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Research Paper

ON $\sim_n$ NOTION OF CONJUGACY IN SEMIGROUPS

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Abstract. In this paper, we study the $\sim_n$ notion of conjugacy in semigroups. After proving some basic results, we characterize this notion in subsemigroups of $\mathcal{P}(T)$ (partial transformation semigroup) and $\mathcal{T}(T)$ (transformation semigroup) through digraphs and their restrictive partial homomorphisms.

1. Introduction and Preliminaries

The concept of conjugacy is essential as far as group theory is concerned. More importantly, most of the famous results on finite groups involve the use of conjugacy in their proofs. Semigroups are a generalization of groups, and the theory of semigroups has evolved as a result of generalizing results of groups to semigroups. Like other notions of groups, it becomes natural to try to generalize the notion of conjugacy from groups to semigroups. Since the definition of conjugacy in a group involves the existence of inverses, the obvious choice for elements

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$a, b \in S$, where S is a semigroup, to be conjugate of each other is the existence of an element $g \in S^1$ (semigroup obtained by adjoining identity 1) such that $ag = gb$. However, unlike groups, this relation is not necessarily transitive in an arbitrary semigroup. This prompted semigroup theorists to search for the notions of conjugacy that are best suitable, and as a result various notions of conjugacy have been studied so far. In this paper, we consider $\sim_n$ notion of conjugacy in partial transformation semigroups and in full transformation semigroups.

Let G be a group and $s, t \in G$, we say s is conjugate to t if $t = x^{-1}sx$ for some $x \in G$ alternatively $sx = xt$ for some $x \in G$. As a result of this, we can extend the notion of conjugacy to a semigroup S. A notion $\sim_l$ of conjugacy on a semigroup S is defined as

$$s \sim_l t \Leftrightarrow \exists\ x \in S^1 \text{ such that } sx = xt$$

where S^1 is $S \cup \{1\}$. The relation $\sim_l$ is an equivalence relation on a free semigroup. However the relation $\sim_l$ is not symmetric in general and it is a universal relation in a semigroup with zero. Lallement in [8] considered $\sim_l$ on a free semigroup S and named it a $\sim_p$ notion of conjugacy. It is defined as

$$s \sim_p t \Leftrightarrow \exists\ p, q \in S^1 \text{ such that } s = pq \text{ and } t = qp$$

The relation $\sim_p$ is an equivalence relation on a free semigroup. However the relation $\sim_p$ is not transitive in general. In order to overcome this problem, Otto in [9] introduced the $\sim_o$ notion of conjugacy in a semigroup S as

$$s \sim_o t \Leftrightarrow \exists\ x, y \in S^1 \text{ such that } sx = xt \text{ and } ty = ys$$

However, the relation $\sim_o$ is also a universal relation $(S \times S)$ on a semigroup with zero. This problem gets resolved with the introduction of $\sim_c$ notion of conjugacy in a semigroup S in [2] by Araujo et al. which is as follows

$$s \sim_c t \Leftrightarrow \exists\ x \in \mathbb{P}^1(s), y \in \mathbb{P}^1(t) \text{ such that } sx = xt \text{ and } ty = ys$$

where for $s \neq 0$, $\mathbb{P}(s) = \{x \in S^1 : (ms)x \neq 0 \text{ for all } ms \in S^1 s \setminus \{0\}\}$ and $\mathbb{P}(0) = \{0\}$. This relation is an equivalence relation on any semigroup S and does not reduce to a universal relation even if S is a semigroup with zero.

Furthermore, J. Konieczny in [5] introduced the $\sim_n$ notion of conjugacy in semigroups. For any s, t in a semigroup S

$$s \sim_n t \Leftrightarrow \exists\ x, y \in S^1 \text{ such that } sx = xt, ty = ys, s = xty \text{ and } t = ysx$$

This relation is always an equivalence relation in any semigroup and does not reduce to a universal relation in a semigroup with zero.

Araujo et al. in [6] characterized $\sim_c$ conjugacy in constant rich subsemigroups of $\mathcal{P}(T)$ (semigroup of all partial maps on a non-empty set T) and $\mathcal{T}(T)$ (semigroup of all maps on a non-empty set T) with the help of rp-homomorphisms of their digraphs. In this paper we prove similar results for $\sim_n$ notion of conjugacy for any subsemigroup of $\mathcal{P}(T)$ without the assumption of constant rich on S. We show that $\sim_n$ is an identity relation on a band S if and only if S is commutative. We also prove some results on comparison of $\sim_n$ notion with notions $\sim_l, \sim_p, \sim_c$ and $\sim_o$.

2. Some results on $\sim_n$ notion of conjugacy in a general semigroup

Definition 2.1. A *band* is a semigroup S whose all elements are idempotents, i.e., $a^2 = a$ for all $a \in S$.

Definition 2.2. A semigroup S satisfying $st = ts$ for any $s, t \in S$ is said to be *commutative.*

For a set T, we denote by Δ_T the identity relation on T.

In the following theorem, we show that $\sim_n$ is an identity relation on a band S if and only if S is commutative.

Theorem 2.3. *The necessary and sufficient condition for $\sim_n = \Delta_S$ on a band S is that S is commutative.*

Proof. Suppose $\sim_n = \Delta_S$. Let $s, t \in S$ then we can write

$$(st)s = s(ts),\ (ts)t = t(st),\ st = s(ts)t \text{ and } ts = t(st)s$$

which implies $st \sim_n ts$ which further implies $st = ts$. So S is commutative.

Conversely, let S is commutative and let $s \sim_n t$ then there exist $x, y \in S^1$ such that

$$sx = xt, ty = ys, s = xty \text{ and } t = ysx.$$

Further $s = xty = xys = ysx = t$. Thus $\sim_n = \Delta_S$. □

Definition 2.4. A semigroup S is said to be a *left zero semigroup* if for all $a, b \in S$, $ab = a$.

Definition 2.5. A semigroup S is said to be a *right zero semigroup* if for all $a, b \in S$, $ab = b$.

