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Research Paper

ON THE POWER GRAPHS OF FINITE GROUPS AND HAMILTON CYCLE

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Abstract. The power graph $\mathcal{P}(G)$ of a finite group G is a graph whose vertex set is the group G and distinct elements $x, y \in G$ are adjacent if one is a power of the other, that is, x and y are adjacent if $x \in \langle y \rangle$ or $y \in \langle x \rangle$. In this paper, we study existence of the Hamilton cycle in the power graph of some finite nilpotent groups G with a cyclic subgroup as direct factor when G is written as direct product Sylow p-subgroups. For this purpose we use of cartesian product a spanning tree and a cycle. Finally, we determined values of n such that $\mathcal{P}(U_n)$ is Hamiltonian, where U_n is a group consist of all positive integers less than n and relatively prime to n under multiplication modulo n.

1. Introduction

The *power graph* $\mathcal{P}(G)$ of a group G is a graph with elements of G as its vertices such that two distinct elements x and y are adjacent if $y = x^m$ or $x = y^m$ for some positive integer m. Clearly,

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two distinct elements x and y are adjacent if and only if $x \in \langle y \rangle$ or $y \in \langle x \rangle$. For a nonempty set S of G, notation $\mathcal{P}(S)$ is induced subgraph of $\mathcal{P}(G)$ with vertex set S. The power graphs of groups were brought up by Kelarev and Quinn [5, 6]. Subsequently Chakrabarty, Ghosh and Sen [3] studied power graphs that are complete or Eulerian or Hamiltonian. A Hamilton path in a graph is a path that meets every vertex, and a Hamilton cycle is a cycle that crosses every vertex. A graph with a Hamilton cycle is called Hamiltonian. The power graph of a finite group is connected, since each non identity element is adjacent to identity element. Also, it is connected with removing one non-identity element. However, the *proper power graph* $\mathcal{P}^*(G)$ obtained by deleting the identity element of G is not connected in general. A *vertex cutset* in a graph Γ is a set of vertices whose deletion increases the number of connected components of Γ. The *vertex connectivity* of a connected graph Γ is the minimum size of a vertex cutset, and will be denoted by $k_0(\Gamma)$. Doostabadi and Farrokhi discussed [4] connectivity proper power graph of some finite groups, particularly, nilpotent groups and show that nilpotent non p-groups have connected proper power graph and hence they are 2-connected. Whether every Hamiltonian graph is 2-connected, in this paper we focus on this type groups and will show that there is Hamilton cycle with special conditions. Chakrabarty, Ghosh and Sen [3] stated this question "Determine the values of n for which $\mathcal{P}(U_n)$ is Hamiltonian?" as a open problem. In the last section, will be answered to this question for a large number of positive integers. We follow [2] for graph-theoretical terminology and notation not defined here.

2. Nilpotent groups with a cyclic subgroup as direct factor

Definition 2.1. The cartesian product $G_1 \square G_2$ of two graphs G_1 and G_2 is a graph such that

* the vertex set of $G_1 \square G_2$ is cartesian product $V(G_1) \times V(G_2)$; and
* any two vertices (u, u') and (v, v') are adjacent in $G_1 \square G_2$ if and only if either
  . $u = v$ and $u'$is adjacent with v'in G_1, or
  . $u' = v'$ and u is adjacent with v in G_2.

Lemma 2.2. *Consider that H, K are finite groups with $gcd(|H|, |K|) = 1$. Then cartesian product $\mathcal{P}(H) \square \mathcal{P}(K)$ is a spanning subgraph of $\mathcal{P}(H \times K)$. Also, vertices (h, k) and $(1, k')$ are adjacent in the graph $\mathcal{P}(H \times K)$.*

Proof. It's enough to show every edge of graph $\mathcal{P}(H) \square \mathcal{P}(K)$ is an edge in the graph $\mathcal{P}(H \times K)$. Assume that $(h, k) \sim (h', k)$ is an edge of $\mathcal{P}(H) \square \mathcal{P}(K)$ for arbitrary $k \in K$, hence $h^m = h'$ for some positive integer m. We replace m with $m_t = m + t|H|$ where $t \in \mathbb{Z}$. Since $gcd(|H|, |K|) = 1$, then congruence equation $m_t \equiv 1(\text{module } |K|)$ is solvable for $t_0 \in \mathbb{Z}$. Thus $(h, k)^{m_{t_0}} = (h', k)$ and $(h, k), (h', k)$ are adjacent in the power graph of group $H \times K$ for every element $k \in K$. For the last part of lemma, suppose that $k^n = k'$ for some positive integer n. We can find positive integer n' such that $k^{n'} = k'$ and $|H|$ divides n'. Put $n' = m|H|$ and the other hand we must should have $|k| \mid n' - n = m|H| - n$, but congruence equation $m|H| \equiv n$(module k) has integer answer for m. Hence $(h,k)^{n'} = (1, k')$. □

The next theorem will characterize connectivity proper power graph of finite nilpotent groups with using structure of Sylow p-subgroups.(see [4])

