

Research Paper

SOME REMARKS ON  $(INC(R))^c$

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ABSTRACT. Let  $R$  be a commutative ring with identity  $1 \neq 0$  which admits at least two maximal ideals. In this article, we have studied simple, undirected graph  $(INC(R))^c$  whose vertex set is the set of all proper ideals which are not contained in  $J(R)$  and two distinct vertices  $I_1$  and  $I_2$  are joined by an edge in  $(INC(R))^c$  if and only if  $I_1 \subseteq I_2$  or  $I_2 \subseteq I_1$ . In this article, we have studied some interesting properties of  $(INC(R))^c$ .

1. INTRODUCTION

The rings considered in this article are commutative with identity  $1 \neq 0$  which admits at least two maximal ideals. The idea of associating a graph with certain subsets of a commutative ring and exploring the interplay between the ring-theoretic properties of a ring and the graph-theoretic properties of the graph associated with it began with the work of I. Beck in [8].

For a commutative ring  $R$ , we denote the set of all maximal ideals of  $R$  by  $Max(R)$ . We denote the cardinality of a set  $A$  using the notation  $|A|$ . Let  $R$  be a ring. Then  $V(I) = \{J \in$

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$I(R) : J \subseteq I$ }; where  $I(R)$  denotes the set of all proper ideals of  $R$ . The graphs considered in this article are undirected. Let  $G = (V, E)$  be a simple graph. Recall from [7] that the *complement of  $G$* , denoted by  $G^c$  is a graph whose vertex set is  $V$  and two distinct  $u, v \in V$  are joined by an edge in  $G^c$  if and only if there exists no edge in  $G$  joining  $u$  and  $v$ . Let  $R$  be a ring with  $|Max(R)| \geq 2$ . Inspired by the research work done on the comaximal graph and comaximal ideal graph of algebraic structures in [2, 12, 13, 14, 15, 18, 19, 23, 25] and the research work done on the annihilating-ideal graph of a ring in [9, 10, 21], Ye and Wu in [26] introduced and investigated an undirected graph associated with  $R$  whose vertex set equals  $\{I \in I(R) : I \not\subseteq J(R)\}$  and distinct vertices  $I_1, I_2$  are joined by an edge if and only if  $I_1 + I_2 = R$ . Ye and Wu called the graph introduced and studied by them in [26] as the comaximal ideal graph of  $R$  and denoted it using the notation  $\mathcal{C}(R)$ .

S. Visweswaran and Jaydeep Parejiya in [22] introduced an undirected graph structure associated with  $R$  denoted by  $INC(R)$ , whose vertex set equals  $\{I \in I(R) : I \not\subseteq J(R)\}$  and distinct vertices  $I_1, I_2$  are joined by an edge if and only if  $I_1$  and  $I_2$  are not comparable under the inclusion relation. Motivated by this research work, we have discussed some properties of  $(INC(R))^c$  in this article.

We give brief of the theorems, proved in this article. In Theorem 3.1, we have proved that if  $R$  is a ring with  $|Max(R)| = 2$  then  $(INC(R))^c$  is a disconnected graph with two components. In Theorem 3.2, we have showed that if  $R$  is a ring with  $|Max(R)| = n$ ;  $n \geq 3, n \in \mathbb{N}$  then  $(INC(R))^c$  is connected and  $diam((INC(R))^c) = 3$ . In Theorem 4.1, we have proved that if  $R$  is a ring with  $|Max(R)| \geq 4$  then  $(INC(R))^c$  is not a bipartite graph. In Theorem 4.2, we have proved that for a ring  $R$  with  $|Max(R)| = 3$ ,  $(INC(R))^c$  is bipartite if and only if  $R \cong F_1 \times F_2 \times F_3$ ; where  $F_1, F_2$  and  $F_3$  are fields. In Theorem 4.3, we have investigated that if  $R$  is a ring with  $|Max(R)| = 2$ , then  $(INC(R))^c$  is a bipartite graph. In Theorem 4.4, we have proved that for a ring  $R$  with  $|Max(R)| \geq 2$ ,  $(INC(R))^c$  is bipartite if and only if (i)  $R \cong F_1 \times F_2 \times F_3$ ; where  $F_1, F_2$  and  $F_3$  are fields or (ii)  $|Max(R)| = 2$ . In Theorem 4.5, we have investigated that for a ring  $R$  with  $|Max(R)| \geq 2$ ,  $(INC(R))^c$  is not a complete bipartite graph. In Theorem 5.1, we have proved that  $(INC(R))^c$  is not a split graph if  $R$  is a ring with  $|Max(R)| \geq 3$ . We have proved in Theorem 6.1 that for a ring  $R$  with  $|Max(R)| \geq 4$ ,  $girth((INC(R))^c) = 3$ . In Theorem 6.2, we have shown that for a ring  $R$  with  $|Max(R)| = 3$ ,  $girth((INC(R))^c) \leq 6$ . Example 1 shows that upper bound of the inequality in Theorem 6.2 is obtained by a ring  $R \cong F_1 \times F_2 \times F_3$ ; where  $F_i$  is a field  $\forall i \in \{1, 2, 3\}$ . In Theorem 6.3, we have investigated that for a ring  $R$  with  $|Max(R)| = 3$ ,  $girth((INC(R))^c) = 6$  if and only if  $R \cong F_1 \times F_2 \times F_3$ ; where  $F_i$  is a field  $\forall i \in \{1, 2, 3\}$ . In Theorem 7.1, we have shown that for a ring  $R$  with  $|Max(R)| = n$ ,  $n \geq 5$  and  $n \in \mathbb{N}$ ,  $(INC(R))^c$  is not planar. Theorem 7.2 (7.3 resp.) gives the characterization of rings  $R$  with  $|Max(R)| = 3$  ( $|Max(R)| = 4$  resp.) for which

$(\text{INC}(R))^c$  is planar. In Theorem 8.1, we have proved that  $(\text{INC}(R))^c$  is not complemented if  $R$  is a ring with  $|\text{Max}(R)| = 4$ . In Theorem 8.2 (and 8.3 resp.), we have characterized rings  $R$  with  $|\text{Max}(R)| = 3$  ( $|\text{Max}(R)| = 2$  resp.) for which  $(\text{INC}(R))^c$  is complemented. Corollary 8.4 gives characterization of ring  $R$  for which  $(\text{INC}(R))^c$  is complemented. Corollary 8.5 depicts that  $(\text{INC}(R))^c$  is not uniquely complemented.

## 2. Preliminaries

It is useful to recall the following definitions and results from graph theory. Let  $G = (V, E)$  be a graph. Let  $a, b \in V$ ,  $a \neq b$ . Recall that the *distance between  $a$  and  $b$* , denoted by  $d(a, b)$  is defined as the length of a shortest path in  $G$  between  $a$  and  $b$  if such a path exists, otherwise  $d(a, b) = \infty$ . We define  $d(a, a) = 0$ . Let  $G$  be a simple graph. Then the complement  $G^c$  of  $G$  is defined by taking  $V(G^c) = V(G)$  and making two vertices  $u$  and  $v$  adjacent in  $G^c$  if and only if they are non-adjacent in  $G$  [7]. A graph  $G$  is said to be *connected* if for any distinct  $a, b \in V$ , there exists a path in  $G$  between  $a$  and  $b$ . Recall from [7] that the *diameter* of a connected graph  $G = (V, E)$  denoted by  $\text{diam}(G)$  is defined as  $\text{diam}(G) = \sup\{d(a, b) | a, b \in V\}$ . Let  $G = (V, E)$  be a connected graph. Let  $a \in V$ . Recall that  $G$  is a *split graph* if  $V(G)$  is the disjoint union of two nonempty subsets  $K$  and  $S$  such that the subgraph of  $G$  induced on  $K$  is complete and  $S$  is an independent set of  $G$ .