Definition 2.6. Let S be a semigroup and $a, b \in S$ then $a\mathcal{J}b$ if and only if there exist $x, y, u, v \in S^1$ such that

$$a = xby \text{ and } b = ubv.$$

In the next theorem we show the relations $\sim_n$, $\sim_p, \sim_o, \sim_l$ and $\mathcal{J}$ coincides on right[left] zero semigroup.

Theorem 2.7. *Let S be any right[left] zero semigroup. Then $\sim_n = \sim_p = \sim_o = \sim_l = \mathcal{J}$ on S.*

Proof. We prove the result by showing that all these notions reduce to a universal relation on S. We take the case of right zero semigroups. The proof in the other case follows dually.

(1) $\sim_n$ reduces to a universal relation: For any $a, b \in S$, there exist $b, a \in S^1$ such that

$$ab = bb, ba = aa, a = bba \text{ and } b = aab.$$

(2) $\sim_p$ reduces to a universal relation: For any $a, b \in S$, there exist $b, a \in S^1$ such that

$$a = ba \text{ and } b = ab.$$

(3) $\sim_o$ reduces to a universal relation: For any $a, b \in S$, there exist $b, a \in S^1$ such that

$$ab = bb, ba = aa.$$

(4) $\sim_l$ reduces to a universal relation: For any $a, b \in S$, there exist $b \in S^1$ such that

$$ab = bb.$$

(5) $\mathcal{J}$ reduces to a universal relation: For any $a, b \in S$, there exist $b, a \in S^1$ such that

$$a = bba \text{ and } b = aab.$$

□

The next proposition follows by [5, Proposition 2.3] and definition of $\sim_n$.

Proposition 2.8. *Let S be a semigroup, then $\sim_n \subseteq \sim_c \subseteq \sim_o$ and $[0]_n = \{0\}$.*

3. $\sim_n$ notion of conjugacy through digraphs in $\mathcal{P}(T)$

Definition 3.1. Let T be any set and let R be a binary relation on T. Then $\Gamma = (T, R)$ is called a *directed graph* (or a *digraph*). We call any $p \in T$ a vertex and any $(p, q) \in R$, an arc of Γ.

For example, Let $T = \{1, 2, 3, 4\}$ and $R = \{(1, 2), (2, 3)\}$, then the digraph Γ is as under,

$$\overset{1}{\bullet} \longrightarrow \overset{2}{\bullet} \longrightarrow \overset{3}{\bullet}\ \overset{4}{\bullet}.$$

Definition 3.2. A vertex $p \in T$ for which there exists no q in T such that $(p, q) \in R$ is called a *terminal vertex* of Γ. A vertex $p \in T$ is said to be *initial vertex* if there is no $q \in T$ for which $(q, p) \in R$ while a vertex $p \in T$ is said to be *non-initial vertex* if $(q, p) \in R$ for some $q \in T$.

For any $\sigma \in \mathcal{P}(T)$, $\Gamma(\sigma) = (T, R_\sigma)$ represents a digraph, where for all $p, q \in T, (p, q) \in R_\sigma$ if and only if $p \in \mathrm{dom}(\sigma)$ and $p\sigma = q$. For example, If $T = \{1, 2, 3\}$ and $R_\sigma = \{(1, 2), (2, 1)\}$, then the digraph $\Gamma(\sigma)$ is represented as

$$\overset{1}{\bullet} \longrightarrow \overset{2}{\bullet} \longrightarrow \overset{1}{\bullet}\,.$$

For a non-empty set T, we fix an element $\diamond \notin T$. For $\sigma \in \mathcal{P}(T)$ and $t \in T$, we will write $t\sigma = \diamond$, if and only if $t \notin \mathrm{dom}(\sigma)$. we also assume that $\diamond\sigma = \diamond$. With this notation, it makes sense to write $s\sigma = t\tau$ or $s\sigma \neq t\tau$ $(\sigma, \tau \in \mathcal{P}(T), s, t \in T)$ even when $s \notin \mathrm{dom}(\sigma)$ or $t \notin \mathrm{dom}(\tau)$. For any $\sigma \in \mathcal{P}(T)$, $\mathrm{span}(\sigma)$ represents $\mathrm{dom}(\sigma) \cup \mathrm{im}(\sigma)$. For any $\alpha \in \mathcal{P}(T)$, by $\alpha \neq 0$ we mean $\mathrm{dom}(\alpha) \neq \emptyset$. Thus $\alpha = 0$ if and only if $\mathrm{dom}(\alpha) = \emptyset$. For semigroups U and S, we write $U \leq S$ to mean that U is a subsemigroup of S.

Definition 3.3. Let $\Gamma_1 = (T_1, R_1)$ and $\Gamma_2 = (T_2, R_2)$ be digraphs. A mapping α from T_1 to T_2 is called a *homomorphism* from Γ_1 to Γ_2 if for all $p, q \in T_1, (p, q) \in R_1$ implies $(p\alpha, q\alpha) \in R_2$. A partial mapping α from Γ_1 to Γ_2 is called a *partial homomorphism* if for all $p, q \in \mathrm{dom}(\alpha)$, $(p, q) \in R_1$ implies $(p\alpha, q\alpha) \in R_2$.

Definition 3.4. A partial homomorphism α from T_1 to T_2 is said to be *restricive partial homomorphism*(or *an rp-homomorphism*) from Γ_1 to Γ_2 if the following hold:

(a) If $(p, q) \in R_1$, then $p, q \in \mathrm{dom}(\alpha)$ and $(p\alpha, q\alpha) \in R_2$.
(b) If p is a terminal vertex in Γ_1 and $p \in \mathrm{dom}(\alpha)$, then $p\alpha$ is a terminal vertex in Γ_2.

We say that Γ_1 is *rp-homomorphic* to Γ_2 if there is an rp-homomorphism from Γ_1 to Γ_2.

Throughout this paper by an *rp-hom* we shall mean an rp-homomorphism between any two digraphs and by a *hom* we shall mean a homomorphism.

The next theorem provides necessary and sufficient conditions for two elements of $\mathcal{P}(T)$ to be $\sim_n$ related.

