U}$ be a uniformity on an MV-algebra A. The uniform continuity of each of the operations $\ominus$, $\rightarrow$ and $\rightsquigarrow$ implies that $(A, \mathcal{U})$ is a uniform MV-algebra.*

Proof. First assume that $\ominus$ is uniformly continuous and $U \in \mathcal{U}$. Since $\ominus$ is uniformly continuous, there exist V_1 and V_2 in $\mathcal{U}$ such that $V_1 \ominus V_2 \subseteq U$. Let $V = V_1 \cap V_2$ and $(x, y) \in V^*$. Then $x^* \in V_1, y^* \in V_2$ and $(x, y) = (1, 1) \ominus (x^*, y^*) \in V_1 \ominus V_2 \subseteq U$. Hence $V^* \subseteq U$, which implies that $*$ is uniformly continuous. By Proposition 4.2, $(A,\mathcal{U})$ is a uniform MV-algebra. Now, if $\rightarrow$ is uniformly continuous and $U \in \mathcal{U}$, then $V_1 \rightarrow V_2 \subseteq U$ for some $V_1, V_2 \in \mathcal{U}$. Put $V = V_1 \cap V_2$ and choose (x,y) from V^*. Then, $(x,y) = (x^*, y^*) \rightarrow (0,0) \in V_1 \rightarrow V_2 \subseteq U$. This allows us to deduce that $*$ is uniformly continuous and so, by Proposition 4.2, $(A,\mathcal{U})$ is a uniform MV-algebra. The proof of other case is similar. $\square$

Example 4.4. Consider $A = \{0, 1/n, 2/n, \ldots, (n-1)/n, 1\}$ as a subalgebra of the standard MV-algebra $[0,1]$. If $U_0 = \triangle$ and $U_i = \{(x,y) : |x-y| < 1/n\}$, then the set $\mathcal{U} = \{U \subseteq X \times X :$ There exists i such that $U_i \subseteq U\}$ is a uniformity on A such that $U_i \ominus U_1 \subseteq U_i, \forall i \geq 1$. Hence, $\ominus$ is uniformly continuous and by Proposition 4.3, $(A,\mathcal{U})$ is a uniform MV-algebra.

Theorem 4.5. *Let I be a compact ideal in a Hausdorff topological MV-algebra (A,τ). If the set $\frac{A}{I}$ is finite and for each $x \in A$, the set $\frac{x}{I}$ has a maximum (minimum), then there exists a uniformity $\mathcal{U}$ on A such that $(A,\mathcal{U})$ is a uniform MV-algebra.*

Proof. Assume that for every $x \in A$, the set $\frac{x}{I}$ has a maximum. Then, by Lemma 3.1, $\frac{x}{I} = y \ominus I$ for each $x \in A$, where y is the maximum of $\frac{x}{I}$. Since $\ominus$ is continuous, $\frac{x}{I}$ is compact. By the fact that $\frac{A}{I}$ is finite, we conclude that A is compact. By [9, Theorem 8.3.13], (A,τ) is a Tychonoff space. Since A is compact, the operations $\oplus$ and $*$ are uniformly continuous. The proof of other case is similar. $\square$

Theorem 4.6. *Let (A,τ) be a Hausdorff topological MV-algebra, and F be a compact filter of A. If $\frac{A}{F}$ is finite and for each $x \in A$, the set $\frac{x}{F}$ has a maximum, then there exists a uniformity $\mathcal{U}$ on A such that $(A,\mathcal{U})$ is a uniform MV-algebra.*

Proof. Since $\frac{A}{F}$ is finite, there exist $x_1, \ldots, x_n \in A$ such that $\frac{A}{F} = \frac{x_1}{F} \bigcup \ldots \bigcup \frac{x_n}{F}$. Let y_i be the maximum of $\frac{x_i}{F}$, for $i = 1, \ldots, n$. If $z \in \frac{x_i}{F}$, then $z = y_i \wedge z = y_i \odot (y_i \rightarrow z) \in y_i \odot F$. Hence, $\frac{x_i}{F} \subseteq y_i \odot F$. This implies that $A = y_1 \odot F \bigcup \ldots \bigcup y_n \odot F$. Since $\odot$ is continuous and F is compact, A is also compact. By [9, Theorem, 8.3.13], there exists a uniformity $\mathcal{U}$ on A such that τ is the topology induced by $\mathcal{U}$. Thus, $\oplus$ and $*$ are uniformly continuous. $\square$