Let  $G = (V, E)$  be a graph such that  $G$  contains a cycle. Recall from [7] that the *girth of  $G$*  denoted by  $\text{girth}(G)$  is defined as the length of a shortest cycle in  $G$ . If a graph  $G$  does not contain any cycle, then we define  $\text{girth}(G) = \infty$ . Let  $n \in \mathbb{N}$ . A complete graph on  $n$  vertices is denoted by  $K_n$ . Let  $G = (V, E)$  be a graph. Then  $G$  is said to be *bipartite* if the vertex set  $V$  of  $G$  can be partitioned into two nonempty subsets  $V_1$  and  $V_2$  such that each edge of  $G$  has one end in  $V_1$  and the other in  $V_2$ . A bipartite graph with vertex partition  $V_1$  and  $V_2$  is said to be *complete*, if each element of  $V_1$  is adjacent to every element of  $V_2$ . Let  $m, n \in \mathbb{N}$ . Let  $G = (V, E)$  be a complete bipartite graph with  $V = V_1 \cup V_2$ . If  $|V_1| = m$  and  $|V_2| = n$ , then  $G$  is denoted by  $K_{m,n}$  [7]. Let  $G = (V, E)$  be a graph. Recall from [4] that two distinct vertices  $u, v$  of  $G$  are said to be *orthogonal*, written  $u \perp v$  if  $u$  and  $v$  are adjacent in  $G$  and there is no vertex of  $G$  which is adjacent to both  $u$  and  $v$  in  $G$ ; that is, the edge  $u - v$  is not an edge of any triangle in  $G$ . Let  $u \in V$ . A vertex  $v$  of  $G$  is said to be a *complement* of  $u$  if  $u \perp v$  [4]. Moreover, we recall from [4] that  $G$  is *complemented* if each vertex of  $G$  admits a complement in  $G$ . Furthermore,  $G$  is said to be *uniquely complemented* if  $G$  is complemented and whenever the vertices  $u, v, w$  of  $G$  are such that  $u \perp v$  and  $u \perp w$ , then a vertex  $x$  of  $G$  is adjacent to  $v$  in  $G$  if and only if  $x$  is adjacent to  $w$  in  $G$ .

Let  $G = (V, E)$  be a graph. Recall from [7, Definition 8.1.1] that  $G$  is said to be *planar* if  $G$  can be drawn in a plane in such a way that no two edges of  $G$  intersect in a point other

than a vertex of  $G$ . Recall that two adjacent edges are said to be in series if their common end vertex is of degree two [11, pg.9]. Two graphs are said to be *homeomorphic* if one graph can be obtained from the other by intersection of vertices of degree two or by the merger of edges in series [11, pg. 100]. It is useful to note from [11, pg. 93] that the graph  $K_5$  is referred to as *Kuratowski's first graph* and  $K_{3,3}$  is referred to as *Kuratowski's second graph*. The celebrated theorem of Kuratowski states that a graph  $G$  is planar if and only if  $G$  does not contain either of Kuratowski's two graphs or any graph homeomorphic to either of them [11, Theorem 5.9].

A ring  $R$  is said to be *local* if it has a unique maximal ideal. Recall that a principal ideal ring  $R$  is said to be a *special principal ring* (SPIR) if  $R$  admits only one prime ideal. If  $\mathfrak{m}$  is the only prime ideal of  $R$ , then  $\mathfrak{m}$  is necessarily nilpotent. If  $R$  is a special principal ideal ring with  $\mathfrak{m}$  as its only prime ideal, then we describe it using the notation that  $(R, \mathfrak{m})$  is a SPIR. Let  $\mathfrak{m}$  be a nonzero maximal ideal of a ring  $R$  such that  $\mathfrak{m}$  is principal and is nilpotent. Let  $n \geq 2$  be least with the property that  $\mathfrak{m}^n = (0)$ . Then it follows from the proof of (iii)  $\Rightarrow$  (i) of [6] that  $\{\mathfrak{m}^i | i \in \{1, \dots, n-1\}\}$  is the set of all nonzero proper ideals of  $R$ . As each ideal of  $R$  is principal with  $\mathfrak{m}$  as its only prime ideal, it follows that  $(R, \mathfrak{m})$  is a SPIR.

### 3. $\text{Diam}((\text{INC}(R))^c)$

**Theorem 3.1.** *Let  $R$  be a ring with  $|\text{Max}(R)| = 2$ . Then  $(\text{INC}(R))^c$  is a disconnected graph with two components.*

*Proof.* Let  $\text{Max}(R) = \{M_1, M_2\}$ . Suppose that  $(\text{INC}(R))^c$  is connected. Consider  $V_i = \{I \in I(R) : I \subseteq M_i \setminus M_j\}$  for  $i \in \{1, 2\}$ ; where  $j \neq i$  and  $j \in \{1, 2\}$ . Observe that  $V_1 \cap V_2 = \emptyset$ . Let  $G_i$  be the subgraph of  $(\text{INC}(R))^c$  induced on  $V_i$ ; for  $i \in \{1, 2\}$ . Let  $I, J \in V(G_1)$ . Note that  $I - M_1 - J$  is a path between  $I$  and  $J$ . So,  $G_1$  is a connected subgraph of  $(\text{INC}(R))^c$ . Similarly,  $G_2$  is a connected subgraph of  $(\text{INC}(R))^c$ . Note that there is no edge in  $(\text{INC}(R))^c$  with one end vertex in  $V(G_1)$  and another end vertex in  $V(G_2)$ . So,  $(\text{INC}(R))^c$  is a disconnected graph with two components,  $G_1$  and  $G_2$ .  $\square$

**Theorem 3.2.** *Let  $R$  be a ring with  $|\text{Max}(R)| = n$ ;  $n \in \mathbb{N}$  and  $n \geq 3$ . Then  $(\text{INC}(R))^c$  is connected and  $\text{diam}(\text{INC}(R))^c = 3$ .*

*Proof.* Let  $\text{Max}(R) = \{M_1, M_2, \dots, M_n\}$ ;  $n \in \mathbb{N}$  and  $n \geq 3$ . Let  $I, J \in V((\text{INC}(R))^c)$  be distinct non-adjacent vertices. Since  $I, J \not\subseteq J(R)$ , there exists  $M_i, M_j \in \text{Max}(R)$  such that  $I \not\subseteq M_i$  and  $J \not\subseteq M_j$ ; for some  $i, j \in \{1, 2, \dots, n\}$ .

**Case (i)**  $M_i = M_j$

Suppose that  $IJ \subseteq J(R)$ . Then  $IJ \subseteq M_i$ . So, either  $I \subseteq M_i$  or  $J \subseteq M_i$ . This is not possible.

So,  $IJ \not\subseteq J(R)$ . If  $IJ = I$  then  $I \subseteq J$ . So,  $I$  and  $J$  are adjacent in  $(\text{INC}(R))^c$  which is a contradiction. Hence,  $IJ \neq I$ . Similarly,  $IJ \neq J$ . So,  $I - IJ - J$  is a path of length two between  $I$  and  $J$  in  $(\text{INC}(R))^c$ .

**Case (ii)**  $M_i \neq M_j$

If  $J \not\subseteq M_i$ , then by Case (i) we have a path of length two between  $I$  and  $J$ . So,  $J \subseteq M_i$ . If  $I \not\subseteq M_j$  then by Case (i), we have a path of length two between  $I$  and  $J$ . So,  $I \subseteq M_j$ . Let  $IM_i \not\subseteq J(R)$ . If  $J = M_i$ , then  $I - IM_i - M_i = J$  is a path of length two between  $I$  and  $J$  in  $(\text{INC}(R))^c$ . If  $J \subsetneq M_i$ , then  $I - IM_i - M_i - J$  is a path of length three between  $I$  and  $J$  in  $(\text{INC}(R))^c$ . Let  $JM_j \not\subseteq J(R)$ . If  $I = M_j$ , then  $I = M_j - JM_j - J$  is a path of length two between  $I$  and  $J$ . If  $I \subsetneq M_j$ , then  $I - M_j - JM_j - J$  is a path of length three between  $I$  and  $J$  in  $(\text{INC}(R))^c$ . So, let  $IM_i \subseteq J(R)$  and  $JM_j \subseteq J(R)$ . So,  $I \subseteq M_1M_2\dots M_{i-1}M_{i+1}\dots M_n$  and  $J \subseteq M_1M_2\dots M_{j-1}M_{j+1}\dots M_n$ . Then  $I - M_k - J$  is a path of length two between  $I$  and  $J$  in  $(\text{INC}(R))^c$ ; where  $M_k \in \text{Max}(R)$  and  $k \in \{1, 2, \dots, n\} \setminus \{i, j\}$ . Hence,  $\text{diam}((\text{INC}(R))^c) \leq 3$ .

Note that  $M_1$  and  $M_2$  are not adjacent in  $(\text{INC}(R))^c$ . So,  $(\text{INC}(R))^c$  is not complete. Thus  $\text{diam}((\text{INC}(R))^c) \neq 1$ . Suppose that  $\text{diam}((\text{INC}(R))^c) = 2$  for some ring  $R$ . Note that  $M_1$  and  $M_2M_3\dots M_n$  are non-adjacent vertices in  $(\text{INC}(R))^c$ . Suppose that there exists a path of length two between  $M_1$  and  $M_2M_3\dots M_n$  say,  $M_1 - I - M_2M_3\dots M_n$ ; for some  $I \in V((\text{INC}(R))^c)$ . Then  $I \subseteq M_1$ . Suppose  $M_2M_3\dots M_n \subseteq I$ . Then  $M_2M_3\dots M_n \subseteq M_1$ . Hence,  $M_i \subseteq M_1$ ; for some  $i \in \{1, 2, \dots, n\}$  which is not possible. So,  $I \subseteq M_2M_3\dots M_n$ . So,  $I \subseteq M_1M_2\dots M_n = J(R)$  which is not possible as  $I \not\subseteq J(R)$ . Hence,  $\text{diam}((\text{INC}(R))^c) \neq 2$ . Therefore,  $\text{diam}((\text{INC}(R))^c) = 3$ .