Theorem 2.3. *Let G be a finite nilpotent group.*

(1) *if G is a p-group, then the number of connected components of $\mathcal{P}^*(G)$ is the same as the number of subgroups of G of order p. In particular, $\mathcal{P}^*(G)$ is connected if and only if G is a cyclic p-group or a generalized quaternion 2-group,*

(2) *if G is not a p-group and each of the Sylow p-subgroups of G is a cyclic p-group or a generalized quaternion 2-group, then $\mathcal{P}^*(G)$ is connected and $\operatorname{diam}(\mathcal{P}^*(G)) = 2$.*

(3) *if G is not a p-group and G has a Sylow p-subgroup, which is neither a cyclic p-group nor a generalized quaternion 2-group, then $\mathcal{P}^*(G)$ is connected and $\operatorname{diam}(\mathcal{P}^*(G)) = 4$.*

The following theorem of Chakrabarty, Ghosh and Sen [3] will be used frequently in this article.

Theorem 2.4. *Let G be a finite group. Then graph $\mathcal{P}(G)$ is complete if and only if G is a cyclic group of order 1 or p^m, for some prime number p.*

The following simple condition is necessary for deciding whether a given graph is Hamiltonian.(see [2])

Theorem 2.5. *Let S be a set of vertices of a Hamiltonian graph Γ. Then $c(\Gamma - S) \leq |S|$, where $c(\Gamma - S)$ is the number of connectivity components.*

As a simple and immediate consequence of the above theoremes, we have:

Corollary 2.6. *Let G be a finite p-group. Then power graph $\mathcal{P}(G)$ is Hamiltonian if and only if G is cyclic.*

Proof. Assume that graph $\mathcal{P}(G)$ is Hamiltonian. Hence proper power graph $\mathcal{P}^*(G)$ is connected and it will be concluded that G is cyclic or a generalized quaternion 2-group Q_{2^n} by theorem 2.3. Let S be a subset of Q_{2^n} consists of elements identity and unique involution. It is easy to see that $c(\mathcal{P}(Q_{2^n}) - S) > 2$ and this is cotradiction by 2.5. Then G is cyclic. Conversely, if G is a cyclic p-group by 2.4, the graph $\mathcal{P}(G)$ is complete and it has a Hamilton cycle. □

In the following, we will pay more attention to finite nilpotent groups with cyclic subgroup as a direct factor. The theorem of V. Batagelj and T. Pisanski in [1] is useful for finding Hamilton cycle.

Theorem 2.7. *Let* $G = T\square C_n$ *be cartesian prouduct of an* n*-cycle* C_n *and a tree* T *with maximum degree* $\Delta(T) \geq 2$. *Then* G *possesses a Hamilton cycle if and only if* $\Delta(T) \leq n$.

Corollary 2.8. *Suppose that* H, K *are groups such that* $gcd(|H|, |K|) = 1$. *If power graphs* $\mathcal{P}(H)$ *and* $\mathcal{P}(K)$ *are Hamiltonian, then graph* $\mathcal{P}(H \times K)$ *is Hamiltonian, also.*

Proof. It's obviously by lemma 2.2 and theorem 2.7. □

Theorem 2.9. *The power graph of generalized quaternion group* $\mathcal{P}(Q_{2^n}), (n \geq 3)$ *has a spanning tree* T *with* $\Delta(T) = 2^{n-3} + 1$.

Proof. Suppose that

$$Q_{2^n} = \langle x, y \mid x^{2^{n-1}} = y^4 = 1, y^2 = x^{2^{n-2}}, x^y = x^{-1} \rangle$$

One can see that

$$Q_{2^n} = \left\{x^t \mid 1 \leq t \leq 2^{n-1}\right\} \bigcup \left(\bigcup_{t=1}^{2^{n-2}} \{yx^t, y^{-1}x^t\} \right)$$

Now $\mathbb{Z}_{2^{n-1}} \cong \langle x \rangle \leq Q_{2^n}$, since the power graph $\mathcal{P}(\langle x \rangle)$ is complete, then we can let the Hamilton path

$$L : 1 \sim x^1 \cdots \sim x^{2^{n-2}} = y^2$$

Moreover for every $1 \leq t \leq 2^{n-3}$, there are the following pathes in the graph $\mathcal{P}(Q_{2^n})$:

$$L_t : 1 \sim yx^t \sim y^{-1}x^t \quad , \quad L'_t : y^2 \sim yx^t x^{2^{n-3}} \sim y^{-1}x^t x^{2^{n-3}}$$

The union pathes L, L_t and L'_t is a spanning tree T with maximum degree $\Delta(T) = 2^{n-3} + 1 = \deg(1) = \deg(y^2)$. □

Theorem 2.10. *The maximum degree every spanning tree in the power graph of generalized quaternion group is at least* $2^{n-3} + 1$.