Theorem 4.7. *Let τ be a topology on an MV-algebra A such that (A,τ) is compact, and assume that $\ominus$ is continuous at (x,x), for every $x \in A$. Let $\mathcal{I}$ be a family of open ideals in A that is closed under intersection and for each $0 \neq x \in A$, there exists $I \in \mathcal{I}$ such that $0 \notin \frac{x}{I}$. Then, there exists a uniformity $\mathcal{U}$ on A such that $(A,\mathcal{U})$ is a uniform MV-algebra.*

Proof. It is easy to prove that the set $B = \{\frac{x}{I} : I \in \mathcal{I}, x \in A\}$ is a base for the topology

$$\tau_0 = \{V \subseteq A : \forall x \in V \ \exists I \in \mathcal{I} \text{ such that } \frac{x}{I} \subseteq V\}.$$

Let $x \in A$, $I \in \mathcal{I}$ and $y = \overline{\frac{x}{I}}$. Then, $\frac{y}{I} \bigcap \frac{x}{I} \neq \varphi$. This implies that $y \in \frac{x}{I}$. So, $\frac{x}{I}$ is both open and closed in (A, τ_0). On the other hand, (A, τ_0) is a T_1-space. In fact, if $x \neq y$, then $x \ominus y \neq 0$ or $y \ominus x \neq 0$. Assume that $x \ominus y \neq 0$. By the hypothesis, $0 \notin \frac{x \ominus y}{I}$ for some $I \in \mathcal{I}$. This implies that $x \notin \frac{y}{I}$ and $y \notin \frac{x}{I}$. Thus, (A, τ_0) is a T_1 space that has a base of open and closed sets. Hence, (A, τ_0) is a Tychonoff space. By [9, Theorem 8.1.2], the topology τ_0 is the topology induced by a uniformity $\mathcal{U}$. The relations $\frac{x}{I} \oplus \frac{y}{I} = \frac{x \oplus y}{I}$ and $(\frac{x}{I})^* = \frac{x^*}{I}$ allow us to deduce that $\oplus$ and $*$ are continuous in (A, τ_0). Now, we show that τ is finer than τ_0. Let $x \in V \in \tau_0$. Then, there exists an ideal $I \in \mathcal{I}$ such that $x \in \frac{x}{I} \subseteq V$. Since $x \ominus x = 0 \in I \in \tau$ and $\ominus$ is continuous at (x, x) in (A, τ), there exists $U \in \tau$ such that $x \in U$ and $U \ominus U \subseteq I$. Let $z \in U$. Then, $z \ominus x$ and $x \ominus z$ are in I. So $z \in \frac{x}{I} \subseteq V$, which implies that $x \in U \subseteq V$. Hence, $V \in \tau$. Now, since $\tau_0 \subseteq \tau$ and (A, τ) is compact, (A, τ_0) is also compact. By [9, Theorem 8.3.13], there exists a uniformity $\mathcal{U}$ on A that induces τ_0, and $(A, \mathcal{U})$ is a uniform MV-algebra. □

Theorem 4.8. *Let τ be a compact topology on MV-algebra A, and suppose that $\rightarrow$ is continuous at (x, x). If $\mathcal{F}$ is a family of open filters that is closed under intersection and for any $1 \neq x \in A$, there exists $F \in \mathcal{F}$ such that $1 \notin \frac{x}{F}$, then there is a uniformity $\mathcal{U}$ on A such that $(A, \mathcal{U})$ is a uniform MV-algebra.*

Proof. Similar to the proof of Theorem 4.7, we can show that the set $\tau_0 = \{U \subseteq A : \forall x \in U \ \exists F \in \mathcal{F},\ x \in \frac{x}{F} \subseteq U\}$ is a topology on A such that (A, τ_0) is a Thychonoff topological MV-algebra and $\tau_0 \subseteq \tau$. Hence, (A, τ_0) is a compact space and by [9, Theorem 8.3.13], there exists a uniformity $\mathcal{U}$ on A that induces τ_0, and $(A, \mathcal{U})$ is a uniform MV-algebra. □

In the sequel, we are going to discuss Tychonoff spaces and uniform MV-algebras. One knows that every Tychonoff space generates a uniformity, and it is clear that any uniform MV-algebra is a completely regular space. We show that a uniform MV-algebra is a Tychonoff space if it satisfies the equivalence conditions of Lemma 4.9. To begin with, we fix our notation as follows.