Theorem 3.5. *Let $S \leq \mathcal{P}(T)$ and $\sigma, \tau \in S$. Then $\sigma \sim_n \tau$ if and only if there are $\alpha, \beta \in S^1$ for which α is an rp-hom from $\Gamma(\sigma)$ to $\Gamma(\tau)$ and β is an rp-hom from $\Gamma(\tau)$ to $\Gamma(\sigma)$ with $q\alpha\beta = q$ for every non-initial vertex q of $\Gamma(\sigma)$ and $k\beta\alpha = k$ for every non-initial vertex k of $\Gamma(\tau)$.*

Proof. Suppose $\sigma \sim_n \tau$ in S. If $\sigma = 0$, then $\tau = 0$ and so $\alpha = \mathrm{id}_T \in S^1$ is an rp-hom from $\Gamma(\sigma)$ to $\Gamma(\tau)$ and $\beta = \mathrm{id}_T \in S^1$ is an rp-hom from $\Gamma(\tau)$ to $\Gamma(\sigma)$ and the given conditions are trivially satisfied. Next suppose $\sigma \neq 0$ and so $\tau \neq 0$ and let $\sigma \sim_n \tau$ in S then there exist $\alpha, \beta \in S^1$ such that

$$\sigma\alpha = \alpha\tau, \tau\beta = \beta\sigma, \sigma = \alpha\tau\beta \text{ and } \tau = \beta\sigma\alpha. \tag{1.1}$$

Let $(p, q) \in \sigma$, i.e., $p\sigma = q$. Then by (1.1), $p\alpha\tau\beta = q$ which implies $(p\alpha)\tau\beta = q$, which implies $p \in \mathrm{dom}\alpha$. Again

$$q\alpha\beta = (p\sigma)\alpha\beta = p\sigma\alpha\beta \overset{1.1}{=} p\alpha\tau\beta \overset{1.1}{=} p\sigma = q \tag{1.2}$$

which implies $q \in \text{dom}\alpha$. Next $(p\alpha)\tau = p\alpha\tau = p\sigma\alpha = q\alpha$. Thus $(p\alpha, q\alpha) \in \Gamma(\tau)$. Again let p be a terminal vertex of $\Gamma(\sigma)$ and let $p \in \text{dom}\alpha$ then as $\sigma\alpha = \alpha\tau$, we have $(p\alpha)\tau = p\alpha\tau = p\sigma\alpha = \diamond\alpha = \diamond$. Thus $p\alpha$ is a terminal vertex in $\Gamma(\tau)$. So α is an rp-hom from $\Gamma(\sigma)$ to $\Gamma(\tau)$. Since q is a non-initial vertex in $\Gamma(\sigma)$, thus by (1.2) we conclude that $q\alpha\beta = q$ for any non-initial vertex $q \in \Gamma(\sigma)$. Symmetrically by using $\tau\beta = \beta\sigma$ and $\tau = \beta\sigma\alpha$ we can prove that β is an rp-hom from $\Gamma(\tau)$ to $\Gamma(\sigma)$ with $k\beta\alpha = k$ for every non-initial vertex k of $\Gamma(\tau)$.

Conversely, let α and β be rp-hom such that $q\alpha\beta = q$ for every non-initial vertex q of $\Gamma(\sigma)$ and $k\beta\alpha = k$ for every non-initial vertex k of $\Gamma(\tau)$. We show that $\sigma \sim_n \tau$.

The following cases arise.

Case 1: Suppose $p \notin \text{dom}\sigma$, then $p\sigma = \diamond$. Thus

$$p(\sigma\alpha) = (p\sigma)\alpha = \diamond\alpha = \diamond.$$

Here two subcases arise;

(i) If $p \notin \text{dom}\alpha$, then

$$p(\alpha\tau) = (p\alpha)\tau = \diamond = p\sigma\alpha.$$

Moreover $p\alpha\tau\beta = \diamond = p\sigma$.

(ii) If $p \in \text{dom}\alpha$, then as p is a terminal vertex of $\Gamma(\sigma)$ and also since α is an rp-hom from $\Gamma(\sigma)$ to $\Gamma(\tau)$ so $p\alpha$ is a terminal vertex in $\Gamma(\tau)$. Therefore we have $p(\alpha\tau) = (p\alpha)\tau = \diamond = p(\sigma\alpha)$. Also $p\alpha\tau\beta = \diamond = p\sigma$.

Case 2: Suppose $p \in \text{dom}\sigma$ and let $q = p\sigma$. Then by definition of an rp-hom $p, q \in \text{dom}\alpha$ and $p(\alpha\tau) = (p\alpha)\tau = q\alpha = (p\sigma)\alpha = p(\sigma\alpha)$. Also since q is a non-initial vertex of $\Gamma(\sigma)$ we have $p\alpha\tau\beta = p\sigma\alpha\beta = q\alpha\beta = q = p\sigma$. Thus in all the cases we have

$$\sigma\alpha = \alpha\tau \text{ and } \sigma = \alpha\tau\beta. \tag{1.3}$$

By symmetry since β is an rp-hom from $\Gamma(\tau)$ to $\Gamma(\sigma)$ with $k\beta\alpha = k$ for every non-initial vertex k of $\Gamma((\tau)$ we have

$$\tau\beta = \beta\sigma \text{ and } \ \tau = \beta\sigma\alpha. \tag{1.4}$$

From (1.3) and (1.4) we have $\sigma \sim_n \tau$.□

If $\sigma, \tau \in \mathcal{T}(T)$. Then every rp-hom from $\Gamma(\sigma)$ to $\Gamma(\tau)$ is a hom. So we have the following corollary.

Corollary 3.6. *Let $S \leq \mathcal{T}(T)$ and $\sigma, \tau \in S$. Then $\sigma \sim_n \tau$ if and only if there are $\alpha, \beta \in S^1$ such that α is a hom from $\Gamma(\sigma)$ to $\Gamma(\tau)$ and β is a hom from $\Gamma(\tau)$ to $\Gamma(\sigma)$ with $q\alpha\beta = q$ for every non-initial vertex q of $\Gamma(\sigma)$ and $k\beta\alpha = k$ for every non-initial vertex k of $\Gamma(\tau)$.*

4. $\sim_n$ NOTION OF CONJUGACY THROUGH CONNECTED PARTIAL MAPS

Let $\ldots, t_{-2}, t_{-1}, t_0, t_1, t_2, \ldots$ be pairwise distinct elements of T. The following transformations (cycle, right ray, double ray, left ray and chain) called as basic partial transformations on T are very important for our study.

