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Research Paper

SOME REMARKS ON $(\text{INC}(R))^c$

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Abstract. Let R be a commutative ring with identity $1 \neq 0$ which admits atleast two maximal ideals. In this article, we have studied simple, undirected graph $(\text{INC}(R))^c$ whose vertex set is the set of all proper ideals which are not contained in $J(R)$ and two distinct vertices I_1 and I_2 are joined by an edge in $(\text{INC}(R))^c$ if and only if $I_1 \subseteq I_2$ or $I_2 \subseteq I_1$. In this article, we have studied some interesting properties of $(\text{INC}(R))^c$.

1. Introduction

The rings considered in this article are commutative with identity $1 \neq 0$ which admits atleast two maximal ideals. The idea of associating a graph with certain subsets of a commutative ring and exploring the interplay between the ring-theoretic properties of a ring and the graph-theoretic properties of the graph associated with it began with the work of I. Beck in [8].

For a commutative ring R, we denote the set of all maximal ideals of R by $Max(R)$. We denote the cardinality of a set A using the notation $|A|$. Let R be a ring. Then $V(I) =$

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$\{J \in I(R) : J \subseteq I\}$; where $I(R)$ denotes the set of all proper ideals of R. The graphs considered in this article are undirected. Let $G = (V, E)$ be a simple graph. Recall from [7] that the *complement of* G, denoted by G^c is a graph whose vertex set is V and two distinct $u, v \in V$ are joined by an edge in G^c if and only if there exists no edge in G joining u and v. Let R be a ring with $|Max(R)| \geq 2$. Inspired by the research work done on the comaximal graph and comaximal ideal graph of algebraic structures in [2, 12, 13, 14, 15, 18, 19, 23, 25] and the research work done on the annihilating-ideal graph of a ring in [9, 10, 21], Ye and Wu [26] introduced and investigated an undirected graph associated with R whose vertex set equals $\{I \in I(R) : I \nsubseteq J(R)\}$ and distinct vertices I_1, I_2 are joined by an edge if and only if $I_1 + I_2 = R$. Ye and Wu called the graph introduced and studied by them in [26] as the comaximal ideal graph of R and denoted it using the notation $\mathscr{C}(R)$.

Visweswaran and Parejiya [22] introduced an undirected graph structure associated with R denoted by INC(R), whose vertex set equals $\{I \in I(R) : I \nsubseteq J(R)\}$ and distinct vertices I_1, I_2 are joined by an edge if and only if I_1 and I_2 are not comparable under the inclusion relation. Motivated by this research work, we have discussed some properties of $(\text{INC}(R))^c$ in this article.

We give brief of the theorems, proved in this article. In Theorem 3.1, we have proved that if R is a ring with $|Max(R)| = 2$ then $(\text{INC}(R))^c$ is a disconnected graph with two components. In Theorem 3.2, we have showed that if R is a ring with $|Max(R)| = n$; $n \geq 3, n \in \mathbb{N}$ then $(\text{INC}(R))^c$ is connected and $diam((\text{INC}(R))^c) = 3$. In Theorem 4.1, we have proved that if R is a ring with $|Max(R)| \geq 4$ then $(\text{INC}(R))^c$ is not a bipartite graph. In Theorem 4.2, we have proved that for a ring R with $|Max(R)| = 3$, $(\text{INC}(R))^c$ is bipartite if and only if $R \cong F_1 \times F_2 \times F_3$; where F_1, F_2 and F_3 are fields. In Theorem 4.3, we have investigated that if R is a ring with $|Max(R)| = 2$, then $(\text{INC}(R))^c$ is a bipartite graph if and only if R is ring isomorphic to one of the following rings:- (i) $F_1 \times F_2$; where F_i is a field for each $i \in \{1, 2\}$. (ii) $R_1 \times F_2$; where (R_1, m_1) is SPIR with $m_1 \neq (0)$ but $m_1^2 = (0)$ and F_2 is a field. (iii) $F_1 \times R_2$; where F_1 is a field and (R_2, m_2) is SPIR with $m_2 \neq (0)$ but $m_2^2 = (0)$. (iv) $R_1 \times R_2$; where (R_i, m_i) is SPIR with $m_i \neq (0)$ but $m_i^2 = (0)$ for each $i \in \{1, 2\}$. In Theorem 4.4, we have proved that for a ring R with $|Max(R)| \geq 2$, $(\text{INC}(R))^c$ is bipartite if and only if one of the following conditions hold:- (i) $R \cong F_1 \times F_2 \times F_3$; where F_1, F_2 and F_3 are fields. (ii) $R \cong F_1 \times F_2$; where F_i is a field for each $i \in \{1, 2\}$. (iii) $R \cong R_1 \times F_2$; where (R_1, m_1) is SPIR with $m_1 \neq (0)$ but $m_1^2 = (0)$ and F_2 is a field. (iv) $R \cong F_1 \times R_2$; where F_1 is a field and (R_2, m_2) is SPIR with $m_2 \neq (0)$ but $m_2^2 = (0)$. (v) $R \cong R_1 \times R_2$; where (R_i, m_i) is SPIR with $m_i \neq (0)$ but $m_i^2 = (0)$ for each $i \in \{1, 2\}$. In Theorem 4.5, we have investigated that for a ring R with $|Max(R)| \geq 2$, $(\text{INC}(R))^c$ is not a complete bipartite graph. In Theorem 5.1, we have proved that $(\text{INC}(R))^c$ is not a split graph if R is a ring with $|Max(R)| \geq 3$. We have proved in Theorem 6.1 that for a ring R with $|Max(R)| \geq 4$, $girth((\mathrm{INC}(R))^c) = 3$. In Theorem 6.2, we have shown that for a ring R with $|Max(R)| = 3$, $girth((\mathrm{INC}(R))^c) \leq 6$. Illustration 1 shows that upper bound of the inequality in Theorem 6.2 is obtained by a ring $R \cong F_1 \times F_2 \times F_3$; where F_i is a field $\forall i \in \{1, 2, 3\}$. In Theorem 6.3, we have investigated that for a ring R with $|Max(R)| = 3$, $girth((\mathrm{INC}(R))^c) = 6$ if and only if $R \cong F_1 \times F_2 \times F_3$; where F_i is a field $\forall i \in \{1, 2, 3\}$. In Theorem 7.1, we have shown that for a ring R with $|Max(R)| = n$, $n \geq 5$ and $n \in \mathbb{N}$, $(\mathrm{INC}(R))^c)$ is not planar. Theorem 7.2 (7.3 resp.) gives the characterization of rings R with $|Max(R)| = 3$ ($|Max(R)| = 4$ resp.) for which $(\mathrm{INC}(R))^c)$ is planar. In Theorem 8.1, we have proved that $(\mathrm{INC}(R))^c$ is not complemented if R is a ring with $|Max(R)| = 4$. In Theorem 8.2 (and 8.3 resp.), we have characterized rings R with $|Max(R)| = 3$ ($|Max(R)| = 2$ resp.) for which $(\mathrm{INC}(R))^c$ is complemented. Corollary 8.4 gives characterization of ring R for which $(\mathrm{INC}(R))^c$ is complemented. Corollary 8.5 depicts that $(\mathrm{INC}(R))^c$ is uniquely complemented if and only if $R \cong R_1 \times R_2$; where (R_i, m_i) is SPIR with $m_i \neq (0)$ but $m_i^2 = (0)$ for each $i \in \{1, 2\}$.

2. Preliminaries

It is useful to recall the following definitions and results from graph theory. Let $G = (V, E)$ be a graph. Let $a, b \in V$, $a \neq b$. Recall that the *distance between a and b*, denoted by $d(a, b)$ is defined as the length of a shortest path in G between a and b if such a path exists, otherwise $d(a, b) = \infty$. We define $d(a, a) = 0$. Let G be a simple graph. Then the complement G^c of G is defined by taking $V(G^c) = V(G)$ and making two vertices u and v adjacent in G^c if and only if they are non-adjacent in G [7]. A graph G is said to be *connected* if for any distinct $a, b \in V$, there exists a path in G between a and b. Recall from [7] that the *diameter* of a connected graph $G = (V, E)$ denoted by $diam(G)$ is defined as $diam(G) = sup\{d(a, b) | a, b \in V\}$. Let $G = (V, E)$ be a connected graph. Let $a \in V$. Recall that G is a *split graph* if $V(G)$ is the disjoint union of two nonempty subsets K and S such that the subgraph of G induced on K is complete and S is an independent set of G.

Let $G = (V, E)$ be a graph such that G contains a cycle. Recall from [7] that the *girth of G* denoted by $girth(G)$ is defined as the length of a shortest cycle in G. If a graph G does not contain any cycle, then we define $girth(G) = \infty$. Let $n \in \mathbb{N}$. A complete graph on n vertices is denoted by K_n. Let $G = (V, E)$ be a graph. Then G is said to be *bipartite* if the vertex set V of G can be partitioned into two nonempty subsets V_1 and V_2 such that each edge of G has one end in V_1 and the other in V_2. A bipartite graph with vertex partition V_1 and V_2 is said to be *complete*, if each element of V_1 is adjacent to every element of V_2. Let $m, n \in \mathbb{N}$. Let $G = (V, E)$ be a complete bipartite graph with $V = V_1 \cup V_2$. If $|V_1| = m$ and $|V_2| = n$, then G is denoted by $K_{m,n}$ [7]. Let $G = (V, E)$ be a graph. Recall from [4] that two distinct vertices u, v of G are said to be orthogonal, written $u \perp v$ if u and v are adjacent in G and there is no vertex of G which is adjacent to both u and v in G; that is, the edge $u - v$ is not an edge of any triangle in G. Let $u \in V$. A vertex v of G is said to be a *complement* of u if $u \perp v$ [4]. Moreover, we recall from [4] that G is *complemented* if each vertex of G admits a complement in G. Furthermore, G is said to be *uniquely complemented* if G is complemented and whenever the vertices u, v, w of G are such that $u \perp v$ and $u \perp w$, then a vertex x of G is adjacent to v in G if and only if x is adjacent to w in G.