Notation. In a uniform MV-algebra $(A, \mathcal{U})$, we denote the closure of $U[x]$ by $\overline{U[x]}$ in the topology induced by $\mathcal{U}$.

Lemma 4.9. *The following conditions are equivalent in a uniform MV-algebra $(A, \mathcal{U})$.*

(i) For each $x \neq 0$, there exists $U \in \mathcal{U}$ such that $0 \notin U[x]$.

(ii) For any $x \neq 1$, there exists $U \in \mathcal{U}$ such that $1 \notin U[x]$.

(iii) For every $x \neq 1$, there exist $U, V \in \mathcal{U}$ such that $\overline{U[x]} \cap \overline{V[1]} = \varphi$.

(iv) For every $x \neq 0$, there exist $U, V \in \mathcal{U}$ such that $\overline{U[x]} \cap \overline{V[0]} = \varphi$.

(v) For any $x \neq y$, there exist $U, V \in \mathcal{U}$ such that $\overline{U[x]} \cap \overline{V[y]} = \varphi$.

Proof. $(i \Rightarrow ii)$ Let $x \neq 1$. Then, $1 \ominus x = x^* \neq 0$. By (i), there exists $U \in \mathcal{U}$ such that $0 \notin U[1 \ominus x]$. Since $\ominus$ is continuous, there exist entourages V and W such that $V[1] \ominus W[x] \subseteq U[1 \ominus x]$. If $z \in V[1] \cap W[x]$, then $0 = z \ominus z \in V[1] \ominus W[x] \subseteq U[1 \ominus x]$, a contradiction. Now, $1 \notin W[x]$ follows from $V[1] \cap W[x] = \varphi$.

$(ii \Rightarrow iii)$ If $x \neq 1$, then by (ii) there exists $U \in \mathcal{U}$ such that $1 \notin U[x]$. From $x \oplus 1^* \in U[x]$ we obtain entourages U_1 and U_2 such that $U_1[x] \oplus U_2[1]^* \subseteq U[x]$. If $z \in U_1[x] \cap U_2[1]$, then $1 = z \oplus z^* \in U_1[x] \oplus U_2[x]^* \subseteq U[x]$, a contradiction. Hence, $U_1[x] \cap U_2[1] = \varphi$. Now, since $1 \to x = x \in U_1[x]$, there exist $V, W \in \mathcal{U}$ such that $V[1] \to W[x] \subseteq U_1[x]$. If $z \in \overline{V[1]} \cap \overline{W[x]}$, then there exist nets $\{x_i : i \in I\}$ and $\{y_i : i \in I\}$ in $V[1]$ and $W[x]$, respectively, that converge to z. Thus, $\{x_i \to y_i : i \in I\}$ is a net in $V[1] \to W[x]$ that converges to 1. This allows us to deduce that $1 \in U_1[x]$, a contradiction. Therefore, $\overline{V[1]} \cap \overline{W[x]} = \varphi$.

$(iii \Rightarrow iv)$ Suppose that $x \neq 0$. Since $x \to 0 \neq 1$, we can find entourages U_1 and U_2 such that $\overline{U_1[x \to 0]} \cap \overline{U_2[1]} = \varphi$. From $x \to 0 \in U_1[x \to 0]$ it follows that $U[x] \to V[0] \subseteq U_1[x \to 0]$, for some $U, V \in \mathcal{U}$. If $z \in \overline{U[x]} \cap \overline{V[0]}$, then there exist nets $\{x_i : i \in I\}$ and $\{y_i : i \in I\}$ in $U[x]$ and $V[0]$, respectively, that converge to z. Now, $\{x_i \to y_i : i \in I\}$ is a net in $U_1[x \to 0]$ that converges to 1. Hence $1 \in \overline{U_1[x \to 0]}$, a contradiction. Therefore, $\overline{U[x]} \cap \overline{V[0]} = \varphi$.