Proof. By structure of the group Q_{2^n} in previous theorem, let $S = \{1, y^2\}$ and $\mathcal{P}(Q_{2^n}) = \Gamma$. It can see that the number of connected componnets of a graph $\Gamma - S$, indeed, $C(\Gamma - S) = 2^{n-2} + 1$. If T is a spanning tree in the graph Γ, then

$$C(T - S) \geq C(\Gamma - S) = 2^{n-2} + 1.$$

Since the subgraph T is connected, hence every component of $T - S$ is adjacent with at least one vertex of S. Thus

$$\deg_T(1) + \deg_T(y^2) \geq C(T - S) \geq 2^{n-2} + 1$$

and this conclude that $\deg_T(1) \geq 2^{n-3}+1$ or $\deg_T(y^2) \geq 2^{n-3}+1$ and therefore $\Delta(T) \geq 2^{n-3}+1$. □

Definition 2.11. For a given finite group G, we may define the directed power graph $\vec{\mathcal{P}}(G)$ as a directed graph with vertex set G, in which there is an arc from x to y ($x \to y$) if $x \neq y$ and $y = x^m$ for some positive integer m.

Theorem 2.12. *Assum that $G = \mathbb{Z}_m \times Q_{2^n}$ and m is odd positive integer. Then the power graph $\mathcal{P}(G)$ is Hamiltonian if and only if $m > 2^{n-3}$.*

Proof. Suppose that $m > 2^{n-3}$. By lemma 2.2, the graph $\mathcal{P}(\mathbb{Z}_m)\Box\mathcal{P}(Q_{2^n})$ is a spanning subgraph of $\mathcal{P}(G)$. It is enough to show that $\mathcal{P}(\mathbb{Z}_m)\Box\mathcal{P}(Q_{2^n})$ is a Hamiltonian graph. The power graph $\mathcal{P}(Q_{2^n})$ has a spanning tree T with $\Delta(T) = 2^{n-3}+1$, and the other hand the graph $\mathcal{P}(\mathbb{Z}_m)$ has a Hamilton cycle C_m. Now, by theorem 2.7 the graph $\mathcal{P}(G)$ is Hamiltonian. Conversely, By structure Q_{2^n} in the previous theorem, for every $1 \leq t \leq 2^{n-2}$ we define subsets of G as the following such that partition the group G:

$$F_t = \mathbb{Z}_m \times \{yx^t, y^{-1}x^t\}, \quad K = \mathbb{Z}_m \times \{1\} \cup \mathbb{Z}_m \times \{y^2\}$$

We claim that for edge $g \sim g'$ in the graph $\mathcal{P}(G)$ and $1 \leq t' \leq 2^{n-2}$, if $g \in F_{t'}$, then $g' \in F_{t'} \cup K$. For proof, consider two cases $g \to g'$ or $g \leftarrow g'$. In the first case the result is clear. For the second case put $g' = (g'_1, g'_2) \in G$. Hence there exists a positve integer l such that $g = g'^l = ({g'_1}^l, {g'_2}^l)$. Where ${g'_2}^l \in \{yx^t, y^{-1}x^t\}$, it can see that $g'_2 \in \{yx^t, y^{-1}x^t\}$.
Now, assume that $\mathcal{P}(G)$ has a Hamilton cycle as following:

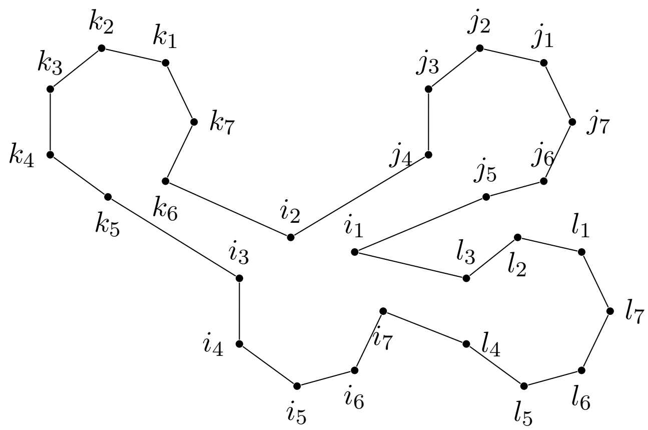
$$C : 1 = c_1 \sim c_2 \sim \cdots \sim c_{2^n m} \sim c_1$$

Put $i_t = \max\{i \mid c_i \in F_t\}, u_t = c_{i_t+1}$ and if $i_{t_0} = 2^n m$ for some $1 \leq t_0 \leq 2^{n-2}$, let $u_{t_0} = c_1$. By above discussion, we have $\{u_t \mid 1 \leq t \leq 2^{n-2}\} \subseteq K$. Thus $2^{n-2} \leq |K| = 2m$ and since m is odd, then $m > 2^{n-3}$. □

Theorem 2.13. *Suppose that $G = \prod_{i=1}^r P_i \times \mathbb{Z}_n$ such that for every $1 \leq i \leq r$, P_i is a finite p_i group where p_i's are distinct primes and $gcd(p_i, n) = 1$. Let T_i be spanning tree of the power graph $\mathcal{P}(P_i)$ with at least maximum degree $\Delta(T_i)$ and moreover $\Delta(T_1) \geq \Delta(T_2) \geq \cdots \geq \Delta(T_r)$. Now, if for every i, $\Delta(T_i) \leq n \times |P_r| \times |P_{r-1}| \times \cdots \times |P_{i+1}|$ then $\mathcal{P}(G)$ is a Hamiltonian graph.*

Proof. By using of induction and 2.7, proof is clear. □

Example 2.14. Assume that $G = \mathbb{Z}_7 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. One can see that power graph $\mathbb{Z}_2 \times \mathbb{Z}_2$ is not Hamiltonian. But, we can find a Hamiltonian cycle in $\mathcal{P}(G)$. Suppose that $G = \{i_t, j_t, k_t, l_t : t = 1, 2 \dots 7\}$ such that $|i_2| = 1, |j_2| = |k_2| = |l_2| = 2$ and for $i \neq 2$, we have $|i_t| = 7, |j_t| = |k_t| = |l_t| = 14$.