Let $G = (V, E)$ be a graph. Recall from [7, Definition 8.1.1] that G is said to be *planar* if G can be drawn in a plane in such a way that no two edges of G intersect in a point other than a vertex of G. Recall that two adjacent edges are said to be in series if their common end vertex is of degree two [11, pg.9]. Two graphs are said to be *homeomorphic* if one graph can be obtained from the other by intersection of vertices of degree two or by the merger of edges in series [11, pg. 100]. It is useful to note from [11, pg. 93] that the graph K_5 is referred to as *Kuratowski's first graph* and $K_{3,3}$ is referred to as *Kuratowski's second graph.* The celebrated theorem of Kuratowski states that a graph G is planar if and only if G does not contain either of Kuratowski's two graphs or any graph homeomorphic to either of them [11, Theorem 5.9].

A ring R is said to be *local* if it has a unique maximal ideal. Recall that a principal ideal ring R is said to be a *special principal ring* (SPIR) if R admits only one prime ideal. If $\mathfrak{m}$ is the only prime ideal of R, then $\mathfrak{m}$ is necessarily nilpotent. If R is a special principal ideal ring with $\mathfrak{m}$ as its only prime ideal, then we describe it using the notation that $(R, \mathfrak{m})$ is a SPIR. Let $\mathfrak{m}$ be a nonzero maximal ideal of a ring R such that $\mathfrak{m}$ is principal and is nilpotent. Let $n \geq 2$ be least with the property that $\mathfrak{m}^n = (0)$. Then it follows from the proof of $(iii) \Rightarrow (i)$ of [6] that $\{\mathfrak{m}^i | i \in \{1, \ldots, n-1\}\}$ is the set of all nonzero proper ideals of R. As each ideal of R is principal with $\mathfrak{m}$ as its only prime ideal, it follows that $(R, \mathfrak{m})$ is a SPIR.

3. $Diam((\mathrm{INC}(R))^c)$

Theorem 3.1. *Let R be a ring with $|Max(R)| = 2$. Then $(\mathrm{INC}(R))^c$ is a disconnected graph with two components.*

Proof. Let $Max(R) = \{M_1, M_2\}$. Suppose that $(\mathrm{INC}(R))^c$ is connected. Consider $V_i = \{I \in I(R) : I \subseteq M_i \text{ but } I \nsubseteq M_j\}$ for $i, j \in \{1, 2\}$; where $j \neq i$. Observe that $V_1 \cap V_2 = \emptyset$. Let G_i be the subgraph of $(\mathrm{INC}(R))^c$ induced on V_i; for $i \in \{1, 2\}$. Let $I, J \in V(G_1)$. Note that $I - M_1 - J$ is a path between I and J. So, G_1 is a connected subgraph of $(\mathrm{INC}(R))^c$. Similarly, G_2 is a connected subgraph of $(\mathrm{INC}(R))^c$. Note that there is no edge in $(\mathrm{INC}(R))^c$ with one end vertex in $V(G_1)$ and another end vertex in $V(G_2)$. So, $(\mathrm{INC}(R))^c$ is a disconnected graph with two components, G_1 and G_2. □

Theorem 3.2. *Let R be a ring with $|Max(R)| = n$; $n \in \mathbb{N}$ and $n \geq 3$. Then $(\mathrm{INC}(R))^c$ is connected and $diam(\mathrm{INC}(R))^c = 3$.*

Proof. Let $Max(R) = \{M_1, M_2, ..., M_n\}$; $n \in \mathbb{N}$ and $n \geq 3$. Let $I, J \in V((\mathrm{INC}(R))^c)$ be distinct non-adjacent vertices. Since $I, J \nsubseteq J(R)$, there exists $M_i, M_j \in Max(R)$ such that $I \nsubseteq M_i$ and $J \nsubseteq M_j$; for some $i, j \in \{1, 2, ..., n\}$.

Case (i) $M_i = M_j$

Suppose that $IJ \subseteq J(R)$. Then $IJ \subseteq M_i$. So, either $I \subseteq M_i$ or $J \subseteq M_i$. This is not possible. So, $IJ \nsubseteq J(R)$. If $IJ = I$ then $I \subseteq J$. So, I and J are adjacent in $(\mathrm{INC}(R))^c$ which is a contradiction. Hence, $IJ \neq I$. Similarly, $IJ \neq J$. So, $I - IJ - J$ is a path of length two between I and J in $(\mathrm{INC}(R))^c$.

Case (ii) $M_i \neq M_j$

If $J \nsubseteq M_i$, then by Case (i) we have a path of length two between I and J. So, $J \subseteq M_i$. If $I \nsubseteq M_j$ then by Case (i), we have a path of length two between I and J. So, $I \subseteq M_j$. Let $IM_i \nsubseteq J(R)$. If $J = M_i$, then $I - IM_i - M_i = J$ is a path of length two between I and J in $(\mathrm{INC}(R))^c$. If $J \subsetneq M_i$, then $I - IM_i - M_i - J$ is a path of length three between I and J in $(\mathrm{INC}(R))^c$. Let $JM_j \nsubseteq J(R)$. If $I = M_j$, then $I = M_j - JM_j - J$ is a path of length two between I and J. If $I \subsetneq M_j$, then $I - M_j - JM_j - J$ is a path of length three between I and J in $(\mathrm{INC}(R))^c$. So, let $IM_i \subseteq J(R)$ and $JM_j \subseteq J(R)$. So, $I \subseteq M_1M_2...M_{i-1}M_{i+1}...M_n$ and $J \subseteq M_1M_2...M_{j-1}M_{j+1}...M_n$. Then $I - M_k - J$ is a path of length two between I and J in $(\mathrm{INC}(R))^c$; where $M_k \in Max(R)$ and $k \in \{1, 2, ..., n\} \setminus \{i, j\}$. Hence, $diam((\mathrm{INC}(R))^c) \leq 3$.

Note that M_1 and M_2 are not adjacent in $(\mathrm{INC}(R))^c$. So, $(\mathrm{INC}(R))^c$ is not complete. Thus $diam((\mathrm{INC}(R))^c) \neq 1$. Suppose that $diam((\mathrm{INC}(R))^c) = 2$ for some ring R. Note that M_1 and $M_2M_3...M_n$ are non-adjacent vertices in $(\mathrm{INC}(R))^c$. Suppose that there exists a path of length two between M_1 and $M_2M_3...M_n$ say, $M_1 - I - M_2M_3...M_n$; for some $I \in V((\mathrm{INC}(R))^c)$. Then $I \subseteq M_1$. Suppose $M_2M_3...M_n \subseteq I$. Then $M_2M_3...M_n \subseteq M_1$. Hence, $M_i \subseteq M_1$; for some $i \in \{1, 2, ..., n\}$ which is not possible. So, $I \subseteq M_2M_3...M_n$. So, $I \subseteq M_1M_2...M_n = J(R)$ which is not possible as $I \nsubseteq J(R)$. Hence, $diam((\mathrm{INC}(R))^c) \neq 2$. Therefore, $diam((\mathrm{INC}(R))^c) = 3$.

□

4. Bipartiteness of $(\mathrm{INC}(R))^c$

Theorem 4.1. *Let R be a ring with $|Max(R)| \geq 4$. Then $(\mathrm{INC}(R))^c$ is not a bipartite graph.*

Proof. Let $M_1, M_2, M_3, M_4 \in Max(R)$. Suppose that $(\mathrm{INC}(R))^c$ is a bipartite graph with V_1 and V_2 as its bipartite sets. Suppose that $M_i \in V_1$; for $M_i \in Max(R)$ and $i \in \{1, 2, 3, 4\}$. Without loss of generality, we may assume that $M_1 \in V_1$. Note that M_1M_2 is adjacent to M_1. So, $M_1M_2 \notin V_1$. Also, M_1M_2 is adjacent to M_2. So, $M_1M_2 \notin V_2$. This is not possible.