$(iv \Rightarrow v)$ Let $x \neq y$ and assume that $x \ominus y \neq 0$. By (iv), there exist $U_1, U_2 \in \mathcal{U}$ such that $\overline{U_1[x \ominus y]} \cap \overline{U_2[0]} = \varphi$. Since $x \ominus y \in U_1[x \ominus y]$, there are entourages W_1 and W_2 such that $W_1[x] \ominus W_2[y] \subseteq U_1[x \ominus y]$. If $z \in \overline{W_1[x]} \cap \overline{W_2[y]}$, then there exist nets $\{x_i : i \in I\}$ and $\{y_i : i \in I\}$ in $W_1[x]$ and $W_2[y]$, respectively, that converge to z. Now, $\{x_i \ominus y_i : i \in I\}$ is a net in $U_1[x \ominus y]$ that converges to 0. Hence $0 \in \overline{U_1[x \ominus y]}$, a contradiction. Therefore, $\overline{W_1[x]} \cap \overline{W_2[y]} = \varphi$.

$(v \Rightarrow i)$ The proof is straightforward. □

Proposition 4.10. *Let $(A, \mathcal{U})$ be a uniform MV-algebra. If τ is the topology induced by $\mathcal{U}$, then (A, τ) is a Tychonoff space if and only if it satisfies one of the conditions of Lemma 4.9.*

Proof. If (A, τ) is a Tychonoff space, it is a T_1 space and so, for any $x \neq 0$ there exists $U \in \mathcal{U}$ such that $0 \notin U[x]$. Hence, $(A, \mathcal{U})$ satisfies (i) of Lemma 4.9. Conversely, let $(A, \mathcal{U})$ satisfy condition (i) of Lemma 4.9 and $x \neq y$. Without loss of generality assume that $x \ominus y \neq 0$. Then $0 \notin U[x \ominus y]$, for some $U \in \mathcal{U}$. Since $\ominus$ is continuous, there exist entourages V and W such that $V[x] \ominus W[y] \subseteq U[x \ominus y]$. Then $V[x] \cap W[x] = \varphi$ because $0 \notin U[x \ominus y]$. Therefore, $(A, \mathcal{U})$ is a T_2-space, which implies that (A, τ) is a Tychonoff space. □

Proposition 4.11. *Every uniform MV-algebra* $(A,\mathcal{U})$ *has a base* B *such that* $U^* = U$, *for any* $U \in B$. *Moreover,* $U^* \in \mathcal{U}$ *for each* $U \in \mathcal{U}$.

Proof. Let B_0 be a base for $\mathcal{U}$ and $B = \{U \cap U^* : U \in B_0\}$. It is clear that $W^* = W$ for every $W \in B$. We show that B is a base for $\mathcal{U}$. Let $U \in \mathcal{U}$. Since $\ominus$ is uniformly continuous, there exists $V \in B_0$ such that $V \subseteq V \ominus V \subseteq U$ and $V^* = (1,1) \ominus V \subseteq V \ominus V \subseteq U$. Hence, $V \cap V^*$ is an element of B which is contained in U. □

Proposition 4.12. *Assume that* $(A,\mathcal{U})$ *is a uniform MV-algebra in which* $r_a(x) = x \ominus a$ *is an open map from* A *to* A, *for any* $a \in A$. *Then, all ideals and filters are closed in* A.

Proof. Let W be a subset of A such that $x \in W$ and $x \leq y$ imply $y \in W$. Let $x \in \overline{W}$ and $V \in \mathcal{U}$. Then $(V[1] \ominus x^*) \cap W$ is a non-empty set, because $V[1] \ominus x^*$ is an open neighborhood of x. Hence, $z \ominus x^* \in W$ for some $z \in V[1]$. By $(M18)$, $z \ominus x^* \leq x$. So, $x \in W$. Thus, $\overline{W} = W$. This implies that every filter is closed in A. Since $*$ is a homeomorphism, every ideal is also closed. □

Proposition 4.13. *Let* $(A,\mathcal{U})$ *be a uniform MV-algebra, and assume that for any* $a \in A$, $l_a(x) = a \ominus x$ *is an open map from* A *to* A. *Let* H *be a subset of* A *which satisfies*

$$\text{if net}\{\mathrm{x_r}\} \text{ converges to a point of H, then } \mathrm{x_r} \text{ is in H, for some r.}$$

If H *is an ideal or a filter, then so is* $\overline{H}$.