This example, guides us to find groups which that whose power graphs have Hamilton cycle by theorem 2.7. In the next theorem we will offer the Hamilton cycle in these power graphs. If G is a finite p-group of exponent p, then the power graph $\mathcal{P}(G)$ is union of complete subgraphs of order p that are intersect only in identity element. Hence the graph $\mathcal{P}(G)$ has a spanning tree T of maximum degree n, where n is the number of cyclic subgroups of order p.

Theorem 2.15. *Suppose that P is a finite p-group of exponent prime number p and $\mathbb{Z}_q$ is cyclic group of prime order q. If q at least equals to the number of disjoint cyclic subgroups of order p, then $\mathcal{P}(P \times \mathbb{Z}_q)$ is Hamiltonian.*

Proof. Put $G = P \times \langle y \rangle, |y| = q$, It is clear that the number of disoint cyclic subgroups of order p and pq is equal and it is $t = (p^m - 1)/(p - 1)$. Also, assume that $\langle (x_1, 1) \rangle, \dots, \langle (x_t, 1) \rangle$ is all of subgroups of order p. For $i = 1, 2, \dots, t$, set $\langle y \rangle = \{y_1, y_2, \dots, y_t, y_{t+1}, \dots, y_q = 1\}$ and $Z_i = \langle (x_i, y_i) \rangle \setminus \langle (x_i, 1) \rangle \cup \langle (1, y_i) \rangle$. Since cyclic subgroup of order q is cyclic, then elements of order p are adjacent to elements of order q. For $i = 1, 2, \dots, t \quad j = 1, 2, \dots q$, induced power graph on the set Z_i and graphs $\mathcal{P}(\langle (x_i, 1) \rangle), \mathcal{P}(\langle (1, y_j) \rangle)$ are complete. We can find the following cycle Hamiltonian:

$$(1, y_1) \sim \mathcal{P}^l(Z_1 \setminus \{z_1\}) \sim (x_1, 1) \sim \mathcal{P}^l(\langle (x_1, 1) \rangle) \sim (x_1, 1)^{-1} \sim z_1$$

$$\sim (1, y_2) \sim \mathcal{P}^l(Z_2 \setminus \{z_2\}) \sim (x_2, 1) \sim \mathcal{P}^l(\langle (x_2, 1) \rangle) \sim (x_2, 1)^{-1} \sim z_2$$

$$\vdots$$

$$\sim (1, y_t) \sim \mathcal{P}^l(Z_t \setminus \{z_t\}) \sim (x_t, 1) \sim \mathcal{P}^l(\langle (x_t, 1) \rangle) \sim (x_t, 1)^{-1} \sim z_t$$

$$\sim (1, y_{t+1}) \sim (1, y_{t+2}) \sim \dots \sim (1, y_q = 1).$$

where $\mathcal{P}^l(\langle(x_1,1)\rangle)$ is a path Hamiltonian between the vertices $(x_1,1)$ and $(x_1,1)^{-1}$ at the graph $\mathcal{P}(\langle(x_1,1)\rangle)$. □

3. Spanning tree in the power graph of finite abelian p-group

As stated in the theorem 2.7 the existance spanning tree is usefull for finding Hamilton cycle in the graph. In this section, we focus on the power graph of finite abelian p-group.

Definition 3.1. Let G be a finite p-group. The subgroup $\Omega_i(G)$ is the subgroup generated by all elemets of order dividing p^i; that is,

$$\Omega_i = \langle g : g^{p^i} = 1\rangle$$

.

Lemma 3.2. *If G is a finite abelian p-group, that is, $G = \prod_{i=1}^{t} \mathbb{Z}_{p^{m_i}}$, then*

(1) $|\Omega_1(G)| = p^t$,

(2) *The number of cyclic subgroup of order p is* $\dfrac{p^t-1}{p-1}$.

Proof. It's clear. □

Notation 1. *For a finite abelian p-group G and every $1 \neq k \in \mathbb{N}$, put $X_k = \Omega_k \setminus \Omega_{k-1}, X_1 = \Omega_1 \setminus \{1\}$. We use of $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ where show the connected componnents of proper power graph $\mathcal{P}^*(G)$. It is proved that the number connected components of $\mathcal{P}^*(G)$ is equal to number of cyclic subgroups of order p (see Theorem 2.6 [4]). Also for each $1 \leq i_1 \leq n$, we state ${X_1}^{(i_1)} = X_1 \cap \Gamma_{i_1}, K_{i_1} = \langle {X_1}^{(i_1)}\rangle$ and $Z_{i_1} = \frac{\Omega_1}{K_{i_1}}$. For cyclic subgroup H of order $p^k, k \geq 1$ containned in Γ_{i_1}, we define the set $Y_{k+1}^{(i_1,H)}$ consists of cyclic subgroups M of order p^{k+1} which have H as a subgroup, that is,*

$$Y_{k+1}^{(i_1,H)} = \{M \leq G : M = \langle y\rangle, H \subseteq M, |M| = p^{k+1}\}$$