Thus, $Max(R) \subseteq V_i$; for some $i \in \{1, 2\}$. Without loss of generality, we may assume that $Max(R) \subseteq V_1$. Now, M_1M_2 is adjacent to M_1. So, $M_1M_2 \notin V_1$. So, $M_1M_2 \in V_2$. Also, $M_1M_2M_3$ is adjacent to M_1. So, $M_1M_2M_3 \notin V_1$. But $M_1M_2M_3$ is also adjacent to M_1M_2. So, $M_1M_2M_3 \notin V_2$. Thus $(\mathrm{INC}(R))^c$ is not a bipartite graph. □

Theorem 4.2. *Let R be a ring with $|Max(R)| = 3$. Then $(\mathrm{INC}(R))^c$ is bipartite if and only if $R \cong F_1 \times F_2 \times F_3$; where F_1, F_2 and F_3 are fields.*

Proof. Let $Max(R) = \{M_1, M_2, M_3\}$. Suppose that $(\mathrm{INC}(R))^c$ is bipartite. Let V_1 and V_2 be its bipartite sets. Suppose that $M_i \in V_1$ and $M_j,\ M_k \in V_2$; for distinct $i, j, k \in \{1, 2, 3\}$. Note that M_iM_j is adjacent to M_i. So, $M_iM_j \notin V_1$. Also, M_iM_j is adjacent to M_j. So, $M_iM_j \notin V_2$. This is not possible. So, $Max(R) \subseteq V_i$; for some $i \in \{1, 2\}$. Without loss of generality, we may assume that $Max(R) \subseteq V_1$. Now, M_iM_j is adjacent to M_i; for $i, j \in \{1, 2, 3\}$ and $j \neq i$. So, $M_iM_j \notin V_1$. Thus $M_iM_j \in V_2$; for all $i, j \in \{1, 2, 3\}$ and $i \neq j$. Suppose that $M_i^2 \neq M_i$; for some $i \in \{1, 2, 3\}$. Without loss of generality, we may assume that $M_1^2 \neq M_1$. Also, $M_1^2M_j$ is adjacent to M_j in $(\mathrm{INC}(R))^c$ for $j \in \{1, 2, 3\}$. So, $M_1^2M_j \notin V_1$; for any $j \in \{1, 2, 3\}$. Observe that $M_1^2M_j$ is adjacent to M_1M_j in $(\mathrm{INC}(R))^c$. So, $M_1^2M_j \notin V_2$. This is not possible. Thus $M_i^2 = M_i$; for each $i \in \{1, 2, 3\}$. Let $x_i \in M_i \setminus (0)$; for some $i \in \{1, 2, 3\}$. Let if possible, $M_i \neq Rx_i$. Note that $(Rx_i)M_j$ is adjacent to M_i; for $j \in \{1, 2, 3\}$ and $j \neq i$. So, $(Rx_i)M_j \notin V_1$. Also, $(Rx_i)M_j$ is adjacent to M_iM_j in $(\mathrm{INC}(R))^c$. So, $(Rx_i)M_j \notin V_2$. This is not possible. Thus $M_i = Rx_i$; for some $x_i \in M_i \setminus (0)$, $i \in \{1, 2, 3\}$. So, $J(R) = M_1M_2M_3 = Rx_1x_2x_3$ is principal and $(J(R))^2 = J(R)$. By Nakayama's lemma [6, Proposition 2.6], $J(R) = (0)$. Hence, by Chinese Remainder Theorem [6, Proposition 1.10 (ii), (iii)], $R \cong \frac{R}{J(R)} \cong \frac{R}{M_1} \times \frac{R}{M_2} \times \frac{R}{M_3} \cong F_1 \times F_2 \times F_3$; where F_1, F_2 and F_3 are fields.
Conversely, assume that $R \cong F_1 \times F_2 \times F_3$; where F_1, F_2 and F_3 are fields. Let $V_1 = \{M_1, M_2, M_3\}$ and $V_2 = \{M_1M_2, M_1M_3, M_2M_3\}$. Then $V_1 \cup V_2 = V((\mathrm{INC}(R))^c)$ and $V_1 \cap V_2 = \emptyset$. Note that V_1 and V_2 form bipartite sets of $(\mathrm{INC}(R))^c$ and hence $(\mathrm{INC}(R))^c$ is a bipartite graph. □

Theorem 4.3. *Let R be a ring with $|Max(R)| = 2$. Then $(\mathrm{INC}(R))^c$ is a bipartite graph if and only if R is isomorphic to one of the following rings:*

(i) *$F_1 \times F_2$; where F_i is a field for each $i \in \{1, 2\}$.*

(ii) *$R_1 \times F_2$; where (R_1, m_1) is SPIR with $m_1 \neq (0)$ but $m_1^2 = (0)$ and F_2 is a field.*

(iii) *$F_1 \times R_2$; where F_1 is a field and (R_2, m_2) is SPIR with $m_2 \neq (0)$ but $m_2^2 = (0)$.*

(iv) *$R_1 \times R_2$; where (R_i, m_i) is SPIR with $m_i \neq (0)$ but $m_i^2 = (0)$ for each $i \in \{1, 2\}$.*

Proof. Let $Max(R) = \{M_1, M_2\}$. Note that there exists $a \in M_1$ and $b \in M_2$ such that $Ra + Rb = R$. It is clear that $a \notin M_2$ and $b \notin M_1$. Hence, $a^n \notin M_2$ and $b^n \notin M_1$; for all $n \geq 1$. Suppose that $(\mathrm{INC}(R))^c$ is a bipartite with vertex partition V_1 and V_2. We can assume without loss of generality that $Ra \in V_1$. Now, either $Ra = Ra^2$ or $Ra \neq Ra^2$. If $Ra = Ra^2$, then $a = ra^2$; for some $r \in R$ and so, $ra = r^2a^2$. This implies that ra is a non-trivial idempotent element of R. If $Ra \neq Ra^2$, then Ra^2 must be in V_2. Observe that $Ra \neq Ra^3$. If $Ra^2 \neq Ra^3$, then Ra^3 can neither be in V_1 nor in V_2. This is impossible and so, we get that $Ra^2 = Ra^3$. This implies that $Ra^2 = Ra^4$. Hence, $a^2 = sa^4$; for some $s \in R$ and so we get that $sa^2 = s^2a^4$ and so sa^2 is a non-trivial idempotent element. Hence, there exists a non-zero local ring R_1 and R_2 such that R is a ring isomorphic to $R_1 \times R_2$. Let us denote the ring $R_1 \times R_2$ by T. Let m_i denote the unique maximal ideal of R_i; for each $i \in \{1, 2\}$. Now, $(\mathrm{INC}(T))^c$ is bipartite by assumption. Let it be bipartite with vertex partition W_1 and W_2. If R_1 is not a field, then $(0) \times R_2,\ m_1 \times R_2$ cannot be in same W_i; for each $i \in \{1, 2\}$. We can assume that $(0) \times R_2 \in W_1$ and $m_1 \times R_2 \in W_2$. Let $x \in m_1 \setminus (0)$. If $m_1 \neq R_1x$, then $R_1x \times R_2$ can neither be in W_1 nor be in W_2. This is impossible and so, $m_1 = R_1x$. As $x \neq 0$, it follows that $R_1x \neq R_1x^2$. If $R_1x^2 \neq (0)$, then $R_1x^2 \times R_2$ can neither be in W_1 nor be in W_2. This is impossible. So, $x^2 = 0$. Thus, either R_1 is a field or (R_1, m_1) is SPIR with $m_1 \neq (0)$ but $m_1^2 = (0)$. Similarly, it follows that either R_2 is a field or (R_2, m_2) is SPIR with $m_2 \neq (0)$ but $m_2^2 = (0)$. Thus, if $(\mathrm{INC}(R))^c$ is a bipartite then R is ring isomorphic to one of the following rings:- (i) $F_1 \times F_2$; where F_i is a field for each $i \in \{1, 2\}$. (ii) $R_1 \times F_2$; where (R_1, m_1) is SPIR with $m_1 \neq (0)$ but $m_1^2 = (0)$ and F_2 is a field. (iii) $F_1 \times R_2$; where F_1 is a field and (R_2, m_2) is SPIR with $m_2 \neq (0)$ but $m_2^2 = (0)$. (iv) $R_1 \times R_2$; where (R_i, m_i) is SPIR with $m_i \neq (0)$ but $m_i^2 = (0)$ for each $i \in \{1, 2\}$.

Conversely, suppose that R is ring isomorphic to one of the following rings:- (i) $F_1 \times F_2$; where F_i is a field for each $i \in \{1, 2\}$. (ii) $R_1 \times F_2$; where (R_1, m_1) is SPIR with $m_1 \neq (0)$ but $m_1^2 = (0)$ and F_2 is a field. (iii) $F_1 \times R_2$; where F_1 is a field and (R_2, m_2) is SPIR with $m_2 \neq (0)$ but $m_2^2 = (0)$. (iv) $R_1 \times R_2$; where (R_i, m_i) is SPIR with $m_i \neq (0)$ but $m_i^2 = (0)$ for each $i \in \{1, 2\}$. If $R \cong F_1 \times F_2$; where F_i is a field for each $i \in \{1, 2\}$ then take $V_1 = \{(0) \times F_2\}$ and $V_2 = \{F_1 \times (0)\}$. If $R \cong R_1 \times F_2$ where (R_1, m_1) is SPIR with $m_1 \neq (0)$ but $m_1^2 = (0)$ and F_2 is a field then take $V_1 = \{(0) \times F_2,\ R_1 \times (0)\}$ and $V_2 = \{m_1 \times F_2\}$. If $R \cong F_1 \times R_2$; where F_1 is a field and (R_2, m_2) is SPIR with $m_2 \neq (0)$ but $m_2^2 = (0)$ then take $V_1 = \{F_1 \times (0),\ (0) \times R_2\}$ and $V_2 = \{F_1 \times m_2\}$. If $R \cong R_1 \times R_2$; where (R_i, m_i) is SPIR with $m_i \neq (0)$ but $m_i^2 = (0)$ for each $i \in \{1, 2\}$ then take $V_1 = \{R_1 \times (0),\ m_1 \times R_2\}$ and $V_2 = \{(0) \times R_2,\ R_1 \times m_2\}$. Note that in all the above cases, $V_1 \cup V_2 = V((\mathrm{INC}(R))^c)$ and $V_1 \cap V_2 = \emptyset$. So, in each of the above cases V_1 and V_2 form bipartite sets of $(\mathrm{INC}(R))^c$ and hence $(\mathrm{INC}(R))^c$ is a bipartite graph. $\square$