Proof. Let H be an ideal. Clearly, 0 is in $\overline{H}$. Let $x \ominus y$ and y be in $\overline{H}$. Then, $U[x] \ominus y$ is an open neighborhood of $x \ominus y$. Hence, there exist $u \in U[x]$ and $h \in H$ such that $h = u \ominus y$. If $\{a_r\}$ is a net in H that converges to y, then the net $\{u \ominus a_r\}$ converges to $u \ominus y = h \in H$. By the hypothesis, $u \ominus a_r \in H$ for some r. Since H is an ideal, $u \in H$. Hence $U[x] \cap H$ is non-empty and so, $x \in \overline{H}$.
Now, let H be a filter. Then, H^* is an ideal. If $\{x_r\}$ is a net that converges a point of H^*, then the net $\{x_r^*\}$ converges a point of H. So $x_r^* \in H$ for some r, which implies that $x_r \in H^*$. By the above paragraph, $\overline{H^*} = \overline{H}^*$ is an ideal. Hence, $\overline{H}$ is a filter. □

Proposition 4.14. *Let* I *be an ideal and* F *be a filter in a uniform MV-algebra* $(A,\mathcal{U})$. *If* 0 *and* 1 *are interior points of* I *and* F *in* $\overline{I}$ *and* $\overline{F}$, *respectively, then* $\overline{I}$ *is an ideal and* $\overline{F}$ *is a filter.*

Proof. Let 1 be an interior point of F and $x \in \overline{F} \setminus F$. Then, there exists $W \in \mathcal{U}$ such that $x \to x = 1 \in \overline{F} \cap W[1] \subseteq F$. Since $\to$ is uniformly continuous, there exists an entourage U such that $\overline{F} \cap U[x] \to \overline{F} \cap U[x] \subseteq \overline{F} \cap W[1] \subseteq F$. If $f \in \overline{F} \cap U[x] \cap F$, then the facts $f \to x \in F$ and $f \in F$ allow us to deduce that x is in F, a contradiction. Hence, $x \in \overline{F} \cap U[x] \subseteq \overline{F} \setminus F$. Therefore, F is closed in $\overline{F}$ and so, $\overline{F} = F$ is a filter in A.
Now, since $*$ is a homeomorphism, the proof of the other case is straightforward. □

Now, we are going to prove that every uniform MV-algebra has at least one closed ideal and one closed filter. To do so, we first recall the definition of submeasures on MV-algebras.
A *submeasure* on an MV-algebra A is an increasing map $N : A \to \mathbb{R}^+$ such that $N(x \oplus y) \leq N(x) + N(y)$, for every $x, y \in A$.[2]
If $P(X)$ is the MV-algebra introduced in Example 3.2, then the map $N : P(X) \to \mathbb{R}^+$ defined by $N(A) = card(A)$ is a submeasure on $P(X)$.
The first part of the proof of the following proposition is similar to [1, Lemma 3.3.10].

Proposition 4.15. *Every uniform MV-algebra has at least one closed ideal and one closed filter.*

Proof. Let $(A,\mathcal{U})$ be a uniform MV-algebra and $U \in \mathcal{U}$. Since $\oplus$ is uniformly continuous, for each $n \geq 0$ there exists an entourage W_n such that $W_{n+1}[0] \oplus W_{n+1}[0] \subseteq W_n[0] \subseteq U[0]$. For any $n \geq 0$, suppose that $U_n = W_n[0] \subseteq A$. Let $V(1) = U_0$, $n \geq 0$ and assume that $V(\frac{m}{2^n})$ are defined for each $m = 1, 2, 3, \ldots, 2^n$ such that $0 \in V(\frac{m}{2^n})$. Then, put $V(\frac{1}{2^{n+1}}) = U_{n+1}$, $V(\frac{2m}{2^{n+1}}) = V(\frac{m}{2^n})$ for $m = 1, 2, 3, \ldots, 2^n$ and for each $m = 1, 2, 3, \ldots, 2^n - 1$, $V(\frac{2m+1}{2^{n+1}}) = V(\frac{m}{2^n}) \oplus U_{n+1} = V(\frac{m}{2^n}) \oplus V(\frac{1}{2^{n+1}})$. Also, we define $V(\frac{m}{2^n}) = A$, when $m > 2^n$. By induction on n we prove that for any $m > 0$ and $n \geq 0$,

$$(*) \quad V(\frac{m}{2^n}) \oplus V(\frac{1}{2^n}) \subseteq V(\frac{m+1}{2^n}).$$