Definition 4.1. An $\alpha \in \mathcal{P}(T)$ is called a *cycle* of length k ($k \geq 1$) if $\alpha = (t_0 t_1 t_2 \cdots t_{k-1})$, i.e., $t_j = t_{j-1}\alpha$, $j = 1, 2, \ldots, k$ and $t_0 = t_{k-1}\alpha$. We write it as

$$t_0 \to t_1 \to t_2 \to \cdots \to t_{k-1} \to t_0.$$

Let $T = \{1, 2, \ldots, 10\}$ and let $\alpha \in \mathcal{P}(T)$ be defined as, $\alpha = (1\ 2\ 3)$. Then α is a cycle of length 3.

Definition 4.2. An $\alpha \in \mathcal{P}(T)$ is called a *right ray* if $\alpha = [t_0\ t_1\ t_2 \cdots >$, i.e., $t_j = t_{j-1}\alpha$, $j \geq 1$. We write it as

$$t_0 \to t_1 \to t_2 \to \cdots.$$

Let $T = \mathbb{N}$ and let $\alpha \in \mathcal{P}(T)$ be defined as, $\alpha = [1\ 2\ 3\ \cdots >$. Then α is a right ray in $\mathcal{P}(T)$.

Definition 4.3. An $\alpha \in \mathcal{P}(T)$ is called a *left ray*, if $\alpha = < \cdots\ t_2\ t_1\ t_0]$, i.e., $t_j\alpha = t_{j-1}$, $j \geq 1$. We write it as

$$\cdots \to t_2 \to t_1 \to t_0.$$

Let $T = \mathbb{Z}$ and let $\alpha \in \mathcal{P}(T)$ be defined as, $\alpha = < \cdots\ 0\ 1\ 2]$. Then α is a left ray in $\mathcal{P}(T)$.

Definition 4.4. An $\alpha \in \mathcal{P}(T)$ is called a *double ray* if $\alpha = < \cdots\ t_{-1}\ t_0\ t_1 \cdots >$, i.e., $t_j = t_{j-1}\alpha$, $j \in \mathbb{Z}$. We write it as

$$\cdots \to t_{-1} \to t_0 \to t_1 \to t_2 \to \cdots\ .$$

Let $T = \mathbb{Z}$ and let $\alpha \in \mathcal{P}(T)$ be defined as, $\alpha = < \cdots -2\ \ -1\ 0\ 1\ 2 \cdots >$. Then α is a double ray in $\mathcal{P}(T)$.

Definition 4.5. An $\alpha \in \mathcal{P}(T)$ is called a *chain* of length k if $\alpha = [t_0 t_1 t_2 \cdots t_k]$, i.e., $t_j = t_{j-1}\alpha$, $j = 1, 2, \ldots, k$. We write it as

$$t_0 \to t_1 \to t_2 \to \cdots \to t_k.$$

Let $T = \{1, 2, \ldots, 10\}$ and let $\alpha \in \mathcal{P}(T)$ be defined as, $\alpha = [1\ 2\ 3]$. Then $\alpha \in \mathcal{P}(T)$ is a chain of length 2.

Definition 4.6. Any element $\kappa \neq 0$ in $\mathcal{P}(T)$ is said to be *connected* if for some non-negative integers m, n, $p\kappa^m = q\kappa^n \neq \diamond$ for all $p, q \in \mathrm{span}(\kappa)$.

For example, let $T = \{1,2,3,4,5\}$. Define $\kappa \in \mathcal{P}(T)$ by $\kappa = \{(1,2),(2,3),(3,4)\}$. Then the diagraph of κ is represented as

$$\overset{1}{\bullet} \longrightarrow \overset{2}{\bullet} \longrightarrow \overset{3}{\bullet} \longrightarrow \overset{4}{\bullet}.$$

Then κ is connected.

Definition 4.7. For $\sigma, \tau \in \mathcal{P}(T)$, if $\mathrm{dom}(\tau) \subseteq \mathrm{dom}(\sigma)$ and $p\tau = p\sigma$ for every $p \in \mathrm{dom}(\tau)$, then τ is said to be *contained* in σ written as $\tau \subseteq \sigma$. They are *disjoint* if $\mathrm{dom}(\sigma) \cap \mathrm{dom}(\tau) = \emptyset$ and *completely disjoint* if $\mathrm{span}(\sigma) \cap \mathrm{span}(\tau) = \emptyset$.

For example, $[p\ q\ r\ s \cdots >$ and $[a\ b\ c\ p]$ in $\mathcal{P}(\mathbb{Z})$ are disjoint while $[a\ b\ \cdots >$ and $[u\ v]$ are completely disjoint.

Definition 4.8. Let C be a set of pairwise disjoint elements of $\mathcal{P}(T)$. Then, for $t \in T$

$$t\Big(\bigcup_{\kappa \in C} \kappa\Big) = \begin{cases} t\kappa \text{ if } t \in \mathrm{dom}(\kappa) \text{ for some } \kappa \in C \\ \diamond \text{ otherwise.} \end{cases}$$

is called the *join* of the elements of C denoted by $\bigcup\limits_{\kappa \in C} \kappa$.

Definition 4.9. Let $\sigma \in \mathcal{P}(T)$ and let ν be a basic partial map with $\nu \subset \sigma$ then ν is *maximal* in σ if $t \notin \mathrm{dom}(\nu)$ implies $t \notin \mathrm{dom}(\sigma)$ and $t \notin \mathrm{im}(\nu)$ implies $t \notin \mathrm{im}(\sigma)$ for every $t \in \mathrm{span}(\nu)$.