Theorem 4.4. *Let R be a ring with $|Max(R)| \geq 2$. Then $(\mathrm{INC}(R))^c$ is bipartite if and only if one of the following conditions hold:*

(i) *$R \cong F_1 \times F_2 \times F_3$; where F_1, F_2 and F_3 are fields.*

(ii) *$R \cong F_1 \times F_2$; where F_i is a field for each $i \in \{1, 2\}$.*

(iii) *$R \cong R_1 \times F_2$; where (R_1, m_1) is SPIR with $m_1 \neq (0)$ but $m_1^2 = (0)$ and F_2 is a field.*

(iv) *$R \cong F_1 \times R_2$; where F_1 is a field and (R_2, m_2) is SPIR with $m_2 \neq (0)$ but $m_2^2 = (0)$.*

(v) *$R \cong R_1 \times R_2$; where (R_i, m_i) is SPIR with $m_i \neq (0)$ but $m_i^2 = (0)$ for each $i \in \{1, 2\}$.*

Proof. Proof follows from Theorems 4.1, 4.2 and 4.3. □

Theorem 4.5. *Let R be a ring with $|Max(R)| \geq 2$. Then $(\mathrm{INC}(R))^c$ is not a complete bipartite graph.*

Proof. Suppose that $(\mathrm{INC}(R))^c$ is a complete bipartite graph. Let V_1 and V_2 be the corresponding bipartite sets. Let $M_1, M_2 \in Max(R)$. Then M_1 and M_2 are not adjacent in $(\mathrm{INC}(R))^c$. So, $Max(R) \subseteq V_1$ or $Max(R) \subseteq V_2$. Without loss of generality, we may assume that $Max(R) \subseteq V_1$. Since $V_2 \neq \emptyset$, there exists $I \in V((\mathrm{INC}(R))^c)$ such that $I \in V_2$. Since we have assumed that $(\mathrm{INC}(R))^c$ is complete bipartite, I is adjacent to M; for each $M \in Max(R)$. So, $I \subseteq J(R)$ which is not possible. So, $(\mathrm{INC}(R))^c$ is not complete bipartite graph. □

5. Splitness of $(\mathrm{INC}(R))^c$

Theorem 5.1. *Let $|Max(R)| = n; n \in \mathbb{N}, n \geq 3$. Then $(\mathrm{INC}(R))^c$ is not a split graph.*

Proof. Let $Max(R) = \{M_1, M_2, ..., M_n\}$; $n \in \mathbb{N}$ and $n \geq 3$. Suppose that $(\mathrm{INC}(R))^c$ is a split graph with $V((\mathrm{INC}(R))^c) = K \cup S$; where the subgraph of $(\mathrm{INC}(R))^c$ induced on K is complete and S is an independent set. Since M_i is not adjacent to M_j; for $i \neq j$ and $i, j \in \{1, 2, ..., n\}$, atmost one M_i can be placed in K. Let $M_1 \in K$ and $M_2, M_3, ..., M_n \in S$. Note that M_2M_3 is adjacent to M_2 in $(\mathrm{INC}(R))^c$. So, $M_2M_3 \notin S$. Now, $M_1 + M_2M_3 = R$. So, $M_2M_3 \notin K$. This is not possible. Hence, $Max(R) \subseteq S$. Note that M_iM_j and M_i are adjacent in $(\mathrm{INC}(R))^c$ and since $M_i \in S$, $M_iM_j \in K$; for every $i, j \in \{1, 2, ..., n\}$ where $i \neq j$. Note that M_1M_2 and M_2M_3 are not adjacent in $(\mathrm{INC}(R))^c$. This is not possible. Hence, $(\mathrm{INC}(R))^c$ is not a split graph. □

6. $Girth((\mathrm{INC}(R))^c)$

Theorem 6.1. *Let R be a ring with $|Max(R)| = n$; $n \in \mathbb{N}, n \geq 4$. Then $girth((\mathrm{INC}(R))^c) = 3$.*

Proof. Let $M_1, M_2, M_3 \in Max(R)$. Suppose that $M_1M_2 \subseteq J(R)$. Then $M_1M_2 \subseteq M_3$. So, $M_1 \subseteq M_3$ or $M_2 \subseteq M_3$ which is not possible. Thus $M_1M_2 \nsubseteq J(R)$. Also, if $M_1M_2 = M_1$ (or $M_1M_2 = M_2$) then $M_1 \subseteq M_2$ (or $M_2 \subseteq M_1$) which is also not possible. So, $M_1M_2 \neq M_i$; for $i \in \{1, 2\}$. Similarly, $M_1M_2M_3 \neq M_1M_2$ and $M_1M_2M_3 \neq M_1$. Also $M_1M_2M_3 \nsubseteq J(R)$. So, we have a cycle $M_1 - M_1M_2 - M_1M_2M_3 - M_1$ of length three in $(\mathrm{INC}(R))^c$. So, $girth((\mathrm{INC}(R))^c) = 3$. □

Theorem 6.2. *Let R be a ring with $|Max(R)| = 3$. Then $girth((\mathrm{INC}(R))^c) \leq 6$.*

Proof. Let $Max(R) = \{M_1, M_2, M_3\}$. Note that $M_1 - M_1M_2 - M_2 - M_2M_3 - M_3 - M_1M_3 - M_1$ is a cycle of length six. Hence, $girth((\mathrm{INC}(R))^c) \leq 6$. □

Illustration 1: Following is an example of a ring R for which $girth((\mathrm{INC}(R))^c)$ is exactly the upper bound of above inequality. i.e. $girth((\mathrm{INC}(R))^c) = 6$.

Let $R \cong F_1 \times F_2 \times F_3$; where F_1, F_2 and F_3 are fields. Let $M_1 = (0) \times F_2 \times F_3$, $M_2 = F_1 \times (0) \times F_3$ and $M_3 = F_1 \times F_2 \times (0)$. Then $(\mathrm{INC}(R))^c$ is itself a cycle $M_1 - M_1M_2 - M_2 - M_2M_3 - M_3 - M_1M_3 - M_1$ of length 6. Hence, $girth((\mathrm{INC}(R))^c) = 6$.

Theorem 6.3. *Let $R \cong R_1 \times R_2 \times R_3$ be a ring; where (R_i, m_i) is a local ring for each $i \in \{1, 2, 3\}$. Then $girth((\mathrm{INC}(R))^c) = 6$ if and only if $R \cong F_1 \times F_2 \times F_3$; where F_1, F_2 and F_3 are fields.*

Proof. Assume that $R \cong F_1 \times F_2 \times F_3$; where F_1, F_2 and F_3 are fields. Then by above Illustration 1, it is clear that $girth((\mathrm{INC}(R))^c) = 6$.

Conversely, assume that $girth((\mathrm{INC}(R))^c) = 6$. Let $R \cong R_1 \times R_2 \times R_3$; where (R_i, m_i) is a local ring for all $i \in \{1, 2, 3\}$.

Case(i) (R_i, m_i) is a local ring which is not a field; for all $i \in \{1, 2, 3\}$.

Note that $Max(R) = \{M_1 = m_1 \times R_2 \times R_3, M_2 = R_1 \times m_2 \times R_3, M_3 = R_1 \times R_2 \times m_3\}$. Let $I = m_1 \times (0) \times R_3$. Here, $M_1M_2 = m_1 \times m_2 \times R_3$. So, $I - M_1M_2 - M_1 - I$ is a cycle of length three. So, $girth((\mathrm{INC}(R))^c) = 3$.

Case(ii) $R \cong R_1 \times R_2 \times F$; where (R_i, m_i) is a local ring which is not a field for all $i \in \{1, 2\}$ and F is a field.

Here, $Max(R) = \{M_1 = m_1 \times R_2 \times F, M_2 = R_1 \times m_2 \times F, M_3 = R_1 \times R_2 \times (0)\}$. Observe that $M_1M_2 = m_1 \times m_2 \times F, M_1M_3 = m_1 \times R_2 \times (0), M_2M_3 = R_1 \times m_2 \times (0)$. Let $I = m_1 \times (0) \times F \subsetneq M_1M_2 \setminus M_3$. Then $I - M_1 - M_1M_2 - I$ is a cycle of length three. So, $girth((\mathrm{INC}(R))^c) = 3$.

Proof is similar if $R \cong F \times R_1 \times R_2$ or $R_1 \times F \times R_2$; where (R_i, m_i) is a local ring which is not a field for all $i \in \{1,2\}$ and F is a field.

Case(iii) $R \cong R_1 \times F_1 \times F_2$; where (R_1, m_1) is a local ring which is not a field and F_1, F_2 are fields.