By the above notation, for every $1 \leq i_1 \leq n$, it's clearly that $|K_{i_1}| = p, |Z_{i_1}| = p^{t-1}$ and if H is cyclic subgroup of G, then $H \subseteq \Gamma_{i_1}$ if and only if $K_{i_1} \subseteq H$. Also, the following theorem states if $Y_{k+1}^{(i_1,H)} \neq \varnothing$, then $|Y_{k+1}^{(i_1,H)}| = p^{t-1}$. Actually, size of the set $Y_{k+1}^{(i_1,H)}$ is not dependent on k and i_1.

Theorem 3.3. *For every $k \geq 1$ and $1 \leq i_1 \leq n$, we have either $|Z_{i_1}| = |Y_{k+1}^{(i_1,H)}| = p^{t-1}$ or $|Y_{k+1}^{(i_1,H)}| = 0$.*

Proof. Suppose that $|Y_{k+1}^{(i_1,H)}| \neq 0$ for some i_1, we show that $|Y_{k+1}^{(i_1,H)}| = p^{t-1}$. Whether $|Y_{k+1}^{(i_1,H)}| \neq 0$, there exists cyclic subgroup $M_0 = \langle y_0 \rangle$ in $Y_{k+1}^{(i_1,H)}$. One can see that $M_0 \subseteq \Gamma_{i_1}$. Let φ be the map defined as follows:

$$\begin{aligned} \varphi : Z_{i_1} &\longrightarrow Y_{k+1}^{(i_1,H)} \\ gK_{i_1} &\longrightarrow \langle gy_0 \rangle, \end{aligned}$$

We show that φ is a well-defined bijection. It's obviously, $(gy_0)^p = y_0^p \in H$ for every $g \in \Omega_1$. Since cyclic subgroups of a cyclic group is unique, then $\varphi(gK_{i_1}) \in Y_{k+1}^{(i_1,H)}$. If $gK_{i_1} = g'K_{i_1}$, then $g^{-1}g' \in K_{i_1}$, since $K_{i_1} \subseteq H \subseteq M_0 = \langle y_0 \rangle$, there exists positve integer r such that $g^{-1}g' = y_0^{rp^k}$ where $gcd(p,r) = 1$. Hence $g'y_0 = gy_0^{rp^k+1} = (gy_0)^{rp^k+1}$ and $g'y_0 \in \langle gy_0 \rangle$. Thus $\langle g'y_0 \rangle = \langle gy_0 \rangle$ and φ is a well-defined map.
For every $\langle y \rangle = M \in Y_{k+1}^{(i_1,H)}$, we have $y^{pr} = y_0^p$ where $\gcd(r,p) = 1$, hence $y^r y_0^{-1} \in \Omega_1$ and so $\varphi(y^r y_0^{-1} K_{i_1}) = \langle y^r y_0^{-1} y_0 \rangle = \langle y^r \rangle = \langle y \rangle$.Therefor the map is onto.
Finally, we prove φ is one-to-one. Assume that $\varphi(gK_{i_1}) = \varphi(g'K_{i_1})$, hence $\langle gy_0 \rangle = \langle g'y_0 \rangle$, then $g'y_0 = (gy_0)^r$ where $gcd(r,p) = 1$. So, $g'g^{-r} = y_0^{r-1}$, but $g'g^{-r} \in \Omega_1$ and we conclude $y_0^{r-1} \in K_{i_1}$ and $g'K_{i_1} = g^r K_{i_1}$,on the other hand $|{y_0}^{r-1}||p$, hence $p|r-1$ and there exists the integer s such that $ps = r-1$. Now, $g^r = g^{ps+1} = g$ and $gK_{i_1} = g'K_{i_1}$. The proof is complete. □

Definition 3.4. By the above theorem for every $1 \leq i_1 \leq n$, if $|Y_2^{(i_1,K_{i_1})}| \neq 0$, then $|Y_2^{(i_1,K_{i_1})}| = p^{t-1}$. Hence put $Y_2^{(i_1,K_{i_1})} = \{M_1, M_2, \ldots, M_{p^{t-1}}\}$. Now, we define $X_2^{(i_1,i_2)}$ as the set of generators of M_{i_2} for every $1 \leq i_2 \leq p^{t-1}$. Put $K_{i_1,i_2} = \langle X_2^{(i_1,i_2)} \rangle$. Similarly, if $Y_3^{(i_1,K_{i_1,i_2})}$ is non-empty set, then $Y_3^{(i_1,K_{i_1,i_2})} = \{M'_1, M'_2, \ldots, M'_{p^{t-1}}\}$ and again we can define $X_3^{(i_1,i_2,i_3)}$ as the set of generators of M'_{i_3} for every $1 \leq i_3 \leq p^{t-1}$. BY induction, the set $X_k^{(i_1,i_2,\ldots,i_k)}$ is defined.