□

4. BIPARTITENESS OF  $(\text{INC}(R))^c$

**Theorem 4.1.** *Let  $R$  be a ring with  $|\text{Max}(R)| \geq 4$ . Then  $(\text{INC}(R))^c$  is not a bipartite graph.*

*Proof.* Let  $M_1, M_2, M_3, M_4 \in \text{Max}(R)$ . Suppose that  $(\text{INC}(R))^c$  is a bipartite graph with  $V_1$  and  $V_2$  as its bipartite sets. Suppose that  $M_i \in V_1$ ; for  $M_i \in \text{Max}(R)$  and  $i \in \{1, 2, 3, 4\}$ . Without loss of generality, we may assume that  $M_1 \in V_1$ . Note that  $M_1M_2$  is adjacent to  $M_1$ . So,  $M_1M_2 \notin V_1$ . Also,  $M_1M_2$  is adjacent to  $M_2$ . So,  $M_1M_2 \notin V_2$ . This is not possible. Thus,  $\text{Max}(R) \subseteq V_i$ ; for some  $i \in \{1, 2\}$ . Without loss of generality, we may assume that  $\text{Max}(R) \subseteq V_1$ . Now,  $M_1M_2$  is adjacent to  $M_1$ . So,  $M_1M_2 \notin V_1$ . So,  $M_1M_2 \in V_2$ . Also,  $M_1M_2M_3$  is adjacent to  $M_1$ . So,  $M_1M_2M_3 \notin V_1$ . But  $M_1M_2M_3$  is also adjacent to  $M_1M_2$ . So,  $M_1M_2M_3 \notin V_2$ . Thus  $(\text{INC}(R))^c$  is not a bipartite graph. □

**Theorem 4.2.** *Let  $R$  be a ring with  $|\text{Max}(R)| = 3$ . Then  $(\text{INC}(R))^c$  is bipartite if and only if  $R \cong F_1 \times F_2 \times F_3$ ; where  $F_1, F_2$  and  $F_3$  are fields.*

*Proof.* Let  $Max(R) = \{M_1, M_2, M_3\}$ . Suppose that  $(INC(R))^c$  is bipartite. Let  $V_1$  and  $V_2$  be its bipartite sets. Suppose that  $M_i \in V_1$  and  $M_j, M_k \in V_2$ ; for distinct  $i, j, k \in \{1, 2, 3\}$ . Note that  $M_i M_j$  is adjacent to  $M_i$ . So,  $M_i M_j \notin V_1$ . Also,  $M_i M_j$  is adjacent to  $M_j$ . So,  $M_i M_j \notin V_2$ . This is not possible. So,  $Max(R) \subseteq V_i$ ; for some  $i \in \{1, 2\}$ . Without loss of generality, we may assume that  $Max(R) \subseteq V_1$ . Now,  $M_i M_j$  is adjacent to  $M_i$ ; for  $i, j \in \{1, 2, 3\}$  and  $j \neq i$ . So,  $M_i M_j \notin V_1$ . Thus  $M_i M_j \in V_2$ ; for all  $i, j \in \{1, 2, 3\}$  and  $i \neq j$ . Suppose that  $M_i^2 \neq M_i$ ; for some  $i \in \{1, 2, 3\}$ . Without loss of generality, we may assume that  $M_1^2 \neq M_1$ . Also,  $M_1^2 M_j$  is adjacent to  $M_j$  in  $(INC(R))^c$  for  $j \in \{1, 2, 3\}$ . So,  $M_1^2 M_j \notin V_1$ ; for any  $j \in \{1, 2, 3\}$ . Observe that  $M_1^2 M_j$  is adjacent to  $M_1 M_j$  in  $(INC(R))^c$ . So,  $M_1^2 M_j \notin V_2$ . This is not possible. Thus  $M_i^2 = M_i$ ; for each  $i \in \{1, 2, 3\}$ . Let  $x_i \in M_i \setminus (0)$ ; for some  $i \in \{1, 2, 3\}$ . Let if possible,  $M_i \neq Rx_i$ . Note that  $(Rx_i)M_j$  is adjacent to  $M_i$ ; for  $j \in \{1, 2, 3\}$  and  $j \neq i$ . So,  $(Rx_i)M_j \notin V_1$ . Also,  $(Rx_i)M_j$  is adjacent to  $M_i M_j$  in  $(INC(R))^c$ . So,  $(Rx_i)M_j \notin V_2$ . This is not possible. Thus  $M_i = Rx_i$ ; for some  $x_i \in M_i \setminus (0)$ ,  $i \in \{1, 2, 3\}$ . So,  $J(R) = M_1 M_2 M_3 = Rx_1 x_2 x_3$  is principal and  $(J(R))^2 = J(R)$ . By Nakayama's lemma [6, Proposition 2.6],  $J(R) = (0)$ . Hence, by Chinese Remainder Theorem [6, Proposition 1.10 (ii), (iii)],  $R \cong \frac{R}{J(R)} \cong \frac{R}{M_1} \times \frac{R}{M_2} \times \frac{R}{M_3} \cong F_1 \times F_2 \times F_3$ ; where  $F_1, F_2$  and  $F_3$  are fields.

Conversely, assume that  $R \cong F_1 \times F_2 \times F_3$ ; where  $F_1, F_2$  and  $F_3$  are fields. Let  $V_1 = \{M_1, M_2, M_3\}$  and  $V_2 = \{M_1 M_2, M_1 M_3, M_2 M_3\}$ . Then  $V_1 \cup V_2 = V((INC(R))^c)$  and  $V_1 \cap V_2 = \emptyset$ . Note that  $V_1$  and  $V_2$  form bipartite sets of  $(INC(R))^c$  and hence  $(INC(R))^c$  is a bipartite graph.  $\square$

**Theorem 4.3.** *Let  $R$  be a ring with  $|Max(R)| = 2$ . Then  $(INC(R))^c$  is a bipartite graph.*

*Proof.* Let  $Max(R) = \{M_1, M_2\}$ . Let  $V_i = \{I \in I(R) : I \subseteq M_i \setminus M_j\}$ ; for  $i \in \{1, 2\}$ ,  $j \neq i$  and  $j \in \{1, 2\}$ . Note that  $V_1 \cap V_2 = \emptyset$  and  $V_1 \cup V_2 = V((INC(R))^c)$ .  $V_1$  and  $V_2$  forms bipartite sets and hence  $(INC(R))^c$  is a bipartite graph.  $\square$

**Theorem 4.4.** *Let  $R$  be a ring with  $|Max(R)| \geq 2$ . Then  $(INC(R))^c$  is bipartite if and only if one of the following conditions hold:-*

- (i)  $R \cong F_1 \times F_2 \times F_3$ ; where  $F_1, F_2$  and  $F_3$  are fields
- (ii)  $|Max(R)| = 2$ .

*Proof.* Proof follows from Theorems 4.1, 4.2 and 4.3.  $\square$

**Theorem 4.5.** *Let  $R$  be a ring with  $|Max(R)| \geq 2$ . Then  $(INC(R))^c$  is not a complete bipartite graph.*

*Proof.* Suppose that  $(\text{INC}(R))^c$  is a complete bipartite graph. Let  $V_1$  and  $V_2$  be the corresponding bipartite sets. Let  $M_1, M_2 \in \text{Max}(R)$ . Then  $M_1$  and  $M_2$  are not adjacent in  $(\text{INC}(R))^c$ . So,  $\text{Max}(R) \subseteq V_1$  or  $\text{Max}(R) \subseteq V_2$ . Without loss of generality, we may assume that  $\text{Max}(R) \subseteq V_1$ . Since  $V_2 \neq \emptyset$ , there exists  $I \in V((\text{INC}(R))^c)$  such that  $I \in V_2$ . Since we have assumed that  $(\text{INC}(R))^c$  is complete bipartite,  $I$  is adjacent to  $M$ ; for each  $M \in \text{Max}(R)$ . So,  $I \subseteq J(R)$  which is not possible. So,  $(\text{INC}(R))^c$  is not complete bipartite graph.  $\square$

### 5. SPLITNESS OF $(\text{INC}(R))^c$

**Theorem 5.1.** *Let  $|\text{Max}(R)| = n; n \in \mathbb{N}, n \geq 3$ . Then  $(\text{INC}(R))^c$  is not a split graph.*