First notice that if $m + 1 > 2^n$, then $(*)$ is obviously true. Let $m < 2^n$. If $n = 1$, then m is also equal to 1. So, $V(\frac{1}{2}) \oplus V(\frac{1}{2}) = U_1 \oplus U_1 \subseteq U_0 = V(1)$. Assume that $(*)$ holds for some n. We verify it for $n+1$. If $m = 2k$, then by the definition of $V(\frac{2m+1}{2^{n+1}})$, $V(\frac{m}{2^{n+1}}) \oplus V(\frac{1}{2^{n+1}}) = V(\frac{2k}{2^{n+1}}) \oplus V(\frac{1}{2^{n+1}}) = V(\frac{k}{2^n}) \oplus V(\frac{1}{2^{n+1}}) = V(\frac{2k+1}{2^n})$. Now, suppose that $m = 2k+1 < 2^{n+1}$ for some $x \geq 0$. Then,

$$V(\frac{m}{2^{n+1}}) \oplus V(\frac{1}{2^{n+1}}) = V(\frac{2k+1}{2^{n+1}}) \oplus U_{n+1} = V(\frac{k}{2^n}) \oplus U_{n+1} \oplus U_{n+1} \subseteq V(\frac{k}{2^n}) \oplus U_n = V(\frac{k}{2^n}) \oplus V(\frac{1}{2^n}).$$

But, by the assumption of induction, $V(\frac{m}{2^{n+1}}) \oplus V(\frac{1}{2^{n+1}}) \subseteq V(\frac{k+1}{2^n}) = V(\frac{m+1}{2^{n+1}})$. If $r \geq 0$, then $\widehat{V(r)} = \{x : \exists y \in V(r) \text{ such that } x \leq y\}$ is a subset of A containing 0 such that for any $x, y \in \widehat{V(r)}$, $x \leq z \leq y$ implies $z \in \widehat{V(r)}$. It is easy to verify that the map $f : A \longrightarrow \mathbb{R}$ defined by $f(x) = inf\{r : x \in \widehat{V(r)}\}$ is a bounded, increasing function with $f(0) \neq f(1) = 1$. Now, the map $N : A \longrightarrow \mathbb{R}^+$ defined by $N(x) = sup\{f(x \oplus z) - f(z) : z \in A\}$ is a non-zero submeasure such that $N(0) = 0$, and the set $I_N = \{x : N(x) = 0\}$ is an ideal of A. To prove that I_N is closed, it is enough to show that N is continuous. First, let us prove that for any $n \geq 0$,

$$\{x : N(x) < \frac{1}{2^n}\} \subseteq \widehat{U_n} \subseteq \{x : N(x) \leq \frac{2}{2^n}\},$$

where $\widehat{U_n} = \{x : \exists y \in U_n$ such that $x \leq y\}$. Notice that $f(0) = 0$. Hence $N(x) < \frac{1}{2^n}$ implies $f(x) = f(x \oplus 0) - f(0) \leq N(x) < \frac{1}{2^n}$. Thus, for some $r \geq 0$, $x \in \widehat{V(r)}$ and $r < \frac{1}{2^n}$. Since $V(r) \subseteq V(\frac{1}{2^n}) = U_n$, it is easy to see that $x \in \widehat{V(r)} \subseteq \widehat{V(\frac{1}{2^n})} = \widehat{U_n}$. Now, let $x \in \widehat{U_n}$. Then, there exists $x' \in U_n$ such that $x \leq x'$. Clearly, for any $z \in A$, there exists $k \geq 0$ such that $\frac{k-1}{2^n} \leq f(z) \leq \frac{k}{2^n}$. Since $z \in \widehat{V(\frac{k}{2^n})}$, there exists $z' \in V(\frac{k}{2^n})$ such that $z \leq z'$. From condition $(*)$ it follows that $z' \oplus x' \in V(\frac{k}{2^n}) \oplus V(\frac{1}{2^n}) \subseteq V(\frac{k+1}{2^n})$, and from $z \oplus x \leq z' \oplus x'$ we deduce that $z \oplus x \in \widehat{V(\frac{k+1}{2^n})}$. Hence, $f(x \oplus z) - f(z) \leq \frac{k+1}{2^n} - \frac{k-1}{2^n} = \frac{2}{2^n}$. This implies that $N(x) \leq \frac{2}{2^n}$. Let $\varepsilon > 0$ be arbitrary. Then, there exists $n \geq 1$ such that $\frac{2}{2^n} < \varepsilon$. Now, the relation $U_n \subseteq \widehat{U_n} \subseteq \{x : N(x) \leq \frac{2}{2^n}\}$ allows us to deduce that N is continuous at 0. To prove the continuity of N on A, take $b \in A$ and assume that $\varepsilon > 0$ is arbitrary. For arbitrary x in A, by $(M33)$, $(b \ominus x) \oplus x = (x \ominus b) \oplus b \geq b$. So, $N(b) \leq N(b \ominus x) + N(x)$. This inequality implies that $\mid N(x) - N(b) \mid \leq max\{N(b \ominus x), N(x \ominus b)\}$. Since N is continuous at 0, there exists $n \geq 0$ such that $N(x) < \varepsilon$, for any $x \in U_n$. Since $\ominus$ is continuous and $b \ominus b = 0 \in U_n$, there exists an open neighborhood V of b such that $b \ominus V \subseteq U_n$ and $V \ominus b \subseteq U_n$. Thus for each $x \in V$, $\mid N(x) - N(b) \mid \leq max\{N(b \ominus x), N(x \ominus b)\} < \varepsilon$, which implies that N is continuous at b. Now, the continuity of N and the identity $I_N = N^{-1}(0)$ imply that I_N is closed in A. Since $*$ is a homeomorphism, the filter I_N^* is a closed filter of A. □