Theorem 3.5. *For every $k \geq 1, 1 \leq i_1 \leq n$ and $1 \leq i_2, i_3, \ldots, i_k \leq p^{t-1}$, the following is established:*

(1) $X_k^{(i_1,i_2,\ldots,i_k)} \subseteq X_k$
(2) $(X_k^{(i_1,i_2,\ldots,i_k)})^p = X_{k-1}^{(i_1,i_2,\ldots,i_{k-1})}$
(3) $X_k^{(i_1,i_2,\ldots,i_k)}$ *is a partition of* X_k
(4) *The power graph* $\mathcal{P}(X_k^{(i_1,i_2,\ldots,i_k)})$ *is complete.*

Proof. It's underestood obviously. □

Theorem 3.6. *Suppose that G is a finite abelian p-group with representation $G = \prod_{i=1}^{t} \mathbb{Z}_{p^{m_i}}$ and put $n = \sum_{i=0}^{t-1} p^i$, we have the following cases:*

(1) *If subgraph T is a spanning tree of $\mathcal{P}(G)$, then $\Delta(T) \geq n$*
(2) *There exists a spanning tree T_0 of $\mathcal{P}(G)$ with $\Delta(T_0) = n$.*

Proof. (1) Since the identity element is a cut-vertex of the graph $\mathcal{P}(G)$ and the number of connectivity components is n, then $\deg_T(1) \geq n$ and this show that $\Delta(T) \geq n$.

(2) We construct the spanning tree T_0 with $\Delta(T_0) = n$. For this purpose, let $G = \{1\} \bigcup_{k \in \mathbb{N}} X_k$. By above theorems, $G = \{1\} \cup X_k^{(i_1,i_2,\ldots,i_k)}$ where the union changes on all indexes $1 \leq i_1 \leq n$ and $1 \leq i_2, i_3, \ldots, i_k \leq p^{t-1}$. Note that it is possible for some cases $X_k^{(i_1,i_2,\ldots,i_k)}$ is empty. Now, if the set $X_k^{(i_1,i_2,\ldots,i_k)} \neq \varnothing$, put

$$X_k^{(i_1,i_2,\ldots,i_k)} = \{g_0^{(i_1,\ldots,i_k)}, g_1^{(i_1,\ldots,i_k)} \ldots, g_r^{(i_1,\ldots,i_k)}\}.$$

In the following, we determine the edges of tree T_0 in the three parts:

(1) Wether, the subgraph $\mathcal{P}(X_k^{(i_1,i_2,\ldots,i_k)})$ is complete, then there exists the path

$$g_0^{(i_1,\ldots,i_k)} \sim g_2^{(i_1,\ldots,i_k)} \sim \cdots \sim g_r^{(i_1,\ldots,i_k)} \sim g_1^{(i_1,\ldots,i_k)}$$

for every $1 \leq i_1 \leq n$ and $1 \leq i_2, i_3, \ldots, i_k \leq p^{t-1}$.

(2) For one edge between the vertex sets X_k and X_{k+1}, consider the edge $g_1^{(i_1,\ldots,i_k)} \sim g_0^{(i_1,\ldots,i_{k+1})}$,

(3) The edges between X_1 and identity element is defined with adjacency $g_0^{i_1}$ and 1, that is, $g_0^{i_1} \sim 1$, for every $1 \leq i_1 \leq n$.

Firstly, subgraph T_0 is connected. Since for every $g \in G$, there exists $k \in \mathbb{N}$ such that $g = g_l^{(i_1,\ldots,i_k)} \in X_k^{(i_1,i_2,\ldots,i_k)}$. We can find the following path that connected vertices $g_l, 1$ in the spanning subgraph T_0:

$$g_l \sim \cdots \sim g_0^{(i_1,\ldots,i_k)} \sim g_1^{(i_1,\ldots,i_{k-1})} \sim \cdots \sim g_0^{(i_1,\ldots,i_{k-1})} \sim \cdots \sim g_0^{i_1} \sim 1.$$

Also, it's obvious that T_0 is without cycle. secondly, we show that $\Delta(T_0) = n$. It's necessary which we know about vertex degrees of T_0. But $deg_{T_0}(1) = n, deg_{T_0}(g_j^{(i_1,\ldots,i_k)}) = 2$ when $j \neq 0, 1$ and it is 1 or 2 if $j = 0$. Finally,

$$deg_{T_0}(g_1^{(i_1,\ldots,i_k)}) = 1 + |Y_{k+1}^{(i_1,K_{(i_1,\ldots,i_k)})}| \leq 1 + p^{t-1}.$$

whether, $1 + p^{t-1} \leq 1 + p + p^2 + \cdots + p^{t-1} = n$, hence for every $g \in G$, $deg_{T_0}(g) \leq n$. Then $\Delta(T_0) = n$ and proof is complete.