*Proof.* Let  $\text{Max}(R) = \{M_1, M_2, \dots, M_n\}; n \in \mathbb{N}$  and  $n \geq 3$ . Suppose that  $(\text{INC}(R))^c$  is a split graph with  $V((\text{INC}(R))^c) = K \cup S$ ; where the subgraph induced on  $K$  is complete and  $S$  is an independent set. Since  $M_i$  is not adjacent to  $M_j$ ; for  $i \neq j$  and  $i, j \in \{1, 2, \dots, n\}$ , atmost one  $M_i$  can be placed in  $K$ . Let  $M_1 \in K$  and  $M_2, M_3, \dots, M_n \in S$ . Note that  $M_2M_3$  is adjacent to  $M_2$  in  $(\text{INC}(R))^c$ . So,  $M_2M_3 \notin S$ . Now,  $M_1 + M_2M_3 = R$ . So,  $M_2M_3 \notin K$ . This is not possible. Hence,  $\text{Max}(R) \subseteq S$ . Note that  $M_iM_j$  and  $M_i$  are adjacent in  $(\text{INC}(R))^c$  and since  $M_i \in S, M_iM_j \in K$ ; for every  $i, j \in \{1, 2, \dots, n\}$  where  $i \neq j$ . Note that  $M_1M_2$  and  $M_2M_3$  are not adjacent in  $(\text{INC}(R))^c$ . This is not possible. Hence,  $(\text{INC}(R))^c$  is not a split graph.  $\square$

### 6. $\text{Girth}((\text{INC}(R))^c)$

**Theorem 6.1.** *Let  $R$  be a ring with  $|\text{Max}(R)| = n; n \in \mathbb{N}, n \geq 4$ . Then  $\text{girth}((\text{INC}(R))^c) = 3$ .*

*Proof.* Let  $M_1, M_2, M_3 \in \text{Max}(R)$ . Suppose that  $M_1M_2 \subseteq J(R)$ . Then  $M_1M_2 \subseteq M_3$ . So,  $M_1 \subseteq M_3$  or  $M_2 \subseteq M_3$  which is not possible. Thus  $M_1M_2 \not\subseteq J(R)$ . Also, if  $M_1M_2 = M_1$  (or  $M_1M_2 = M_2$ ) then  $M_1 \subseteq M_2$  (or  $M_2 \subseteq M_1$ ) which is also not possible. So,  $M_1M_2 \neq M_i$ ; for  $i \in \{1, 2\}$ . Similarly,  $M_1M_2M_3 \neq M_1M_2$  and  $M_1M_2M_3 \neq M_1$ . Also  $M_1M_2M_3 \not\subseteq J(R)$ . So, we have a cycle  $M_1 - M_1M_2 - M_1M_2M_3 - M_1$  of length three in  $(\text{INC}(R))^c$ . So,  $\text{girth}((\text{INC}(R))^c) = 3$ .  $\square$

**Theorem 6.2.** *Let  $R$  be a ring with  $|\text{Max}(R)| = 3$ . Then  $\text{girth}((\text{INC}(R))^c) \leq 6$ .*

*Proof.* Let  $\text{Max}(R) = \{M_1, M_2, M_3\}$ . Note that  $M_1 - M_1M_2 - M_2 - M_2M_3 - M_3 - M_1M_3 - M_1$  forms a cycle of length six. Hence,  $\text{girth}((\text{INC}(R))^c) \leq 6$ .  $\square$

**Example 1:** Following is an example of a ring for which  $girth((\text{INC}(R))^c)$  is exactly the upper bound of above inequality. i.e.  $girth((\text{INC}(R))^c) = 6$ .

Let  $R \cong F_1 \times F_2 \times F_3$ ; where  $F_1, F_2$  and  $F_3$  are fields. Let  $M_1 = (0) \times F_2 \times F_3$ ,  $M_2 = F_1 \times (0) \times F_3$  and  $M_3 = F_1 \times F_2 \times (0)$ . Then  $(\text{INC}(R))^c$  is itself a cycle  $M_1 - M_1M_2 - M_2 - M_2M_3 - M_3 - M_1M_3 - M_1$  of length 6. Hence,  $girth((\text{INC}(R))^c) = 6$ .

**Theorem 6.3.** *Let  $R \cong R_1 \times R_2 \times R_3$  be a ring; where  $(R_i, m_i)$  is a local ring for each  $i \in \{1, 2, 3\}$ . Then  $girth((\text{INC}(R))^c) = 6$  if and only if  $R \cong F_1 \times F_2 \times F_3$ ; where  $F_1, F_2$  and  $F_3$  are fields.*

*Proof.* Assume that  $R \cong F_1 \times F_2 \times F_3$ . Then by above Example 1, it is clear that  $girth((\text{INC}(R))^c) = 6$ .

Conversely, assume that  $girth((\text{INC}(R))^c) = 6$ . Let  $R \cong R_1 \times R_2 \times R_3$ ; where  $(R_i, m_i)$  is a local ring for all  $i \in \{1, 2, 3\}$ .

**Case(i)  $(R_i, m_i)$  is a local ring which is not a field; for all  $i \in \{1, 2, 3\}$ .**

Note that  $\text{Max}(R) = \{M_1 = m_1 \times R_2 \times R_3, M_2 = R_1 \times m_2 \times R_3, M_3 = R_1 \times R_2 \times m_3\}$ . Let  $I = m_1 \times (0) \times R_3$ . Here,  $M_1M_2 = m_1 \times m_2 \times R_3$ . So,  $I - M_1M_2 - M_1 - I$  forms a cycle of length three. So,  $girth((\text{INC}(R))^c) = 3$ .

**Case(ii)  $R \cong R_1 \times R_2 \times F$ ; where  $(R_i, m_i)$  is a local ring which is not a field for all  $i \in \{1, 2\}$  and  $F$  is a field.**

Here,  $\text{Max}(R) = \{M_1 = m_1 \times R_2 \times F, M_2 = R_1 \times m_2 \times F, M_3 = R_1 \times R_2 \times (0)\}$ . Observe that  $M_1M_2 = m_1 \times m_2 \times F, M_1M_3 = m_1 \times R_2 \times (0), M_2M_3 = R_1 \times m_2 \times (0)$ . Let  $I = m_1 \times (0) \times F \subsetneq M_1M_2 \setminus M_3$ . Then  $I - M_1 - M_1M_2 - I$  forms a cycle of length three. So,  $girth((\text{INC}(R))^c) = 3$ . Proof is similar if  $R \cong F \times R_1 \times R_2$  or  $R_1 \times F \times R_2$ ; where  $(R_i, m_i)$  is a local ring which is not a field for all  $i \in \{1, 2\}$  and  $F$  is a field.

**Case(iii)  $R \cong R_1 \times F_1 \times F_2$ ; where  $(R_1, m_1)$  is a local ring which is not a field and  $F_1, F_2$  are fields.**

Here,  $\text{Max}(R) = \{M_1 = m_1 \times F_1 \times F_2, M_2 = R_1 \times (0) \times F_2, M_3 = R_1 \times F_1 \times (0)\}$ . Also,  $M_1M_2 = m_1 \times (0) \times F_2, M_2M_3 = R_1 \times (0) \times (0), M_1M_3 = m_1 \times F_1 \times (0)$ . Let  $I = (0) \times (0) \times F_2 \subsetneq M_1M_2 \setminus M_3$ . So,  $I - M_1M_2 - M_1 - I$  forms a cycle of length three. So,  $girth((\text{INC}(R))^c) = 3$ . Proof runs similar for  $R \cong F_1 \times R_1 \times F_2$  or  $F_1 \times F_2 \times R_1$ ; where  $F_i$  is a field for all  $i \in \{1, 2\}$  and  $(R_1, m_1)$  is a local ring which is not a field. Hence,  $R \cong F_1 \times F_2 \times F_3$ ; where  $F_1, F_2$  and  $F_3$  are fields.  $\square$

## 7. PLANARITY OF $(\text{INC}(R))^c$

**Theorem 7.1.** *Let  $R$  be a ring with  $|\text{Max}(R)| \geq 5$ . Then  $(\text{INC}(R))^c$  is not planar.*



*Proof.* Let  $M_1, M_2, M_3, M_4, M_5 \in \text{Max}(R)$ . Observe that  $M_1M_2M_3, M_1M_2M_3M_4, M_1M_2M_3M_5$  are distinct vertices in  $(\text{INC}(R))^c$ . Note that a subgraph of  $(\text{INC}(R))^c$  induced on  $V_1 \cup V_2$  contains  $K_{3,3}$ ; where  $V_1 = \{M_1, M_2, M_3\}$  and  $V_2 = \{M_1M_2M_3, M_1M_2M_3M_4, M_1M_2M_3M_5\}$ . Hence,  $(\text{INC}(R))^c$  is not planar.  $\square$

**Theorem 7.2.** *Let  $R$  be a ring with  $|\text{Max}(R)| = 3$ . Then  $(\text{INC}(R))^c$  is planar if and only if  $R$  is isomorphic to one of the following rings:-*

- (i)  $R_1 \times F_2 \times F_3$ ; where  $(R_1, M_1)$  is SPIR with  $M_1 \neq (0)$  but  $M_1^2 = (0)$  and  $F_2, F_3$  are fields.
- (ii)  $F_1 \times F_2 \times F_3$ ; where  $F_1, F_2, F_3$  are fields.