For example, Let $\sigma = [p\ q\ r\ s \cdots > \cup\ [a\ b\ c\ p] \in \mathcal{P}(\mathbb{Z})$. Then σ contains infinitely many right rays. For example, $[c\ p\ q\ r \cdots >$ but only two of them, namely $[p\ q\ r\ s \cdots >$ and $[a\ b\ c\ p\ q\ r\ s \cdots >$ are maximal in σ.

Proposition 4.10. [1] *Let $\sigma \in \mathcal{P}(T)$ with $\sigma \neq 0$. Then there exists a unique set C of pairwise completely disjoint, connected maps contained in σ such that $\sigma = \bigcup\limits_{\kappa \in C} \kappa$.*

Example 4.11. Let $T = \{1,2,3,4,5\}$ and let $\sigma \in \mathcal{P}(T)$ be defined as

$$\sigma = \{(1,2),(2,3),(4,5)\}.$$

Clearly σ has a unique representation in terms of of pairwise completely disjoint, connected maps contained in σ. i.e., $\sigma = \cup_{\sigma_i \in C}\ \sigma_i$ where $C = \{\sigma_1, \sigma_2\}$ and $\sigma_1 = \{(1,2),(2,3)\}$ and $\sigma_2 = \{(4,5)\}$.

The connected maps of C in Proposition 4.10 are called *connected components* of σ. By c-components of σ, we shall always mean connected components of σ.

The next lemma gives a relationship between span of two partial maps which are n-related and the domain of their conjugators .

Lemma 4.12. *Let $S \leq \mathcal{P}(T)$ and $\sigma, \tau \in S$ with $\sigma \sim_n \tau$ then there exist $\alpha, \beta \in S^1$ such that $dom(\alpha) = span(\sigma)$ and $dom(\beta) = span(\tau)$.*

Proof. Let $\sigma \sim_n \tau$ then there exist $\alpha, \beta \in S^1$ such that $\sigma\alpha = \alpha\tau, \tau\beta = \beta\sigma, \sigma = \alpha\tau\beta$ and $\tau = \beta\sigma\alpha$. By Theorem 3.5, α is an rp-hom from $\Gamma(\sigma)$ to $\Gamma(\tau)$ and β is an rp-hom from $\Gamma(\tau)$ to $\Gamma(\sigma)$. We have to show that $\text{dom}(\alpha) = \text{span}(\sigma)$. Let $t \in \text{span}(\sigma)$, which means $t \in \text{dom}(\sigma) \cup \text{im}(\sigma)$. If $t \in \text{dom}(\sigma)$, then there exists $u \in T$ such that $(t, u) \in \sigma$. Since α is an rp-hom from $\Gamma(\sigma)$ to $\Gamma(\tau)$. Therefore $t, u \in \text{dom}(\alpha)$. So in this case $\text{span}(\sigma) \subseteq \text{dom}(\alpha)$. Similarly if $t \in \text{im}(\sigma)$, then $\text{span}(\sigma) \subseteq \text{dom}(\alpha)$. Next we have to show $\text{dom}(\alpha) \subseteq \text{span}(\sigma)$. Since $\sigma = \alpha\tau\beta$, $\text{dom}(\alpha) = \text{dom}(\sigma) \subseteq \text{span}(\sigma)$ implies $\text{dom}(\alpha) \subseteq \text{span}(\sigma)$. By similar arguments, we can show that $\text{dom}(\beta) = \text{span}(\tau)$. □

Notation 1. Let $\sigma \in \mathcal{P}(T)$ and κ be a c-component of σ. Then by σ_κ we mean the *restriction* of σ on $\text{Span}(\kappa)$.

The next proposition is the interconnection of *c*-components and $\sim_n$ notion of conjugacy.

Proposition 4.13. *Let $S \leq \mathcal{P}(T)$ and $\sigma, \tau \in S$. Then, $\sigma \sim_n \tau$ if and only if*

(1) *(i) For every c-component κ of σ there exist a c-component λ of τ and an rp-hom $\alpha_\kappa \in \mathcal{P}(T)$ from $\Gamma(\kappa)$ to $\Gamma(\lambda)$ with $dom(\alpha_\kappa) = span(\kappa)$.*
(ii) For every c-component κ' of τ there exist a c-component λ' of σ and an rp-hom $\alpha'_{\kappa'} \in \mathcal{P}(T)$ from $\Gamma(\kappa')$ to $\Gamma(\lambda')$ with $dom(\alpha'_{\kappa'}) = span(\kappa')$.

(2) *(i) $\bigcup_{\kappa \in C} \alpha_\kappa \in S^1$, where C is the collection of c-components of σ.*
(ii) $\bigcup_{\kappa' \in C'} \alpha'_{\kappa'} \in S^1$, where C' is the collection of c-components of τ.

(3) *There are $\alpha, \beta \in S^1$ such that $q\alpha\beta = q$ for any non-initial vertex q of $\Gamma(\sigma)$ and $k\beta\alpha = k$ for every non-initial vertex k of $\Gamma(\tau)$.*

Proof. If $\sigma = 0$, then $\tau = 0$, and the result follows trivially. Suppose $\sigma \neq 0$ then $\tau \neq 0$ and let $\sigma \sim_n \tau$, then there are $\alpha, \beta \in S^1$ such that $\sigma\alpha = \alpha\tau, \tau\beta = \beta\sigma, \sigma = \alpha\tau\beta$ and $\tau = \beta\sigma\alpha$ and so by Theorem 3.5, α is an rp-hom from $\Gamma(\sigma)$ to $\Gamma(\tau)$ and β is an rp-hom from $\Gamma(\tau)$ to $\Gamma(\sigma)$ with $q\alpha\beta = q$ for any non-initial vertex q of $\Gamma(\sigma)$ and $k\beta\alpha = k$ for every non-initial vertex k of $\Gamma(\tau)$ which is (3). Now we have to prove only (1) and (2).