Here, $Max(R) = \{M_1 = m_1 \times F_1 \times F_2, M_2 = R_1 \times (0) \times F_2, M_3 = R_1 \times F_1 \times (0)\}$. Also, $M_1M_2 = m_1 \times (0) \times F_2, M_2M_3 = R_1 \times (0) \times (0), M_1M_3 = m_1 \times F_1 \times (0)$. Let $I = (0) \times (0) \times F_2 \subsetneq M_1M_2 \setminus M_3$. So, $I - M_1M_2 - M_1 - I$ is a cycle of length three. So, $girth((\text{INC}(R))^c) = 3$. Proof runs similar for $R \cong F_1 \times R_1 \times F_2$ or $F_1 \times F_2 \times R_1$; where F_i is a field for all $i \in \{1,2\}$ and (R_1, m_1) is a local ring which is not a field. Hence, $R \cong F_1 \times F_2 \times F_3$; where F_1, F_2 and F_3 are fields. □

7. Planarity of $(\text{INC}(R))^c$

Theorem 7.1. *Let R be a ring with $|Max(R)| \geq 5$. Then $(\text{INC}(R))^c$ is not planar.*

Proof. Let $M_1, M_2, M_3, M_4, M_5 \in Max(R)$. Observe that $M_1M_2M_3, M_1M_2M_3M_4$, $M_1M_2M_3M_5$ are distinct vertices in $(\text{INC}(R))^c$. Note that a subgraph of $(\text{INC}(R))^c$ induced on $V_1 \cup V_2$ contains $K_{3,3}$; where $V_1 = \{M_1, M_2, M_3\}$ and $V_2 = \{M_1M_2M_3, M_1M_2M_3M_4, M_1M_2M_3M_5\}$. Hence, $(\text{INC}(R))^c$ is not planar. □

Theorem 7.2. *Let R be a ring with $|Max(R)| = 3$. Then $(\text{INC}(R))^c$ is planar if and only if R is isomorphic to one of the following rings:*

(i) *$R_1 \times F_2 \times F_3$; where (R_1, m_1) is SPIR with $m_1 \neq (0)$ but $m_1^2 = (0)$ and F_2, F_3 are fields.*

(ii) *$F_1 \times F_2 \times F_3$; where F_1, F_2, F_3 are fields.*

Proof. Let $Max(R) = \{M_1, M_2, M_3\}$. Suppose that $(\text{INC}(R))^c$ is planar. Let if possible, $M_i^2 \neq M_i$ and $M_j^2 \neq M_j$; for some distinct $i, j \in \{1,2,3\}$. Then a subgraph of $(\text{INC}(R))^c$ induced on $V_1 \cup V_2$ contains $K_{3,3}$; where $V_1 = \{M_i, M_j, M_iM_j\}$ and $V_2 = \{M_i^2M_j, M_iM_j^2, M_i^2M_j^2\}$. So, $(\text{INC}(R))^c$ is not planar which is not possible. So, we have following two possibilities:- (i)$M_i^2 \neq M_i$ and $M_j^2 = M_j$; for $i \in \{1,2,3\}$ and for all $j \in \{1,2,3\} \setminus \{i\}$. (ii) $M_i^2 = M_i$; for each $i \in \{1,2,3\}$. Suppose that $M_i^2 \neq M_i$ and $M_j^2 = M_j$; for $i \in \{1,2,3\}$ and for all $j \in \{1,2,3\} \setminus \{i\}$. Without loss of generality, we may assume that $M_1^2 \neq M_1, M_2^2 = M_2$ and $M_3^2 = M_3$. Suppose that $M_1^3 = M_1^2$. Let $x_1 \in M_1 \setminus (M_2 \cup M_3 \cup M_1^2)$. If $Rx_1 \neq M_1$ then $(\text{INC}(R))^c$ contains a subgraph homeomorphic to $K_{3,3}$ as shown in following Figure 1. This is not possible. So, $M_1 = Rx_1$.

Let $x_2 \in M_2 \setminus (M_1 \cup M_3)$. If $M_2 \neq Rx_2$ then a subgraph of $(\text{INC}(R))^c$ induced on $V_1 \cup V_2$ contains $K_{3,3}$; where $V_1 = \{M_1, M_2, M_1M_2\}$ and $V_2 = \{M_1^2M_2, M_1Rx_2, M_1^2(Rx_2)\}$.

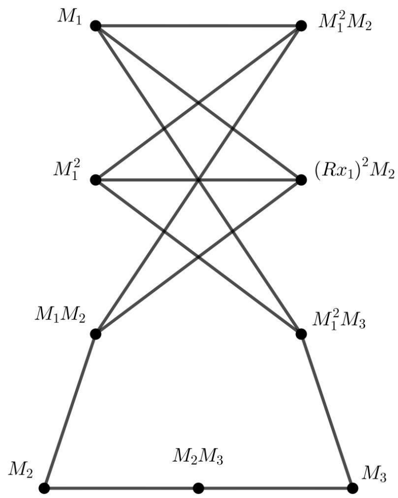

Figure 1. $K_{3,3}$.

So, $M_2 = Rx_2$. By a similar argument, $M_3 = Rx_3$; for some $x_3 \in M_3 \setminus (M_1 \cup M_2)$. So, $J(R) = M_1M_2M_3 = Rx_1x_2x_3$ is principal. Now, $M_1^3M_2^2M_3^2 = M_1^2M_2M_3$. By Nakayama's lemma [6, Proposition 2.6], $M_1^2M_2M_3 = (0)$. Thus by Chinese Remainder Theorem [6, Proposition 1.10(ii), (iii)], $R \cong \frac{R}{M_1^2} \times \frac{R}{M_2} \times \frac{R}{M_3} \cong R_1 \times F_2 \times F_3$; where (R_1, m_1) is a local ring and F_2, F_3 are fields. Note that $m_1^2 = (0)$. Let P be any prime ideal of R_1. Then $P \subseteq m_1$. Now, $m_1^2 = (0) \subseteq P$. So, $P = m_1$. Thus (R_1, m_1) is SPIR with $m_1^2 = (0)$. Hence, $R \cong R_1 \times F_2 \times F_3$; where (R_1, m_1) is SPIR with $m_1^2 = (0)$ and F_2, F_3 are fields.

Suppose that $M_1^3 \neq M_1^2$. Let $x_1 \in M_1 \setminus (M_2 \cup M_3 \cup M_1^2)$. If $Rx_1 \neq M_1$ then a subgraph of $(\mathrm{INC}(R))^c$ induced on $V_1 \cup V_2$ contains $K_{3,3}$; where$V_1 = \{M_1, M_2, M_1M_2\}$ and $V_2 = \{M_1^2M_2, M_1^3M_2, (Rx_1)M_2\}$. So, $(\mathrm{INC}(R))^c$ is non-planar. So, $Rx_1 = M_1$. Also, if $M_1^4 \neq M_1^3$, then a subgraph of $(\mathrm{INC}(R))^c$ induced on $V_1 \cup V_2$ contains $K_{3,3}$; where $V_1 = \{M_1, M_2, M_1M_2\}$ and $V_2 = \{M_1^2M_2, M_1^3M_2, M_1^4M_2\}$. So it is non-planar which is not possible. Hence, $M_1^4 = M_1^3$. Let $x_2 \in M_2 \setminus (M_3 \cup M_1)$. If $Rx_2 \neq M_2$ then a subgraph of $(\mathrm{INC}(R))^c$ induced on $V_1 \cup V_2$ contains $K_{3,3}$; where $V_1 = \{M_1, M_2, M_1M_2\}$ and $V_2 = \{M_1^2M_2, M_1^3M_2, M_1(Rx_2)\}$. Hence, it is non-planar. This is not possible. So, $Rx_2 = M_2$. Similarly, $Rx_3 = M_3$; for some $x_3 \in M_3 \setminus (M_1 \cup M_2)$. Thus $M_i = Rx_i; \forall i \in \{1, 2, 3\}$. Observe that $J(R) = M_1M_2M_3$. Let $I = M_1M_2M_3$ and $M = M_1^3M_2M_3$. Now, $IM = M_1^4M_2^2M_3^2 = M_1^3M_2M_3 = M$. By Nakayama's lemma [6, Proposition 2.6], $M = M_1^3M_2M_3 = (0)$. Thus by Chinese Remainder Theorem [6, Proposition 1.10 (ii), (iii)], $R \cong R_1 \times F_2 \times F_3$; where (R_1, m_1) is a local ring and F_1, F_2 are fields. If $m_1^2 \neq (0)$ then $(\mathrm{INC}(R))^c$ contains a subgraph homeomorphic to $K_{3,3}$ as shown in the following Figure 2 and so it is non-planar which is not possible.

Thus $m_1^2 = (0)$. Let P be any prime ideal of R_1. Then $P \subseteq m_1$. Now, $m_1^2 = (0) \subseteq P$. So, $P = m_1$. Thus (R_1, m_1) is SPIR with $m_1 \neq (0)$ but $m_1^2 = (0)$. Suppose, $M_i^2 = M_i$;

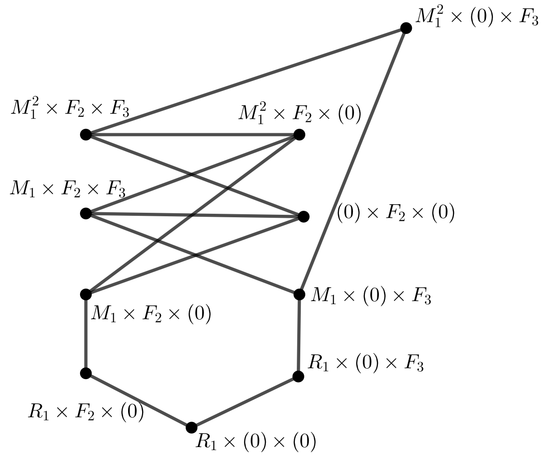

Figure 2.