*Proof.* Let  $\text{Max}(R) = \{M_1, M_2, M_3\}$ . Suppose that  $(\text{INC}(R))^c$  is planar. Let if possible,  $M_i^2 \neq M_i$  and  $M_j^2 \neq M_j$ ; for some distinct  $i, j \in \{1, 2, 3\}$ . Then a subgraph of  $(\text{INC}(R))^c$  induced on  $V_1 \cup V_2$  contains  $K_{3,3}$ ; where  $V_1 = \{M_i, M_j, M_iM_j\}$  and  $V_2 = \{M_i^2M_j, M_iM_j^2, M_i^2M_j^2\}$ . So,  $(\text{INC}(R))^c$  is not planar which is not possible. So, we have following two possibilities:-  
 (i)  $M_i^2 \neq M_i$  and  $M_j^2 = M_j$ ; for one  $i \in \{1, 2, 3\}$  and for all  $j \in \{1, 2, 3\} \setminus \{i\}$ . (ii)  $M_i^2 = M_i$ ; for each  $i \in \{1, 2, 3\}$ . Suppose that  $M_i^2 \neq M_i$  and  $M_j^2 = M_j$ ; for one  $i \in \{1, 2, 3\}$  and for all  $j \in \{1, 2, 3\} \setminus \{i\}$ . Without loss of generality, we may assume that  $M_1^2 \neq M_1, M_2^2 = M_2$  and  $M_3^2 = M_3$ . Suppose that  $M_1^3 = M_1^2$ . Let  $x_1 \in M_1 \setminus (M_2 \cup M_3 \cup M_1^2)$ . If  $Rx_1 \neq M_1$  then  $(\text{INC}(R))^c$  contains a subgraph homeomorphic to  $K_{3,3}$  as shown in following Figure 1. This is not possible. So,  $M_1 = Rx_1$ .

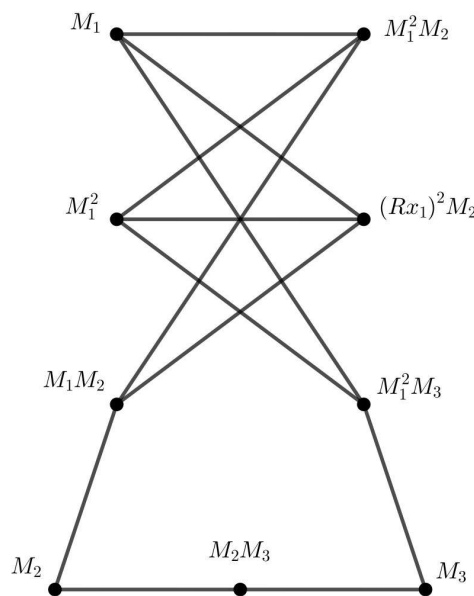


FIGURE 1.  $K_{3,3}$

Let  $x_2 \in M_2 \setminus (M_1 \cup M_3)$ . If  $M_2 \neq Rx_2$  then a subgraph of  $(\text{INC}(R))^c$  induced on  $V_1 \cup V_2$  contains  $K_{3,3}$ ; where  $V_1 = \{M_1, M_2, M_1M_2\}$  and  $V_2 = \{M_1^2M_2, M_1Rx_2, M_1^2(Rx_2)\}$ . So,  $M_2 = Rx_2$ . By a similar argument,  $M_3 = Rx_3$ ; for some  $x_3 \in M_3 \setminus (M_1 \cup M_2)$ . So,  $J(R) = M_1M_2M_3 = Rx_1x_2x_3$  is principal. Now,  $M_1^3M_2^2M_3^2 = M_1^2M_2M_3$ . By Nakayama's lemma [6, Proposition 2.6],  $M_1^2M_2M_3 = (0)$ . Thus by Chinese Remainder Theorem [6, Proposition 1.10(ii), (iii)],  $R \cong \frac{R}{M_1^2} \times \frac{R}{M_2} \times \frac{R}{M_3} \cong R_1 \times F_2 \times F_3$ ; where  $(R_1, M_1)$  is a local ring and  $F_2, F_3$  are fields. Note that  $M_1^2 = (0)$ . Let  $P$  be any prime ideal of  $R_1$ . Then  $P \subseteq M_1$ . Now,  $M_1^2 = (0) \subseteq P$ . So,  $P = M_1$ . Thus  $(R_1, M_1)$  is SPIR with  $M_1^2 = (0)$ . Hence,  $R \cong R_1 \times F_2 \times F_3$ ; where  $(R_1, M_1)$  is SPIR with  $M_1^2 = (0)$  and  $F_2, F_3$  are fields. Suppose that  $M_1^3 \neq M_1^2$ . Let  $x_1 \in M_1 \setminus (M_2 \cup M_3 \cup M_1^2)$ . If  $Rx_1 \neq M_1$  then a subgraph of  $(\text{INC}(R))^c$  induced on  $V_1 \cup V_2$  contains  $K_{3,3}$ ; where  $V_1 = \{M_1, M_2, M_1M_2\}$  and  $V_2 = \{M_1^2M_2, M_1^3M_2, (Rx_1)M_2\}$ . So,  $(\text{INC}(R))^c$  is non-planar. So,  $Rx_1 = M_1$ . Also, if  $M_1^4 \neq M_1^3$ , then a subgraph of  $(\text{INC}(R))^c$  induced on  $V_1 \cup V_2$  contains  $K_{3,3}$ ; where  $V_1 = \{M_1, M_2, M_1M_2\}$  and  $V_2 = \{M_1^2M_2, M_1^3M_2, M_1^4M_2\}$ . So it is non-planar which is not possible. Hence,  $M_1^4 = M_1^3$ . Let  $x_2 \in M_2 \setminus (M_3 \cup M_1)$ . If  $Rx_2 \neq M_2$  then a subgraph of  $(\text{INC}(R))^c$  induced on  $V_1 \cup V_2$  contains  $K_{3,3}$ ; where  $V_1 = \{M_1, M_2, M_1M_2\}$  and  $V_2 = \{M_1^2M_2, M_1^3M_2, M_1(Rx_2)\}$ . Hence, it is non-planar. This is not possible. So,  $Rx_2 = M_2$ . Similarly,  $Rx_3 = M_3$ ; for some  $x_3 \in M_3 \setminus (M_1 \cup M_2)$ . Thus  $M_i = Rx_i; \forall i \in \{1, 2, 3\}$ . Observe that  $J(R) = M_1M_2M_3$ . Let  $I = M_1M_2M_3$  and  $M = M_1^3M_2M_3$ . Now,  $IM = M_1^4M_2^2M_3^2 = M_1^3M_2M_3 = M$ . By Nakayama's lemma [6, Proposition 2.6],  $M = M_1^3M_2M_3 = (0)$ . Thus by Chinese Remainder Theorem [6, Proposition 1.10(ii), (iii)],  $R \cong R_1 \times F_2 \times F_3$ ; where  $(R_1, M_1)$  is a local ring and  $F_1, F_2$  are fields. If  $M_1^2 \neq (0)$ , then  $(\text{INC}(R))^c$  contains a subgraph homeomorphic to  $K_{3,3}$  as shown in the following Figure 2 and so it is non-planar which is not possible.

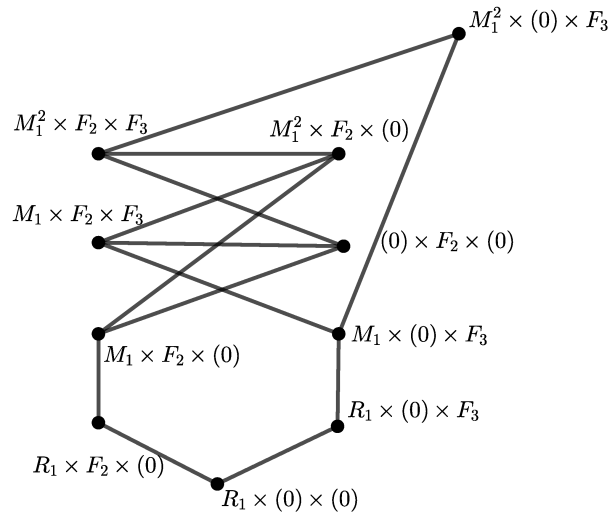


FIGURE 2.

Thus  $M_1^2 = (0)$ . Let  $P$  be any prime ideal of  $R_1$ . Then  $P \subseteq M_1$ . Now,  $M_1^2 = (0) \subseteq P$ . So,  $P = M_1$ . Thus  $(R_1, M_1)$  is SPIR with  $M_1 \neq (0)$  but  $M_1^2 = (0)$ . Suppose,  $M_i^2 = M_i$ ;  $\forall i \in \{1, 2, 3\}$ . Note that  $(J(R))^2 = J(R)$ . So by Nakayama's lemma [6, Proposition 2.6],  $J(R) = (0)$ . Thus by Chinese Remainder Theorem [6, Proposition 1.10 (ii), (iii)],  $R \cong \frac{R}{M_1} \times \frac{R}{M_2} \times \frac{R}{M_3} \cong F_1 \times F_2 \times F_3$  where  $F_1, F_2$  and  $F_3$  are fields.