(1) (i) Let κ be a *c*-component of σ and let $p \in \text{span}(\kappa)$, since α is an rp-hom this means $p\alpha \in \lambda$ for some *c*-component λ of τ. We claim that $(\text{span}(\kappa))\alpha \subseteq \text{span}(\lambda)$. Let $z \in \text{span}(\kappa)$ then by definition of connectedness there exist $r, s \geq 0$ such that $p\sigma^r = p\kappa^r = z\kappa^s = z\sigma^s \neq \diamond$. Since $\sigma\alpha = \alpha\tau$, we have $(z\alpha)\tau^s = (z\sigma^s)\alpha = (p\sigma^r)\alpha = (p\alpha)\tau^r \neq \diamond$ which implies $p\alpha$ and $z\alpha$ are in the span of same *c*-component of τ. So $z\alpha \in \text{span}(\lambda)$. Therefore $(\text{span}(\kappa))\alpha \subseteq \text{span}(\lambda)$. Thus we have proved the claim. Let

$\alpha_\kappa = \alpha|_{span(\kappa)}$. Then $\alpha_\kappa = \alpha|_{span(\kappa)}$ is an rp-hom from $\Gamma(\kappa)$ to $\Gamma(\lambda)$ (by the claim and the fact that α is an rp-homomorphism from $\Gamma(\sigma)$ to $\Gamma(\tau)$), $\mathrm{dom}(\alpha_\kappa) = \mathrm{span}(\kappa)$ (by definition of α_κ).

(ii) The proof follows dually by part (i).

(2) (i) $\bigcup_{\kappa\in C} \alpha_\kappa = \alpha \in S^1$ (by definition of α_κ) and Lemma 4.12.

(ii) This follows similarly as part (i).

Conversely, suppose that (1), (2) and (3) are satisfied. Let $\alpha = \bigcup_{\kappa\in C} \alpha_\kappa$. Note that α is well defined since α_{κ_1} and α_{κ_2} are disjoint if $\kappa_1 \neq \kappa_2$. Suppose $(q, z) \in \sigma$. Then $q, z \in \mathrm{span}(\kappa)$ for some c-component κ of σ. Thus $q, z \in \mathrm{dom}(\alpha_\kappa)$ and $q\alpha = q\alpha_\kappa \xrightarrow{\lambda} z\alpha_\kappa = z\alpha$, implying $q\alpha \xrightarrow{\tau} z\alpha$. Suppose q is a terminal vertex in $\Gamma(\sigma)$ and $q \in \mathrm{dom}(\alpha)$. Then there is a unique c-component κ of σ such that q is a terminal vertex in $\Gamma(\kappa)$. Then $q\alpha = q\alpha_\kappa$ is a terminal vertex in $\Gamma(\lambda)$ and so a terminal vertex in $\Gamma(\tau)$. Hence α is an rp-hom from $\Gamma(\sigma)$ to $\Gamma(\tau)$. By condition (2) $\alpha \in S^1$. By symmetry, we can similarly prove $\beta \in S^1$ is an rp-hom from $\Gamma(\tau)$ to $\Gamma(\sigma)$. Then by condition (3) and Theorem 3.5 we have $\sigma \sim_n \tau$.□

The next definition is from [2] and is useful for further results of this section.

Definition 4.14. Let T be a non-empty subset of the set $\mathbb{Z}^+$ of positive integers. Then T is partially ordered by the relation $|$(divides). Order the elements of T according to usual less than relation as $t_1 < t_2 < t_3 \cdots$, we define a subset $\mathrm{sac}(T)$ of T as follows : for every integer n, $1 \leq n < |T| + 1$,

$$\mathrm{sac}(T) = \{t_n \in T\colon \text{ for all } i < n, t_n \text{ is not a multiple of } t_i\}.$$

The set $\mathrm{sac}(T)$ is a maximal anti-chain of the poset $(T, |)$. We will call $\mathrm{sac}(T)$, the standard anti-chain of T.

For example, if $T = \{2, 4, 7\}$ then $\mathrm{sac}(T) = \{2, 7\}$.

Definition 4.15. Let σ be in $\mathcal{P}(T)$ such that σ contains a cycle. Let T denote the set of lengths of cycles in σ. The standard anti-chain of $(T, |)$ is called the *cycle set* of σ and it is denoted by $\mathrm{cs}(\sigma)$.

Definition 4.16. A c-component $\kappa \in \mathcal{P}(T)$ is of *rro type* (right rays only) if it has a maximal right ray but no cycles, double rays, left rays or maximal chains, and is of *cho type* (chains only) if it has a maximal chain but no cycles or rays.

Lemma 4.17. [2, Lemma 4.11] *Let $\kappa \in \mathcal{P}(T)$ such that κ contains a maximal left ray or it is of cho type. Then κ contains a unique terminal vertex.*

Definition 4.18. Let $\kappa \in \mathcal{P}(T)$ be connected such that κ has a maximal left ray or is of cho type. The unique terminal vertex of κ established by Lemma 4.17 will be called the *root* of κ.

Definition 4.19. A relation R on a non-empty set E is called *well-founded* if every non-empty subset $D \subseteq E$ contains an R-minimal element that is, $q \in D$ exists such that there is no $q \in D$ with $(q, p) \in R$.

Definition 4.20. If R be a well-founded relation on a set E, then a unique function π defined on E with ordinals as values as,

$$\pi(p) = \sup\{\pi(q) + 1 : (q, p) \in R\}.$$

for every $p \in E$ is called the *rank* of p in $< E, R >$.

Example 4.21. Let $T = \{a, b, c, \ldots, a_1, b_1, c_1 \ldots\}$ and let $\kappa = [a, b, c, \ldots >\in \mathcal{P}(T)$. Then $\pi(a) = 0, \pi(b) = 1, \pi(c) = 2$ and so on.