At the end of this paper, we are going to show that a contravariant functor exists from the category of uniform MV-algebras to the category of monoids. Let C be the category whose objects are uniform MV-algebras, and $hom(A, B)$ denote the set of all MV-homomorphisms $f : A \to B$ that are also uniformly continuous. We also use M to denote the category of monoids.

Proposition 4.16. *There exists a contravariant functor from the category of uniform MV-algebras to the category of monoids.*

Proof. Let $(A,\mathcal{U})$ be a uniform MV-algebra, and $N(A)$ be the set of all submeasures on A. Then, $N(A)$ is a monoid with $0(x) = 0$ as the identity element under the operation $(N_1 + N_2)(x) = N_1(x) + N_2(x)$. Now, the map $\mathcal{N} : \mathsf{C} \to \mathsf{M}$ that assigns to any uniform MV-algebra A the monoid $(N(A), +, 0)$ is a contravarint functor. In fact, if $f : (A,\mathcal{U}) \to (B, \mathcal{V})$ is a morphism in C, then it is easy to prove that the map $\mathcal{N}(f)(N_2) = N_2 \circ f$ is a morphism from $(N(B), +, 0)$

to $(N(A), +, 0)$ in M. Let $f : (A, \mathcal{U}) \to (B, \mathcal{V})$ and $g : (B, \mathcal{V}) \to (C, \mathcal{W})$ be morphisms in C. If $N \in N(C)$, then $\mathcal{N}(g \circ f)(N) = N \circ g \circ f = \mathcal{N}(f)(N \circ g) = \mathcal{N}(f) \circ \mathcal{N}(g)(N)$. Hence $\mathcal{N}(g \circ f) = \mathcal{N}(f) \circ \mathcal{N}(g)$. If $I : (A, \mathcal{U}) \to (A, \mathcal{U})$ is the identity map, then for any $N \in N(A)$, $\mathcal{N}(I)(N) = N \circ I = N$, which implies that $\mathcal{N}(I)$ is the identity morphism in M. $\square$

Conclusion

In this paper, we studied the relationship betweenTychonoff spaces, uniform spaces and MV-algebras. In the third section, we provided conditions under which an MV-algebra could be made into a Tychonoff space. In Section 4, after studying the uniform continuity of the operations, we examined the relationship between MV-algebras and uniform spaces. Since a close relationship exists between uniform and quasi-metric spaces, researchers can study the relationship between MV-algebras and quasi-metric spaces.

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Farzaneh Rajabisotudeh

Department of Mathematics, Faculty of Mathematics, Statistics and Computer Science,

University of Sistan and Baluchestan,

Zahedan, Iran.

Rajabisotudeh@gmail.com

Nader Kouhestani

Department of Mathematics, Faculty of Mathematics, Statistics and Computer Science,

University of Sistan and Baluchestan,

Zahedan, Iran.

kouhestani@math.usb.ac.ir

Gholam Reza Rezaei

Department of Mathematics, Faculty of Mathematics, Statistics and Computer Science,

University of Sistan and Baluchestan,

Zahedan, Iran.

grezaei@math.usb.ac.ir