Conversely, assume that  $R \cong R_1 \times F_2 \times F_3$ ; where  $(R_1, M_1)$  is SPIR with  $M_1 \neq (0)$  but  $M_1^2 = (0)$  and  $F_2, F_3$  are fields. Then clearly by Figure 3,  $(\text{INC}(R))^c$  is planar.

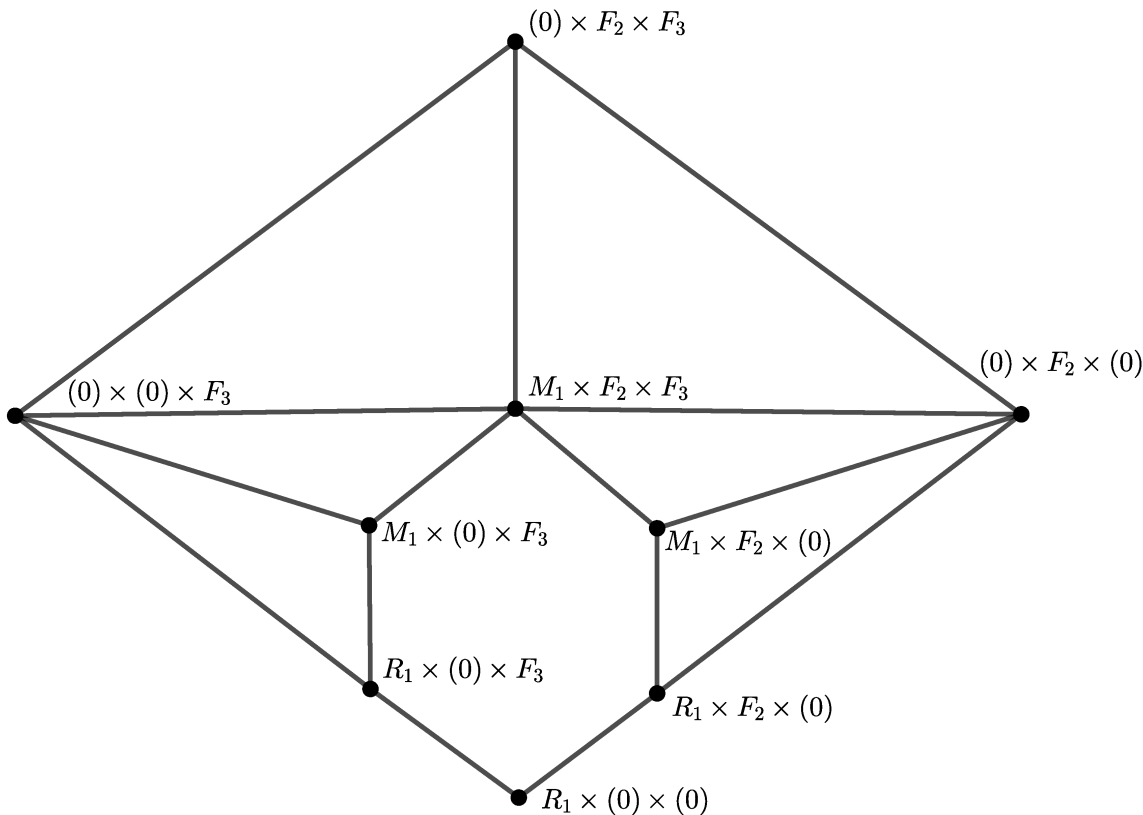


FIGURE 3.  $(\text{INC}(R_1 \times F_2 \times F_3))^c$

If  $R \cong F_1 \times F_2 \times F_3$ ; where  $F_1, F_2$  and  $F_3$  are fields. Then  $V((\text{INC}(F_1 \times F_2 \times F_3))^c) = \{M_1, M_2, M_3, M_1M_2, M_1M_3, M_2M_3\}$ . Clearly  $(\text{INC}(R))^c$  is a cycle  $M_1 - M_1M_2 - M_2 - M_2M_3 - M_3 - M_1M_3 - M_1$ . Hence,  $(\text{INC}(R))^c$  is planar.  $\square$

**Theorem 7.3.** *Let  $R$  be a ring with  $|\text{Max}(R)| = 4$ . Then  $(\text{INC}(R))^c$  is planar if and only if  $R \cong F_1 \times F_2 \times F_3 \times F_4$ ; where  $F_1, F_2, F_3$  and  $F_4$  are fields.*

*Proof.* Let  $\text{Max}(R) = \{M_1, M_2, M_3, M_4\}$ . Suppose  $M_i^2 \neq M_i$ ; for some  $i \in \{1, 2, 3, 4\}$ . Without loss of generality, we may assume that  $M_1^2 \neq M_1$ . Let  $V_1 = \{M_1, M_2, M_1M_2\}$  and

$V_2 = \{M_1M_2M_3, M_1M_2M_4, M_1^2M_2M_3\}$ . Then the subgraph of  $(\text{INC}(R))^c$  induced by  $V_1 \cup V_2$  contains  $K_{3,3}$ . So,  $(\text{INC}(R))^c$  is not planar which is a contradiction. Thus  $M_i^2 = M_i$ ; for all  $i \in \{1, 2, 3, 4\}$ . Let  $x_1 \in M_1 \setminus (M_2 \cup M_3 \cup M_4)$ . Suppose  $M_1 \neq Rx_1$ . Let  $V_1 = \{M_1, M_2, M_1M_2\}$  and  $V_2 = \{(Rx_1)M_2M_3, (Rx_1)M_2M_4, (Rx_1)M_2\}$ . Then the subgraph of  $(\text{INC}(R))^c$  induced on  $V_1 \cup V_2$  contains  $K_{3,3}$ . So,  $M_1 = Rx_1$ . Similarly, we can say that  $M_i = Rx_i$ ; for some  $x_i \in (M_i \setminus \bigcup_{j \neq i}^{j=1}^4 M_j)$ ,  $i \in \{1, 2, 3, 4\}$ . Thus  $J(R) = M_1M_2M_3M_4 = Rx_1x_2x_3x_4$ . Note that  $J(R)$  is principal and  $(J(R))^2 = J(R)$ . So, by Nakayama's lemma [7, Proposition 2.6],  $J(R) = (0)$ . Thus by Chinese Remainder Theorem [6, Proposition 1.10 (ii), (iii)],  $R \cong \frac{R}{J(R)} \cong \frac{R}{M_1} \times \frac{R}{M_2} \times \frac{R}{M_3} \cong F_1 \times F_2 \times F_3 \times F_4$ ; where  $F_i$  is a field for all  $i \in \{1, 2, 3, 4\}$ .

Conversely, assume that  $R \cong F_1 \times F_2 \times F_3 \times F_4$ ; where  $F_i$  is a field for all  $i \in \{1, 2, 3, 4\}$ . Then clearly from the following Figure 4,  $(\text{INC}(R))^c$  is planar.