$\forall i \in \{1,2,3\}$. Note that $(J(R))^2 = J(R)$. So by Nakayama's lemma [6, Proposition 2.6], $J(R) = (0)$. Thus by Chinese Remainder Theorem [6, Proposition 1.10 (ii), (iii)], $R \cong \frac{R}{M_1} \times \frac{R}{M_2} \times \frac{R}{M_3} \cong F_1 \times F_2 \times F_3$ where F_1, F_2 and F_3 are fields.

Conversely, assume that $R \cong R_1 \times F_2 \times F_3$; where (R_1, m_1) is SPIR with $m_1 \neq (0)$ but $m_1^2 = (0)$ and F_2, F_3 are fields. Then clearly by Figure 3, $(\mathrm{INC}(R))^c$ is planar.

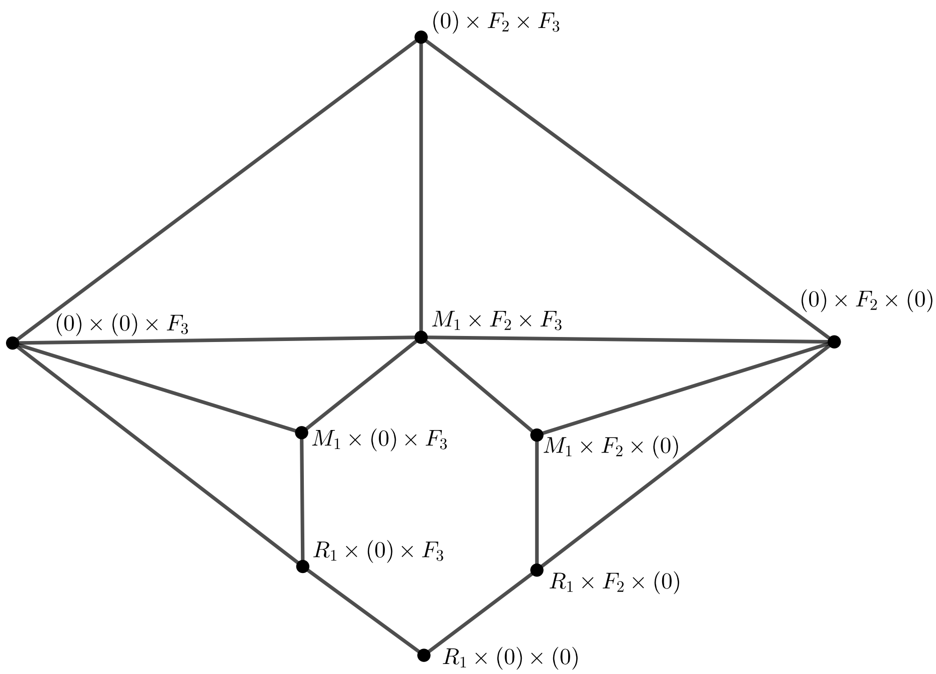

Figure 3. $(\mathrm{INC}(R_1 \times F_2 \times F_3))^c$.

If $R \cong F_1 \times F_2 \times F_3$;where F_1, F_2 and F_3 are fields. Then $V((\mathrm{INC}(F_1 \times F_2 \times F_3))^c) = \{M_1, M_2, M_3, M_1M_2, M_1M_3, M_2M_3\}$. Clearly $(\mathrm{INC}(R))^c$ is a cycle $M_1 - M_1M_2 - M_2 - M_2M_3 - M_3 - M_1M_3 - M_1$. Hence, $(\mathrm{INC}(R))^c$ is planar. □

Theorem 7.3. *Let R be a ring with $|Max(R)| = 4$. Then* $(\mathrm{INC}(R))^c$ *is planar if and only if* $R \cong F_1 \times F_2 \times F_3 \times F_4$*; where* F_1, F_2, F_3 *and* F_4 *are fields.*

Proof. Let $Max(R) = \{M_1, M_2, M_3, M_4\}$. Suppose that $M_i^2 \neq M_i$; for some $i \in \{1, 2, 3, 4\}$.

Without loss of generality, we may assume that $M_1^2 \neq M_1$. Let $V_1 = \{M_1, M_2, M_1M_2\}$ and $V_2 = \{M_1M_2M_3, M_1M_2M_4, M_1^2M_2M_3\}$. Then the subgraph of $(\mathrm{INC}(R))^c$ induced by $V_1 \cup V_2$ contains $K_{3,3}$. So, $(\mathrm{INC}(R))^c$ is not planar which is a contradiction. Thus $M_i^2 = M_i$; for all $i \in \{1, 2, 3, 4\}$. Let $x_1 \in M_1 \setminus (M_2 \cup M_3 \cup M_4)$. Suppose that $M_1 \neq Rx_1$. Let $V_1 = \{M_1, M_2, M_1M_2\}$ and $V_2 = \{(Rx_1)M_2M_3, (Rx_1)M_2M_4, (Rx_1)M_2\}$. Then the subgraph of $(\mathrm{INC}(R))^c$ induced on $V_1 \cup V_2$ contains $K_{3,3}$. So, $M_1 = Rx_1$. Similarly, we can say that $M_i = Rx_i$; for all $x_i \in \left(M_i \setminus \bigcup_{\substack{j=1\\ j\neq i}}^{4} M_j\right)$, $i \in \{1, 2, 3, 4\}$. Thus $J(R) = M_1M_2M_3M_4 = Rx_1x_2x_3x_4$. Note that $J(R)$ is principal and $(J(R))^2 = J(R)$. So, by Nakayama's lemma [7, Proposition 2.6], $J(R) = (0)$. Thus by Chinese Remainder Theorem [6, Proposition 1.10 (ii), (iii)], $R \cong \frac{R}{J(R)} \cong \frac{R}{M_1} \times \frac{R}{M_2} \times \frac{R}{M_3} \cong F_1 \times F_2 \times F_3 \times F_4$; where F_i is a field for all $i \in \{1, 2, 3, 4\}$. Conversely, assume that $R \cong F_1 \times F_2 \times F_3 \times F_4$; where F_i is a field for all $i \in \{1, 2, 3, 4\}$. Then

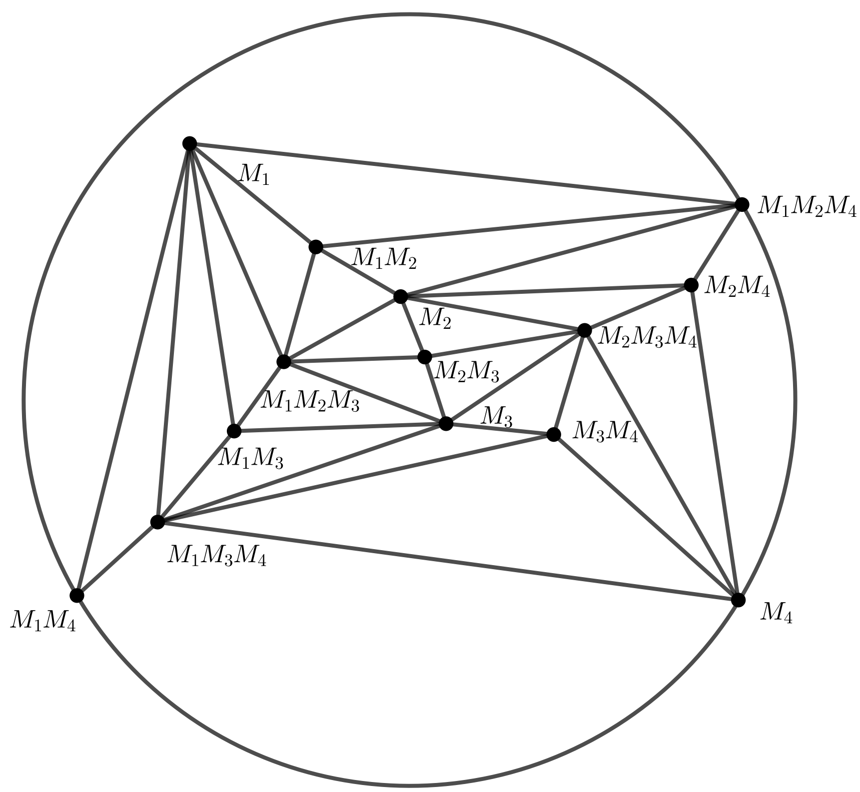

FIGURE 4. $(\mathrm{INC}(F_1 \times F_2 \times F_3 \times F_4))^c$.

clearly from the following Figure 4, $(\mathrm{INC}(R))^c$ is planar. □

8. Complementedness of $(\mathrm{INC}(R))^c$

Theorem 8.1. *Let R be a ring with $|Max(R)| = 4$. Then $(\mathrm{INC}(R))^c$ is not complemented.*

Proof. Let $Max(R) = \{M_1, M_2, M_3, M_4\}$. Suppose that $(\mathrm{INC}(R))^c$ is complemented. So, every vertex in $(\mathrm{INC}(R))^c$ has a complement in $(\mathrm{INC}(R))^c$. Let $I = M_1M_2$. Then there exists $J \in V((\mathrm{INC}(R))^c)$ such that $I \perp J$. So, I and J are adjacent in $(\mathrm{INC}(R))^c$. So, either $I \subseteq J$ or $J \subseteq I$. If $I \subseteq J$, then $I - J - M_1M_2M_3 - I$ is a triangle in $(\mathrm{INC}(R))^c$ which is not possible. If $J \subseteq I$, then $I - J - M_1 - I$ is a triangle in $(\mathrm{INC}(R))^c$ which is not possible. Hence, $(\mathrm{INC}(R))^c$ is not complemented. $\square$