□

4. Hamilton cycle in the graph $\mathcal{P}(U_{n'})$

The subset $U_{n'}$ of $\mathbb{Z}_{n'}$ consists of the elements of $\mathbb{Z}_{n'}$ which are relatively prime to n'; it is a group under multiplication mod n'. Chakrabarty, Ghosh and Sen [3] considered the power graph $\mathcal{P}(U_{n'})$ and described properties of it such as complete and planar graph, also they stated open problem about existence Hamilton cycle in it. Now, in this section we will try to answer this question. The structure of the group $U_{n'}$ is as following:
Let $n' = 2^m p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n}$ be a natural number, where p_i's are distinct odd primes, $\alpha_i \geq 0$ for all $i = 1, 2, \dots, n$. Then

$$(U_{n'}, .) \cong (U_{2^m} \times U_{p_1^{\alpha_1}} \times \cdots \times U_{p_n^{\alpha_n}}).$$

Also, group $(U_n, .)$ enables us to consider it as a direct prouduct of additive cyclic groups.

$$(U_{n'}, .) \cong \begin{cases} (\mathbb{Z}_{\varphi(p_1^{\alpha_1})} \times \mathbb{Z}_{\varphi(p_2^{\alpha_2})} \times \ldots \times \mathbb{Z}_{\varphi(p_n^{\alpha_n})}, +) & m = 0, 1 \\ (\mathbb{Z}_2 \times \mathbb{Z}_{\varphi(p_1^{\alpha_1})} \times \mathbb{Z}_{\varphi(p_2^{\alpha_2})} \times \ldots \times \mathbb{Z}_{\varphi(p_n^{\alpha_n})}, +) & m = 2 \\ (\mathbb{Z}_2 \times \mathbb{Z}_{2^{m-2}} \times \mathbb{Z}_{\varphi(p_1^{\alpha_1})} \times \mathbb{Z}_{\varphi(p_2^{\alpha_2})} \times \ldots \times \mathbb{Z}_{\varphi(p_n^{\alpha_n})}, +) & m > 2 \end{cases}$$

Notation 2. *Suppose that $G = \mathbb{Z}_{\varphi(p_1^{\alpha_1})} \times \mathbb{Z}_{\varphi(p_2^{\alpha_2})} \times \ldots \times \mathbb{Z}_{\varphi(p_n^{\alpha_n})}$ where p_i's are prime and $2 < p_1 < \ldots < p_n$. We introduce the following symbols for the next theorems:*

(1) *S_q is Sylow q-subgroup G with orders q^{n_q} when q divide $|G|$.*
(2) *Let $\pi(G)$ be the set all prime numbers that dividing $|G|$ and we define For every $q \in \pi(G)$:*
(3) *$I_q = \{p \in \pi(G) : p > q\}$.*
(4) *$z_q = |\{1 \leq j \leq n : q | p_j - 1\}|$.*
(5) *If $I_q = \varnothing$, then put $t_q = 0$ otherwise, $t_q = |\prod_{p \in I_q} S_p| - 1$.*
(6) *If $q = p_i$ and $\alpha_i \geq 2$, then put $l_q = \dfrac{q^{z_q+1} - 1}{q - 1} = 1 + q + \cdots + q^{z_q}$ otherwise,$l_q = \dfrac{q^{z_q} - 1}{q - 1} = 1 + q + \cdots + q^{z_q - 1}$.*

Now, we are ready to state the following theorem:

Theorem 4.1. *Suppose that $G = \mathbb{Z}_{\varphi(p_1^{\alpha_1})} \times \mathbb{Z}_{\varphi(p_2^{\alpha_2})} \times \ldots \times \mathbb{Z}_{\varphi(p_n^{\alpha_n})}$ where p_i's are distinct odd prime numbers and for every $q \in \pi(G)$ we have $t_q \geq l_q - 1$, then*

(1) *the power graph $\mathcal{P}(G)$ is Hamiltonian.*
(2) *if $t_2 \geq 2^{n+1} - 2$, then power graph $\mathcal{P}(\mathbb{Z}_2 \times G)$ is Hamiltonian.*
(3) *if $t_2 \geq 2^{n+2} - 2$, then power graph $\mathcal{P}(\mathbb{Z}_2 \times \mathbb{Z}_{2^m} \times G)$ is Hamiltonian.*

Proof. (1) Let $q' = \max \pi(G)$, it can see that $I_{q'} = \varnothing, t_{q'} = 0$ and hence $z_{q'} = 0$ or 1. If $z_{q'} = 0$, then $q' = p_i$ with $\alpha_i \geq 2$ for some $1 \leq i \leq n$. Thus, $S_{q'} = S_{p_i} = \mathbb{Z}_{p_i^{\alpha_i - 1}}$. Now, assume that $z_{q'} = 1$, then there exists unique prime number p_j such that $q' | p_j - 1$. If $q' \neq p_i$ for every i, then $S_{q'} = \mathbb{Z}_{q'^{n_{q'}}}$, otherwise $q' = p_i$ for some i and we must be have $\alpha_i = 1$. Again we get $S_{q'} = \mathbb{Z}_{q'^{n_{q'}}}$. Hence in each case, the power graph $\mathcal{P}(S'_q)$ is complete and it's Hamiltonian.