Notation 2. *Let $\kappa \in \mathcal{P}(T)$ be connected of rro type or cho then $\pi_\kappa(p)$ denotes the rank of p under the relation κ.*

Definition 4.22. Let $< u_q >_{q\geq 0}$ and $< v_q >_{q\geq 0}$ be sequences of ordinals. Then we say that $< v_q >$ *dominates* $< u_q >$ if

$$v_{q+r} \geq u_q \text{ for every } q \geq 0 \text{ and for some } r \geq 0.$$

Notation 3. *Let $\kappa \in \mathcal{P}(T)$ be connected of rro type, and $\mu = [p_0 p_1 p_2 \ldots >$ be a maximal right ray in κ. We denote by $< \mu_q^\kappa >_{q\geq 0}$ the sequence of ordinals with*

$$\mu_q^\kappa = \pi_\kappa(p_q) \text{ for every } q \geq 0.$$

Example 4.23. Let $T = \{p_0, p_1, p_2, \ldots, q_0, q_1, q_2, \ldots\}$ and let

$$\kappa = [p_0 p_1 p_2 p_3 \cdots > \cup [q_0 p_2] \cup [q_1 q_2 p_2] \cup [q_3 q_4 q_5 p_2] \cup [q_6 q_7 q_8 q_9 p_2] \cup \cdots \in \mathcal{P}(T)$$

and the right ray $\mu = [p_0 p_1 p_2 \cdots >$ in κ, then the sequence $< \mu_q^\kappa >$ is

$$< 0, 1, \omega, \omega + 1, \omega + 2, \omega + 3, \ldots > .$$

Definition 4.24. For $\sigma \in \mathcal{P}(T)$, we define

$$s(\sigma) = \sup\{\pi_\kappa(a_0) : \kappa \text{ is a c-component of } \sigma \text{ of type cho with root } a_0\},$$

where we agree that $s(\sigma) = 0$ if σ has no c-component of cho type.

The next results (Proposition 4.25 to Theorem 4.31) are from Araujo et al. [2] and are required to prove Theorem 4.32.

Proposition 4.25. *Let $\kappa, \lambda \in \mathcal{P}(T)$ be connected such that κ has a cycle $(p_0 p_1 \cdots p_{k-1})$. Then $\Gamma(\kappa)$ is rp-hom to $\Gamma(\lambda)$ if and only if λ has a cycle $(q_0 q_1 \cdots q_{m-1})$ such that $m|k$.*

Lemma 4.26. *Let $\kappa, \lambda \in \mathcal{P}(T)$ be connected such that λ has a cycle $(q_0\ q_1\ \cdots\ q_{m-1})$. Suppose κ has a double ray or is of rro type. Then $\Gamma(\kappa)$ is rp-hom to $\Gamma(\lambda)$.*

Lemma 4.27. *Let $\kappa,\ \lambda \in \mathcal{P}(T)$ be connected. Suppose that λ has a double ray and κ either has a double ray or has rro type. Then $\Gamma(\kappa)$ is rp-hom to $\Gamma(\lambda)$.*

Lemma 4.28. *Let $\kappa, \lambda \in \mathcal{P}(T)$ be connected. Suppose that λ has a maximal left ray and κ either has a maximal left ray or is of cho type. Then $\Gamma(\kappa)$ is rp-hom $\Gamma(\lambda)$.*

Proposition 4.29. *Let $\kappa, \lambda \in \mathcal{P}(T)$ be connected of cho type with roots p_0 and q_0, respectively. Then $\Gamma(\kappa)$ is rp-hom to $\Gamma(\lambda)$ if and only if $\pi(x_0) \leq \pi(y_0)$.*

Proposition 4.30. *Let $\kappa, \lambda \in \mathcal{P}(T)$ be connected of rro type. Then $\Gamma(\kappa)$ is rp-hom to $\Gamma(\lambda)$ if and only if there are maximal right ray μ in κ and η in λ such that $< \eta_n^{\lambda} >$ dominates $< \mu_n^{\kappa} >$.*

Theorem 4.31. *Let $\sigma,\ \tau \in \mathcal{P}(T)$. Then $\sigma \sim_c \tau$ in $\mathcal{P}(T)$ if and only if the following conditions hold:*

(1) *$cs(\sigma) = cs(\tau)$.*
(2) *σ contains a double ray but no cycle if and only if τ contains a double ray but no cycle.*
(3) *If σ contains a c-component κ of rro type but no cycles or double rays then τ contains a c-component λ of rro type but no cycles or double rays and $<\eta_p^{\lambda}>$ dominates $<\mu_p^{\kappa}>$ for some maximal right rays μ in κ and η in λ.*
(4) *If τ contains a c-component λ of rro type but no cycles or double rays then σ contains a c-component κ of rro type but no cycles or double rays and $<\mu_p^{\kappa}>$ dominates $<\eta_p^{\lambda}>$ for some maximal right rays η in λ and μ in κ.*
(5) *σ contains a maximal left ray if and only if τ contains a maximal left ray.*
(6) *If σ contains a c-component κ of cho type with root p_0 but no maximal left rays then τ contains a c-component λ of cho type with root q_0 but no maximal left rays, and $\pi_\kappa(p_0) \leq \pi_\lambda(q_0)$.*
(7) *If τ contains a c-component λ of cho type with root q_0 but no maximal left ray then σ contains a c-component κ of cho type with root p_0 but no maximal left rays, and $\pi_\lambda(q_0) \leq \pi_\kappa(p_0)$.*

Now we are ready to prove our main result of the section on $\sim_n$ notion of conjugacy in partial transformation semigroup.

Theorem 4.32. *Let $\sigma,\ \tau \in \mathcal{P}(T)$. Then $\sigma \sim_n \tau$ in $\mathcal{P}(T)$ if and only the following conditions hold:*

(1) *There are* $\alpha, \beta \in \mathcal{P}(T)$ *such that* $q\alpha\beta = q$ *for any non-initial vertex* q *of* $\Gamma(\sigma)$ *and* $k\beta\alpha = k$ *for every non-initial vertex* k *of* $\Gamma(\tau)$.

(2) $cs(\sigma) = cs(\tau)$.

(3) σ *contains a double ray but no cycle if and only if* τ *contains a double ray but no cycle.*

(4) *If* σ *contains a c-component* κ *of rro type but no cycles or double rays then* τ *contains a c-component* λ *of rro type but no cycles or double rays and* $<\eta_p^\lambda>$ *dominates* $<\mu_p^\kappa>$ *for some maximal right rays* μ *in* κ *and* η *in* λ.

(5) *If* τ *contains a c-component* λ *of rro type but no cycles or double rays then* σ *contains a c-component* κ *of rro type but no cycles or double rays and* $<\mu_p^\kappa>$ *dominates* $<\eta_p^\lambda>$ *for some maximal right rays* η *in* λ *and* μ *in* κ.