Theorem 8.2. *Let R be a ring with $|Max(R)| = 3$. Then $(\mathrm{INC}(R))^c$ is complemented if and only if $R \cong F_1 \times F_2 \times F_3$; where F_1, F_2 and F_3 are fields.*

Proof. Let $Max(R) = \{M_1, M_2, M_3\}$. Suppose that $(\mathrm{INC}(R))^c$ is complemented. Let $I \in (\mathrm{INC}(R))^c$. Since $(\mathrm{INC}(R))^c$ is complemented, there exists $J \in V((\mathrm{INC}(R))^c)$ such that $I \perp J$. So, I and J are adjacent in $(\mathrm{INC}(R))^c$ and there is no $K \in V((\mathrm{INC}(R))^c)$ which is adjacent to both, I and J. As I and J are adjacent in $(\mathrm{INC}(R))^c$, $I \subseteq J$ or $J \subseteq I$. Without loss of generality, we may assume that $I \subseteq J$. Let $M_1 \in Max(R)$ be such that $I \subseteq J \subseteq M_1$. If $J \neq M_1$ then $I - J - M_1 - I$ is a triangle in $(\mathrm{INC}(R))^c$ which is not possible. So, $J = M_1$. Suppose that $M_i^2 \neq M_i$; for some $i \in \{1, 2, 3\}$. Let $I = M_iM_j$; $j \in \{1, 2, 3\}$ and $j \neq i$. Now, $J = M_i$ or $J = M_j$. Note that $I = M_iM_j - J - M_i^2M_j$ is a triangle in $(\mathrm{INC}(R))^c$ which is not possible. So, $M_i^2 = M_i$; for each $i \in \{1, 2, 3\}$. Let $x_i \in M_i \setminus (0)$; for $i \in \{1, 2, 3\}$. If $Rx_i \neq M_i$ then $I = M_iM_j - J - (Rx_i)M_j - I$ is a triangle in $(\mathrm{INC}(R))^c$ which is not possible. So, $Rx_i = M_i$; for each $i \in \{1, 2, 3\}$ and for $x_i \in M_i \setminus (0)$. So, M_i is principal; for each $i \in \{1, 2, 3\}$. Thus $J(R) = M_1M_2M_3 = Rx_1x_2x_3$ is also principal. Moreover, $(J(R))^2 = J(R)$. So, by Nakayama's lemma [6, Proposition 2.6] $J(R) = (0)$. Hence, by Chinese Remainder Theorem [6, Proposition 1.10(ii),(iii)], $R \cong \frac{R}{J(R)} \cong \frac{R}{M_1} \times \frac{R}{M_2} \times \frac{R}{M_3} \cong F_1 \times F_2 \times F_3$; as a rings where F_1, F_2 and F_3 are fields.

Conversely, assume that $R \cong F_1 \times F_2 \times F_3$; where F_1, F_2 and F_3 are fields. From Figure.4 , it is clear that $(\mathrm{INC}(R))^c$ is complemented. $\square$

Theorem 8.3. *Let R be a ring with $|Max(R)| = 2$. Then $(\mathrm{INC}(R))^c$ is complemented if and only if $R \cong R_1 \times R_2$; where (R_i, m_i) is SPIR with $m_i \neq (0)$ but $m_i^2 = (0)$ for each $i \in \{1, 2\}$.*

Proof. Let $Max(R) = \{M_1, M_2\}$. Suppose that $(\mathrm{INC}(R))^c$ is complemented. Suppose that $M_i^3 \neq M_i^2$; for some $i \in \{1, 2\}$. Without loss of generailty, let $M_1^3 \neq M_1^2$. Let $I = M_1^3$. Let J be a complement of I in $(\mathrm{INC}(R))^c$. If $J \neq M_1$ then $I = M_1^3 - J - M_1 - I$ is a triangle in $(\text{INC}(R))^c$ which is not possible. So, $J = M_1$. Now, $I = M_1^3 - J = M_1 - M_1^2 - I$ is also a triangle in $(\text{INC}(R))^c$ which is not possible. Thus $M_1^3 = M_1^2$. Similarly we can show that $M_2^3 = M_2^2$.

Case(i) $M_i^2 \neq M_i$**; for each** $i \in \{1, 2\}$**.**

Let $I = M_i^2$ and let J be complement of I. Let $J \subseteq M_i$, for some $i \in \{1, 2\}$. If $J \neq M_i$, then $I = M_i^2 - J - M_i - I$ is a triangle in $(\text{INC}(R))^c$ which is not possible. So, $J = M_i$. Let $x_i \in M_i \setminus M_j$; where $i, j \in \{1, 2\}$ and $i \neq j$. Suppose that $M_i \neq Rx_i$. If $Rx_i^2 \neq M_i^2$ then $I = M_i^2 - J = M_i - Rx_i^2 - I$ is a triangle in $(\text{INC}(R))^c$ which is not possible. So, $M_i^2 = Rx_i^2$. Let $y_i \in M_i \setminus (M_i^2 \cup M_j)$; for distinct $i, j \in \{1, 2\}$. Suppose that $Ry_i \neq M_i$; for $i \in \{1, 2\}$. Note that $Ry_i \nsubseteq M_i^2$. Then either $M_i^2 \subsetneq Ry_i$ or $M_i^2 \nsubseteq Ry_i$. Suppose that $M_i^2 \subsetneq Ry_i$. Then $M_i^2 = I - J = M_i - Ry_i - I$ is a triangle in $(\text{INC}(R))^c$ which is not possible. Suppose that $M_i^2 \nsubseteq Ry_i$. Then $I = M_i^2 - M_i^2 Ry_i - J = M_i - I$ is a triangle in $(\text{INC}(R))^c$ which is not possible. Thus $M_i = Ry_i$; for each $i \in \{1, 2\}$. Thus, $M_i = Rx_i$; for each $i \in \{1, 2\}$. Note that $J(R) = M_1M_2 = Rx_1x_2$ is principal and $(J(R))^3 = (J(R))^2$. So by Nakayama's lemma [6, Proposition 2.6], $(J(R))^2 = (0)$. Hence, by Chinese Remainder Theorem [6, Proposition 1.10(ii),(iii)], $R \cong \frac{R}{J(R)} \cong \frac{R}{M_1^2} \times \frac{R}{M_2^2} \cong R_1 \times R_2$; where (R_1, m_1) and (R_2, m_2) are local rings which are not fields. Observe that $m_i^2 = (0)$; for each $i \in \{1, 2\}$. Let P_i be any prime ideal of R_i. Then $m_i^2 = (0) \subseteq P_i$. So $P_i = m_i$. Thus, (R_i, m_i) is a SPIR with $m_i \neq (0)$ but $m_i^2 = (0)$; for each $i \in \{1, 2\}$.

Case(ii) $M_1^2 = M_1$ **and** $M_2^2 \neq M_2$**.**

As $M_2^2 \neq M_2$, by previous Case(i) there exists $x \in M_2 \setminus M_1$ such that $M_2 = Rx$. Let $I = M_1$ and J be a complement of I in $(\text{INC}(R))^c$. Let $x_1 \in M_1 \setminus (J \cup M_2)$. Suppose that $M_1 \neq Rx_1$. Note that $JRx_1 \nsubseteq J(R)$. If $JRx_1 \neq J$ and $JRx_1 \neq Rx_1$ then $I - J - JRx_1 - I$ is a triangle in $(\text{INC}(R))^c$ which is not possible. So, $JRx_1 = J$ or $JRx_1 = Rx_1$. Suppose that $JRx_1 = Rx_1$. Then $Rx_1 \subseteq J$ which is not possible. So, $JRx_1 = J$. Now, $I = M_1 - J - Rx_1 - I$ is a triangle in $(\text{INC}(R))^c$ which is not possible. So, $M_1 = Rx_1$. Note that $J(R) = M_1M_2 = Rx_1x$ is principal and $(J(R))^3 = (J(R))^2$. By Nakayama's lemma [6, Proposition 2.6], $(J(R))^2 = (0)$. Thus by Chinese Remainder Theorem [6, Proposition 1.10(ii),(iii)], $R \cong \frac{R}{M_1^2 = M_1} \times \frac{R}{M_2^2} \cong F_1 \times R_2$; where F_1 is a field and (R_2, m_2) is a local ring which is not a field. Note that $m_2^2 = (0)$ as $J(R)^2 = (0)$. Let P_2 be any prime ideal of R_2. Then $m_2^2 = (0) \subseteq P_2$. So, $P_2 = m_2$. Thus (R_2, m_2) is SPIR with $m_2 \neq (0)$ but $m_2^2 = (0)$.