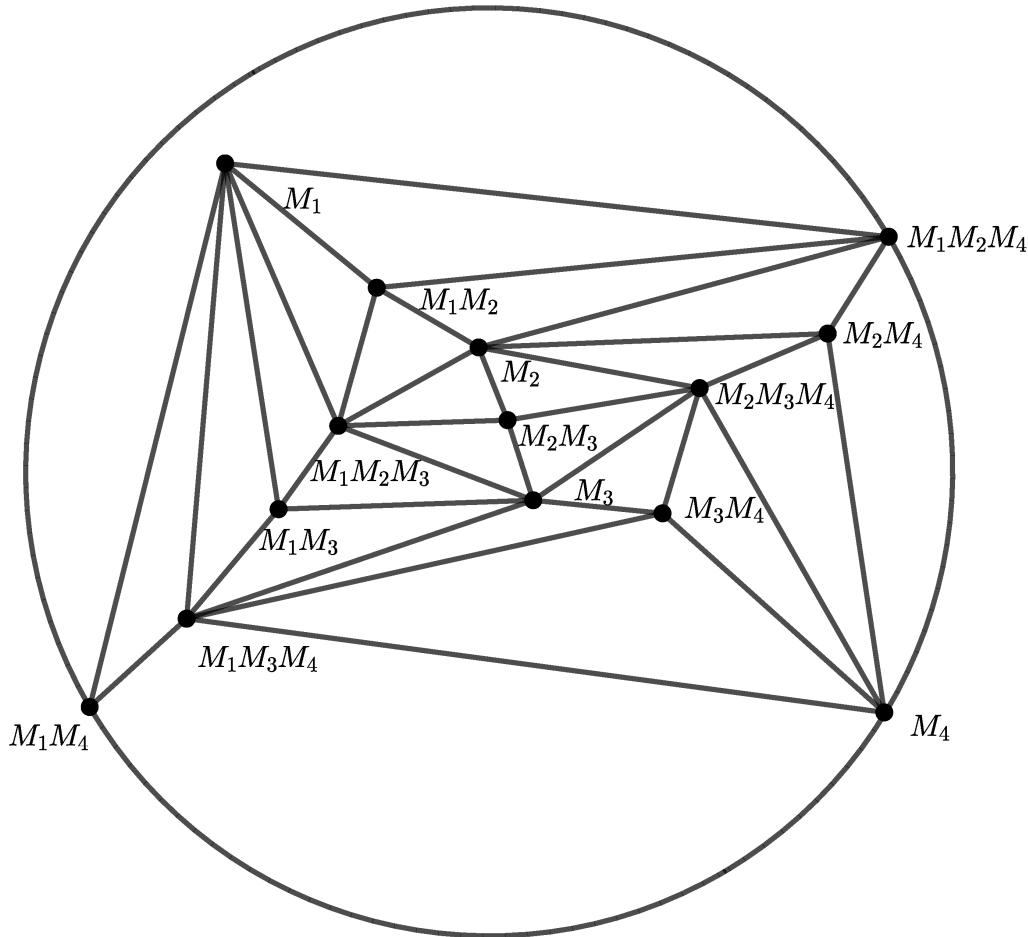


FIGURE 4.  $(\text{INC}(F_1 \times F_2 \times F_3 \times F_4))^c$

□

8. COMPLEMENTEDNESS OF  $(\text{INC}(R))^c$

**Theorem 8.1.** *Let  $R$  be a ring with  $|\text{Max}(R)| = 4$ . Then  $(\text{INC}(R))^c$  is not complemented.*

*Proof.* Let  $\text{Max}(R) = \{M_1, M_2, M_3, M_4\}$ . Suppose that  $(\text{INC}(R))^c$  is complemented. So, every vertex in  $(\text{INC}(R))^c$  has a complement in  $(\text{INC}(R))^c$ . Let  $I = M_1M_2$ . Then there exists  $J \in V((\text{INC}(R))^c)$  such that  $I \perp J$ . So,  $I$  and  $J$  are adjacent in  $(\text{INC}(R))^c$ . If  $J \neq M_i$ ; for any  $i \in \{1, 2\}$  then  $I = M_1M_2 - J - M_1 - M_1M_2$  forms a triangle in  $(\text{INC}(R))^c$  which is not possible. So,  $J = M_i$ ; for some  $i \in \{1, 2\}$ . Without loss of generality, we may assume that  $J = M_1$ . Note that  $M_1M_2M_3 \neq M_1M_2$  and  $M_1M_2M_3 \neq M_1$ . Also,  $M_1M_2M_3 \not\subseteq J(R)$ . So,  $I = M_1M_2 - J = M_1 - M_1M_2M_3 - I$  forms a triangle in  $(\text{INC}(R))^c$  which is not possible. Hence,  $(\text{INC}(R))^c$  is not complemented.  $\square$

**Theorem 8.2.** *Let  $R$  be a ring with  $|\text{Max}(R)| = 3$ . Then  $(\text{INC}(R))^c$  is complemented if and only if  $R \cong F_1 \times F_2 \times F_3$ ; where  $F_1, F_2$  and  $F_3$  are fields.*

*Proof.* Let  $\text{Max}(R) = \{M_1, M_2, M_3\}$ . Suppose that  $(\text{INC}(R))^c$  is complemented. Let  $I \in (\text{INC}(R))^c$ . Since  $(\text{INC}(R))^c$  is complemented, there exists  $J \in V((\text{INC}(R))^c)$  such that  $I \perp J$ . So,  $I$  and  $J$  are adjacent in  $(\text{INC}(R))^c$  and there is no  $K \in V((\text{INC}(R))^c)$  which is adjacent to both,  $I$  and  $J$ . As  $I$  and  $J$  are adjacent in  $(\text{INC}(R))^c$ ,  $I \subseteq J$  or  $J \subseteq I$ . Without loss of generality, we may assume that  $I \subseteq J$ . Let  $M_1 \in \text{Max}(R)$  be such that  $I \subseteq J \subseteq M_1$ . If  $J \neq M_1$  then  $I - J - M_1 - I$  forms a triangle in  $(\text{INC}(R))^c$  which is not possible. So,  $J = M_1$ . Suppose that  $M_i^2 \neq M_i$ ; for some  $i \in \{1, 2, 3\}$ . Let  $I = M_iM_j$ ;  $j \in \{1, 2, 3\}$  and  $j \neq i$ . Now,  $J = M_i$  or  $J = M_j$ . Without loss of generality, we may assume that  $J = M_i$ . Then  $I = M_iM_j - J = M_i - M_i^2M_j$  forms a triangle in  $(\text{INC}(R))^c$  which is not possible. So,  $M_i^2 = M_i$ ; for each  $i \in \{1, 2, 3\}$ . Let  $x_i \in M_i \setminus (0)$ ; for  $i \in \{1, 2, 3\}$ . If  $Rx_i \neq M_i$  then  $I = M_iM_j - J = M_i - (Rx_i)M_j - I$  forms a triangle in  $(\text{INC}(R))^c$  which is not possible. So,  $Rx_i = M_i$ ; for some  $x_i \in M_i \setminus (0)$ , where  $i \in \{1, 2, 3\}$ . So,  $M_i$  is principal; for each  $i \in \{1, 2, 3\}$ . Thus  $J(R) = M_1M_2M_3 = Rx_1x_2x_3$  is also principal. Moreover,  $(J(R))^2 = J(R)$ . So, by Nakayama's lemma [6, Proposition 2.6]  $J(R) = (0)$ . Hence, by Chinese Remainder Theorem [6, Proposition 1.10(ii),(iii)],  $R \cong \frac{R}{J(R)} \cong \frac{R}{M_1} \times \frac{R}{M_2} \times \frac{R}{M_3} \cong F_1 \times F_2 \times F_3$ ; where  $F_1, F_2$  and  $F_3$  are fields.

Conversely, assume that  $R \cong F_1 \times F_2 \times F_3$ ; where  $F_1, F_2$  and  $F_3$  are fields. From Figure 4, it is clear that  $(\text{INC}(R))^c$  is complemented.  $\square$

**Theorem 8.3.** *Let  $R$  be a ring with  $|\text{Max}(R)| = 2$ . Then  $(\text{INC}(R))^c$  is complemented if and only if  $R \cong R_1 \times R_2$ ; where  $(R_i, M_i)$  is SPIR with  $M_i \neq (0)$  but  $M_i^2 = (0)$  for each  $i \in \{1, 2\}$ .*

*Proof.* Let  $Max(R) = \{M_1, M_2\}$ . Suppose that  $(INC(R))^c$  is complemented. Suppose that  $M_i^3 \neq M_i^2$ ; for some  $i \in \{1, 2\}$ . Without loss of generality, let  $M_1^3 \neq M_1^2$ . Let  $I = M_1^3$ . Let  $J$  be a complement of  $I$  in  $(INC(R))^c$ . If  $J \neq M_1$  then  $I = M_1^3 - J - M_1 - I$  forms a triangle in  $(INC(R))^c$  which is not possible. So,  $J = M_1$ . Now,  $I = M_1^3 - J = M_1 - M_1^2 - I$  also forms a triangle in  $(INC(R))^c$  which is not possible. Thus  $M_1^3 = M_1^2$ . Hence,  $M_i^3 = M_i^2$ ; for each  $i \in \{1, 2\}$ .