In the following, suppose that $q \in \pi(G) \setminus \{q'\}$ and graph $\mathcal{P}(\prod_{p \in I_q} S_p)$ is Hamiltonian. Sylow q-subgroup S_q has form $S_q = \mathbb{Z}_{p_i^{\alpha_i - 1}} \times \mathbb{Z}_{q^{\beta_1}} \times \ldots \times \mathbb{Z}_{q^{\beta_{z_q}}}$ when $q = p_i$ and $\alpha_i \geq 2$ for some $1 \leq i \leq n$, otherwise it is $\mathbb{Z}_{q^{\beta_1}} \times \ldots \times \mathbb{Z}_{q^{\beta_{z_q}}}$.

By theorem 3.6, power graph $\mathcal{P}(S_q)$ has a spanning tree with $\Delta(T) = l_q$ respectively. On the other hand,

$|\prod_{p \in I_q} S_p| = t_q + 1 \geq l_q - 1 + 1 = l_q = \Delta(T)$ Hence $\Delta(T) \leq |\mathcal{P}(\prod_{p \in I_q} S_p|$ and by theorem 2.7, the spanning subgraph $\mathcal{P}(S_q) \square \mathcal{P}(\prod_{p \in I_q} S_p)$ of power graph $\mathcal{P}(S_q \times \prod_{p \in I_q} S_p)$ is Hamiltonian. Also, it is easy that conclude the power graph of finite group G is Hamiltonian, by induction.

(2,3) If $q = 2$, then $z_q = n$ and $S_2 = \mathbb{Z}_{2^{\beta_1}} \times \ldots \times \mathbb{Z}_{2^{\beta_n}}$. Therefore Sylow 2-subgroups of $\mathbb{Z}_2 \times G$ and $\mathbb{Z}_2 \times \mathbb{Z}_{2^m} \times G$ are $H = \mathbb{Z}_2 \times S_2$ and $K = \mathbb{Z}_2 \times \mathbb{Z}_{2^m} \times S_2$,respectively. On the other hand, the power graphs $\mathcal{P}(H)$ and $\mathcal{P}(K)$ have spanning trees with maximum degrees $2^{n+1} - 1$ and $2^{n+2} - 1$, respectively. Now, if $t_2 \geq 2^{n+1} - 2$ or $2^{n+2} - 2$, then it is easy to see $|\prod_{p \in I_2} S_p| = t_2 + 1 \geq 2^{n+1} - 2 + 1$ or $2^{n+2} - 2 + 1$ and this proof cases $(2), (3)$.

□

Corollary 4.2. *Suppose that* $G = \mathbb{Z}_{\varphi(p_1^{\alpha_1})} \times \mathbb{Z}_{\varphi(p_2^{\alpha_2})} \times \ldots \times \mathbb{Z}_{\varphi(p_n^{\alpha_n})}$ *where* p_i*'s are prime and* $2 < p_1 < \ldots < p_n$. *If for every* $1 \leq i \leq n$ *we had* $\alpha_i \geq 2$ *, then power graphs* $\mathcal{P}(G)$ *and* $\mathcal{P}(\mathbb{Z}_2 \times G)$ *are Hamiltonian.*

Proof. For every $q \in \pi(G)$, there exists $0 \leq i \leq n$ such that $p_i \leq q < p_{i+1}$ where $p_0 = 2$. Thus, $z_q \leq n - i$ and

$$l_q \leq \frac{q^{n-i+1} - 1}{q - 1} = 1 + q + \cdots + q^{n-i}.$$

On the other hand, since $\alpha_i \geq 2$ we have $|I_q| \geq n - i$. Hence

$$t_q = |\prod_{p \in I_q} S_p| - 1 \geq -1 + \prod_{j=i+1}^{n-i} P_j^{n_j} \geq -1 + (1+q)^{n-i} \geq -1 + (1 + q + \cdots + q^{n-i}) \geq -1 + l_q.$$

Therfore condition's before theorem is hold and $\mathcal{P}(G)$ is Hamiltonian. For the second part,

$$t_2 \geq \prod_{i=1}^{n} p_i - 1 \geq -1 + (1+2)^n \geq -1 + (2^n + 2^n + \cdots + 1) \geq 2^{n+1} - 1.$$

Then $\mathcal{P}(\mathbb{Z}_2 \times G)$ is Hamiltonian. □

Corollary 4.3. *Suppose that* $G = U_{n'}$,

(1) *If* $n' = m^2$ *for some* $m \in \mathbb{N}$, *then* $\mathcal{P}(G)$ *is Hamiltonian.*

(2) *There is* $d \in \mathbb{N}$ *such that* $U_{\frac{n'}{d}}$ *and* $U_{n'd}$ *are Hamiltonian.*

Proof. (1) It is clear that for every prime divisor q of n', q^2 divides n'. Then by corollary 4.2, $\mathcal{P}(G)$ is Hamiltonian.

(2) $n' = p_1^{\alpha_1} \times \cdots \times p_n^{\alpha_n}$ where p_i's are distinct primes. Put $d = \prod_{j \in I} p_j$ where $I = \{1 \leq i \leq n \mid \alpha_i = 1\}$. It can see that $U_{\frac{n'}{d}}$ and $U_{n'd}$ are Hamiltonian by corollary 4.2.

□

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