Case(iii) $M_i^2 = M_i$**; for each** $i \in \{1, 2\}$**.**

By Case(ii), there exists $x_i \in M_i \setminus M_j$ such that $M_i = Rx_i$; for each $i \in \{1, 2\}$. Note that $(J(R))^2 = J(R)$. So by Nakayama's lemma [6, Proposition 2.6], $J(R) = (0)$. Hence, by Chinese Remainder Theorem [6, Proposition 1.10(ii),(iii)], $R \cong \frac{R}{J(R)} \cong \frac{R}{M_1} \times \frac{R}{M_2} \cong F_1 \times F_2$; where F_1 and F_2 are fields. So, if $(\text{INC}(R))^c$ is complemented then R is isomorphic to one of the following rings:- (i)$R_1 \times R_2$; where (R_i, m_i) is SPIR with $m_i \neq (0)$ but $m_i^2 = (0)$ for each $i \in \{1, 2\}$. (ii)$F_1 \times R_2$; where F_1 is a field and (R_2, m_2) is SPIR with $m_2 \neq (0)$ but $m_2^2 = (0)$. (iii)$F_1 \times F_2$; where F_1 and F_2 are fields. Now, suppose that $R \cong R_1 \times F_2$; where (R_1, m_1) is SPIR with $m_1 \neq (0)$ but $m_1^2 = (0)$ and F_2 is a field. Note that $V((\mathrm{INC}(R))^c) = \{m_1 \times F_2, (0) \times F_2, R_1 \times (0)\}$. Observe that $R_1 \times (0)$ is an isolated vertex in $(\mathrm{INC}(R))^c$. So, $(\mathrm{INC}(R))^c$ is not complemented. Suppose that $R \cong F_1 \times F_2$; where F_1 and F_2 are fields. Note that $V((\mathrm{INC}(R))^c) = \{F_1 \times (0), (0) \times F_2\}$ and both these vertices are isolated in $(\mathrm{INC}(R))^c$. So, $(\mathrm{INC}(R))^c$ is not complemented. Thus, $R_1 \times R_2$; where (R_i, m_i) is SPIR with $m_i \neq (0)$ but $m_i^2 = (0)$ for each $i \in \{1, 2\}$.

Conversely, assume that $R \cong R_1 \times R_2$; where (R_i, m_i) is SPIR with $m_i \neq (0)$ but $m_i^2 = (0)$ for each $i \in \{1, 2\}$. Here, $V((\mathrm{INC}(R))^c) = \{m_1 \times R_2, R_1 \times m_2, R_1 \times (0), (0) \times R_2\}$. Here, $m_1 \times R_2$ and $(0) \times R_2$ are complement of each other. Also, $R_1 \times (0)$ and $R_1 \times m_2$ are complements of each other. Thus, $(\mathrm{INC}(R))^c$ is complemented. □

Corollary 8.4. *Let R be a ring. Then $(\mathrm{INC}(R))^c$ is complemented if and only if R is isomorphic to one of the following rings:*

(i) *$R \cong F_1 \times F_2 \times F_3$; where F_1, F_2 and F_3 are fields.*

(ii) *$R \cong R_1 \times R_2$; where (R_i, m_i) is SPIR with $m_i \neq (0)$ but $m_i^2 = (0)$ for each $i \in \{1, 2\}$.*

Proof. Proof follows from Theorems 8.1, 8.2 and 8.3. □

Corollary 8.5. *Let R be a ring. Then $(\mathrm{INC}(R))^c$ is uniquely complemented if and only if $R \cong R_1 \times R_2$; where (R_i, m_i) is SPIR with $m_i \neq (0)$ but $m_i^2 = (0)$ for each $i \in \{1, 2\}$.*

Proof. Note that if $(\mathrm{INC}(R))^c$ is uniquely complemented then it is complemented. So, by Corollary 7.4, $R \cong F_1 \times F_2 \times F_3$; where F_1, F_2 and F_3 are fields or $R \cong R_1 \times R_2$; where (R_i, m_i) is SPIR with $m_i \neq (0)$ but $m_i^2 = (0)$ for each $i \in \{1, 2\}$. Suppose that $R \cong F_1 \times F_2 \times F_3$; where F_1, F_2 and F_3 are fields. Note that $F_1 \times (0) \times F_3$ and $F_1 \times F_2 \times (0)$ are complements of $F_1 \times (0) \times (0)$ in $(\mathrm{INC}(R))^c$. Observe that $(0) \times (0) \times F_3 \in N(F_1 \times (0) \times F_3)$ but $(0) \times (0) \times F_3 \notin N(F_1 \times F_2 \times (0))$. So, $N(F_1 \times (0) \times F_3) \neq N(F_1 \times F_2 \times (0))$. Thus $(\mathrm{INC}(R))^c$ is not uniquely complemented. Suppose that $R \cong R_1 \times R_2$; where (R_i, m_i) is SPIR with $m_i \neq (0)$ but $m_i^2 = (0)$ for each $i \in \{1, 2\}$. Here, $V((\mathrm{INC}(R))^c) = \{m_1 \times R_2, R_1 \times m_2, R_1 \times (0), (0) \times R_2\}$. Note that $m_1 \times R_2$ and $(0) \times R_2$ are the only complements of each other. Also, $R_1 \times (0)$ and $R_1 \times m_2$ are the only complements of each other. Thus, $(\mathrm{INC}(R))^c$ is uniquely complemented. □

9. Open Problems

Let R be a ring with $|Max(R)| = 2$. Then one can attempt the problems to classify the rings R for which

(i) $(\text{INC}(R))^c$ is split.

(ii) $(\text{INC}(R))^c$ is planar.

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References

[1] G. Aalipour, S. Akbari, R. Nikandish, M. J. Nikmehr and F. Shaiveisi, *On the coloring of the annihilating-ideal graph of a commutative ring*, Discrete Math., **312** (2012) 2620-2625.

[2] S. Akbari, B. Miraftab and R. Nikandish, *Co-maximal graph of subgroups of groups*, Can. Math. Bull., **60** No. 1 (2017) 12-25.

[3] S. Akbari, R. Nikandish and M. J. Nikmehr, *Some results on the intersection graphs of ideals of rings*, J. Algebra Appl., **12** No. 04 (2013) 1250200.

[4] D. F. Anderson, R. Levy and J. Shapiro, *Zero-divisor graphs, von Neumann regular rings and Boolean Algebras*, J. Pure Appl. Algebra, **180** (2003) 221-241.

[5] D. F. Anderson and P. S. Livingston, *The zero-divisor graph of a commutative ring*, J. Algebra, **217** (1999) 434-447.

[6] M. F. Atiyah and I. G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley, Reading, Massachusetts, 1969.

[7] R. Balakrishnan and K. Ranganathan, *A Textbook of Graph Theory*, Universitext, Springer, 2000.

[8] I. Beck, *Coloring of commutative rings*, J. Algebra, **116** No. 1 (1988) 208-226.

[9] M. Behboodi and Z. Rakeei, *The annihilating-ideal graphs of commutative rings I*, J. Algebra Appl., **10** No. 4 (2011) 727-739.

[10] M. Behboodi and Z. Rakeei, *The annihilating-ideal graphs of commutative rings II*, J. Algebra Appl., **10** No. 4 (2011) 741-753.

[11] N. Deo, *Graph Theory with Applications to Engineering and Computer Science*, Prentice Hall of India Private Limited, New Delhi, 1994.

[12] M. I. Jinnah and S. C. Mathew, *When is the comaximal graph split?*, Comm. Algebra, **40** No. 7 (2012) 2400-2404.

[13] H. R. Maimani, M. Salimi, A. Sattari and S. Yassemi, *Comaximal graph of commutative rings*, J. Algebra, **319** No. 4 (2008) 1801-1808.

[14] B. Miraftab, R. Nikandish, *Co-maximal graphs of two generator groups*, J. Algebra its Appl., **18** No. 04 (2019) 1950068.

[15] S. M. Moconja and Z. Z. Petrovic, *On the structure of comaximal graphs of commutative rings with identity*, Bull. Aust. Math. Soc., **83** (2011) 11-21.

[16] K. Nazzal and M. Ghanem, *On the Line Graph of the Zero Divisor Graph for the Ring of Gaussian Integers Modulo*, Int. J. Comb, **2012** (2012).

[17] R. Nikandish and H. R. Maimani, *Dominating sets of the annihilating-ideal graphs*, Electron. Notes Discrete Math., **45** (2014) 17-22.

[18] K. Samei, *On the comaximal graph of a commutative ring*, Can. Math. Bull., **57** No. 2 (2014) 413-423.

[19] P. K. Sharma and S. M. Bhatwadekar, *A note on graphical representation of rings*, J. Algebra, **176** (1995) 124-127.

[20] A. Sharma and A. Gaur, *Line Graphs associated to the Maximal graph*, J. Algebra Relat. Top., **3** No. 1 (2015) 1-11.

[21] S. Visweswaran and J. Parejiya, *Annihilating -ideal graphs with independence number at most four*, Cogent Math., **3** No. 1 (2016).

[22] S. Vishweswaran and J. Parejiya, *Some results on a supergraph of the comaximal ideal graph of a commutative ring*, Commun. Comb. Optim., **3** No. 2 (2018) 1-22.

[23] S. Visweswaran and J. Parejiya, *When is the complement of the comaximal graph of a commutative ring planar?*, ISRN Algebra, **8** No. 2014 (2014).

[24] S. Visweswaran and H. D. Patel, *Some results on the complement of the annihilating ideal graph of a commutative ring*, J. Algebra Appl., **14** No. 07 (2015), 1550099.

[25] H. J. Wang, *Graphs associated to Co-maximal ideals of commutative rings*, J. Algebra, **320** No. 7 (2008) 2917-2933.

[26] M. Ye and T. Wu, *Co-maximal ideal graphs of commutative rings*, J. Algebra Appl., **11** No. 06 (2012), 1250114.

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