**Case(i)**  $M_i^2 \neq M_i$ ; for each  $i \in \{1, 2\}$ .

Let  $I = M_i^2$  and let  $J$  be complement of  $I$ . If  $J \neq M_i$  then  $I = M_i^2 - J - M_i - I$  forms a triangle in  $(INC(R))^c$  which is not possible. So,  $J = M_i$ . Let  $x_i \in M_i \setminus M_j$ ; where  $i, j \in \{1, 2\}$  and  $i \neq j$ . Suppose that  $M_i \neq Rx_i$ . If  $Rx_i^2 \neq M_i^2$  then  $I = M_i^2 - J = M_i - Rx_i^2 - I$  forms a triangle in  $(INC(R))^c$  which is not possible. So,  $M_i^2 = Rx_i^2$ . Let  $y_i \in M_i \setminus (M_i^2 \cup M_j)$ ; for distinct  $i, j \in \{1, 2\}$ . Suppose that  $Ry_i \neq M_i$ ; for  $i \in \{1, 2\}$ . Note that  $Ry_i \not\subseteq M_i^2$ . Then either  $M_i^2 \subsetneq Ry_i$  or  $M_i^2 \not\subseteq Ry_i$ . Suppose that  $M_i^2 \subsetneq Ry_i$ . Then  $M_i^2 = I - J = M_i - Ry_i - I$  forms a triangle in  $(INC(R))^c$  which is not possible. Suppose that  $M_i^2 \not\subseteq Ry_i$ . Then  $I = M_i^2 - M_i^2 Ry_i - J = M_i - I$  forms a triangle in  $(INC(R))^c$  which is not possible. Thus  $M_i = Ry_i$ ; for each  $i \in \{1, 2\}$ . Thus,  $M_i = Rx_i$ ; for each  $i \in \{1, 2\}$ . Note that  $J(R) = M_1 M_2 = Rx_1 x_2$  is principal and  $(J(R))^3 = (J(R))^2$ . So by Nakayama's lemma [6, Proposition 2.6],  $(J(R))^2 = (0)$ . Hence, by Chinese Remainder Theorem [6, Proposition 1.10(ii),(iii)],  $R \cong \frac{R}{J(R)} \cong \frac{R}{M_1^2} \times \frac{R}{M_2^2} \cong R_1 \times R_2$ ; where  $R_1$  and  $R_2$  are local rings which are not fields. Observe that  $M_i^2 = (0)$ ; for each  $i \in \{1, 2\}$ . Let  $P_i$  be any prime ideal of  $R_i$ . Then  $M_i^2 = (0) \subseteq P_i$ . So  $P_i = M_i$ . Thus,  $(R_i, M_i)$  is a SPIR with  $M_i \neq (0)$  but  $M_i^2 = (0)$ ; for each  $i \in \{1, 2\}$ .

**Case(ii)**  $M_1^2 = M_1$  and  $M_2^2 \neq M_2$ .

As  $M_2^2 \neq M_2$ , by previous Case(i) there exists  $x \in M_2 \setminus M_1$  such that  $M_2 = Rx$ . Let  $I = M_1$  and  $J$  be a complement of  $I$  in  $(INC(R))^c$ . Let  $x_1 \in M_1 \setminus (J \cup M_2)$ . Suppose that  $M_1 \neq Rx_1$ . Note that  $JRx_1 \not\subseteq J(R)$ . If  $JRx_1 \neq J$  and  $JRx_1 \neq Rx_1$  then  $I - J - JRx_1 - I$  forms a triangle in  $(INC(R))^c$  which is not possible. So,  $JRx_1 = J$  or  $JRx_1 = Rx_1$ . Suppose that  $JRx_1 = Rx_1$ . Then  $Rx_1 \subseteq J$  which is not possible. So,  $JRx_1 = J$ . Now,  $I = M_1 - J - Rx_1 - I$  forms a triangle in  $(INC(R))^c$  which is not possible. So,  $M_1 = Rx_1$ . Note that  $J(R) = M_1 M_2 = Rx_1 x$  is principal and  $(J(R))^3 = (J(R))^2$ . By Nakayama's lemma [6, Proposition 2.6],  $(J(R))^2 = (0)$ . Thus by Chinese Remainder Theorem [6, Proposition 1.10(ii),(iii)],  $R \cong \frac{R}{M_1^2 = M_1} \times \frac{R}{M_2^2} \cong F_1 \times R_2$ ; where  $F_1$  is a field and  $R_2$  is a local ring which is not a field. Note that  $M_2^2 = (0)$  as  $J(R)^2 = (0)$ . Let  $P_2$  be any prime ideal of  $R_2$ . Then  $M_2^2 = (0) \subseteq P_2$ . So,  $P_2 = M_2$ . Thus  $(R_2, M_2)$  is SPIR with  $M_2 \neq (0)$  but  $M_2^2 = (0)$ .

**Case(iii)**  $M_i^2 = M_i$ ; for each  $i \in \{1, 2\}$ .

By Case(ii), there exists  $x_i \in M_i \setminus M_j$  such that  $M_i = Rx_i$ ; for each  $i \in \{1, 2\}$ . Note that  $(J(R))^2 = J(R)$ . So by Nakayama's lemma [6, Proposition 2.6],  $J(R) = (0)$ . Hence, by

Chinese Remainder Theorem [6, Proposition 1.10(ii),(iii)],  $R \cong \frac{R}{J(R)} \cong \frac{R}{M_1} \times \frac{R}{M_2} \cong F_1 \times F_2$ ; where  $F_1$  and  $F_2$  are fields. So, if  $(\text{INC}(R))^c$  is complemented then  $R$  is isomorphic to one of the following rings:- (i)  $R_1 \times R_2$ ; where  $(R_i, M_i)$  is SPIR with  $M_i \neq (0)$  but  $M_i^2 = (0)$  for each  $i \in \{1, 2\}$ . (ii)  $F_1 \times R_2$ ; where  $F_1$  is a field and  $(R_2, M_2)$  is SPIR with  $M_2 \neq (0)$  but  $M_2^2 = (0)$ . (iii)  $F_1 \times F_2$ ; where  $F_1$  and  $F_2$  are fields. Now, suppose that  $R \cong R_1 \times F_2$ ; where  $(R_1, M_1)$  is SPIR with  $M_1 \neq (0)$  but  $M_1^2 = (0)$  and  $F_2$  is a field. Note that  $V((\text{INC}(R))^c) = \{M_1 \times F_2, (0) \times F_2, R_1 \times (0)\}$ . Observe that  $R_1 \times (0)$  is an isolated vertex in  $(\text{INC}(R))^c$ . So,  $(\text{INC}(R))^c$  is not complemented. Suppose that  $R \cong F_1 \times F_2$ ; where  $F_1$  and  $F_2$  are fields. Note that  $V((\text{INC}(R))^c) = \{F_1 \times (0), (0) \times F_2\}$  and both these vertices are isolated in  $(\text{INC}(R))^c$ . So,  $(\text{INC}(R))^c$  is not complemented. Thus,  $R_1 \times R_2$ ; where  $(R_i, M_i)$  is SPIR with  $M_i \neq (0)$  but  $M_i^2 = (0)$  for each  $i \in \{1, 2\}$ .

Conversely, assume that  $R \cong R_1 \times R_2$ ; where  $(R_i, M_i)$  is SPIR with  $M_i \neq (0)$  but  $M_i^2 = (0)$  for each  $i \in \{1, 2\}$ . Here,  $V((\text{INC}(R))^c) = \{M_1 \times R_2, R_1 \times M_2, R_1 \times (0), (0) \times R_2\}$ . Here,  $M_1 \times R_2$  and  $(0) \times R_2$  are complement of each other. Also,  $R_1 \times (0)$  and  $R_1 \times M_2$  are complements of each other. Thus,  $(\text{INC}(R))^c$  is complemented.

□

**Corollary 8.4.** *Let  $R$  be a ring. Then  $(\text{INC}(R))^c$  is complemented if and only if  $R$  is isomorphic to one of the following rings:-*

(i)  $R \cong F_1 \times F_2 \times F_3$ ; where  $F_1, F_2$  and  $F_3$  are fields.

(ii)  $R \cong R_1 \times R_2$ ; where  $(R_i, M_i)$  is SPIR with  $M_i \neq (0)$  but  $M_i^2 = (0)$  for each  $i \in \{1, 2\}$ .

*Proof.* Proof follows from Theorems 8.1, 8.2 and 8.3. □

**Corollary 8.5.** *Let  $R$  be a ring. Then  $(\text{INC}(R))^c$  is not uniquely complemented.*

*Proof.* Note that if  $(\text{INC}(R))^c$  is uniquely complemented then it is complemented. So, by Corollary 7.4,  $R \cong F_1 \times F_2 \times F_3$ ; where  $F_1, F_2$  and  $F_3$  are fields or  $R \cong R_1 \times R_2$ ; where  $(R_i, M_i)$  is SPIR with  $M_i \neq (0)$  but  $M_i^2 = (0)$  for each  $i \in \{1, 2\}$ . Suppose that  $R \cong F_1 \times F_2 \times F_3$ ; where  $F_1, F_2$  and  $F_3$  are fields. Note that  $F_1 \times (0) \times F_3$  and  $F_1 \times F_2 \times (0)$  are complements of  $F_1 \times (0) \times (0)$  in  $(\text{INC}(R))^c$ . Observe that  $(0) \times (0) \times F_3 \in N(F_1 \times (0) \times F_3)$  but  $(0) \times (0) \times F_3 \notin N(F_1 \times F_2 \times (0))$ . So,  $N(F_1 \times (0) \times F_3) \neq N(F_1 \times F_2 \times (0))$ . Thus  $(\text{INC}(R))^c$  is not uniquely complemented. □

## 9. OPEN PROBLEMS

Let  $R$  be a ring with  $|Max(R)| = 2$ . Then one can attempt the problems to classify the rings  $R$  for which

- (i)  $(INC(R))^c$  is split.
- (ii)  $(INC(R))^c$  is planar.

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