(6) σ *contains a maximal left ray if and only if* τ *contains a maximal left ray.*

(7) *If* σ *contains a c-component* κ *of cho type with root* p_0 *but no maximal left rays then* τ *contains a c-component* λ *of cho type with root* q_0 *but no maximal left rays, and* $\pi_\kappa(p_0) \leq \pi_\lambda(q_0)$.

(8) *If* τ *contains a c-component* λ *of cho type with root* q_0 *but no maximal left ray then* σ *contains a c-component* κ *of cho type with root* p_0 *but no maximal left rays, and* $\pi_\lambda(q_0) \leq \pi_\kappa(p_0)$.

Proof. Let $\sigma \sim_n \tau$. Then by Theorem 3.5 there exist $\alpha, \beta \in \mathcal{P}(T)$ such that α is an rp-hom from $\Gamma(\sigma)$ to $\Gamma(\tau)$ and β is an rp-hom from $\Gamma(\tau)$ to $\Gamma(\sigma)$ with $q\alpha\beta = q$ for any non-initial vertex q of $\Gamma(\sigma)$ and $k\beta\alpha = k$ for every non-initial vertex k of $\Gamma(\tau)$. Since $\sim_n \subseteq \sim_c$, then by Theorem 4.31, (2) to (8) hold.

Conversely, if $\sigma = \tau = 0$, then trivially $\sigma \sim_n \tau$. Suppose $\sigma, \tau \neq 0$ and all the conditions from (1) to (8) hold. Let κ be a c-component of σ. We will prove that $\Gamma(\kappa)$ is an rp-hom to $\Gamma(\lambda)$ for some c-component λ of τ. The result then follows by Proposition 4.13.

Suppose κ has a cycle of length r, since by (2), $cs(\sigma) = cs(\tau)$, τ has a cycle υ of length s such that $s|r$. Let λ be the c-component of τ containing υ. Then $\Gamma(\kappa)$ is an rp-hom to $\Gamma(\lambda)$ by Proposition 4.25.

Suppose κ has a double ray. If some c-component λ of τ has a cycle, then $\Gamma(\kappa)$ is rp-hom to $\Gamma(\lambda)$ by Lemma 4.26. Suppose τ does not have a cycle. Then, by (2) and (3), both σ and τ have a double ray but not a cycle. Let λ be a c-component of τ containing a double ray. Then $\Gamma(\kappa)$ is an rp-hom to $\Gamma(\lambda)$ by Lemma 4.27.

Suppose κ is of rro type. If τ has some c-component λ with a cycle or a double ray, then $\Gamma(\kappa)$ is an rp-hom to $\Gamma(\lambda)$ by Lemma 4.26 and Lemma 4.27. Suppose τ does not have a cycle or a double ray. Then by (4), there is a c-component λ in τ of rro type such that $< \eta_p^\lambda >$ dominates $< \mu_p^\kappa >$ for some maximal right rays μ in κ and η in λ. Hence $\Gamma(\kappa)$ is an rp-hom to $\Gamma(\lambda)$ by Proposition 4.30.

Suppose κ has a maximal left ray. Then by (5) there is some c-component λ of τ having a maximal left ray. Then $\Gamma(\kappa)$ is an rp-hom to $\Gamma(\lambda)$ by Lemma 4.28.

Suppose κ is of cho type with root p_0. If τ has some c-component λ having a maximal left ray then $\Gamma(\kappa)$ is an rp-hom to $\Gamma(\lambda)$ by Lemma 4.28. Suppose τ does not have a maximal left ray. Then by (6), σ does not have a maximal left ray, and so by (7), there is a c-component λ in τ of cho type with root q_0 such that $\pi_\kappa(p_0) \leq \pi_\kappa(q_0)$. Hence $\Gamma(\kappa)$ is an rp-hom to $\Gamma(\lambda)$, by Proposition 4.30.

We have proved that for every c-component κ of σ there exists a c-component λ of τ and an rp-hom $\alpha_\kappa \in \mathcal{P}(T)$ from $\Gamma(\kappa)$ to $\Gamma(\lambda)$. We may assume that for every c-component κ of σ, $\mathrm{dom}(\alpha_\kappa) = \mathrm{span}(\kappa)$. Then by Proposition 4.13, $\Gamma(\sigma)$ is an rp-hom to $\Gamma(\tau)$. By symmetry, $\Gamma(\tau)$ is an rp-hom to $\Gamma(\sigma)$. Then by condition (1) and Theorem 3.5 we get $\sigma \sim_n \tau$.□

Corollary 4.33. [2, Corollary 5.6] *Let $\sigma, \tau \in \mathcal{P}(T)$ where T is finite. Then $\sigma \sim_c \tau$ if and only if $cs(\sigma) = cs(\tau)$ and $s(\sigma) = s(\tau)$.*

If T is finite then any $\sigma \in \mathcal{P}(T)$ has no left[right] or a double ray. Hence by Theorem 4.32 and Corollary 4.33, we have the following corollary.

Corollary 4.34. *Let $\sigma, \tau \in \mathcal{P}(T)$ where T is finite. Then $\sigma \sim_n \tau$ if and only if $cs(\sigma) = cs(\tau), s(\sigma) = s(\tau)$ and there are $\alpha, \beta \in \mathcal{P}(T)$ such that $q\alpha\beta = q$ for every non-initial vertex q of $\Gamma(\sigma)$ and $k\beta\alpha = k$ for every non-initial vertex k of $\Gamma(\tau)$.*

Proof. Let $\sigma \sim_n \tau$ then by Theorem 3.5 there are $\alpha, \beta \in \mathcal{P}(T)$ such that $q\alpha\beta = q$ for every non-initial vertex q of $\Gamma(\sigma)$ and $k\beta\alpha = k$ for every non-initial vertex k of $\Gamma(\tau)$. Since $\sim_n \subseteq \sim_c$, so the other conditions follow by the Corollary 4.33.

The converse follows on the similar lines as of Theorem 4.32.□

Theorem 4.35. [2, Theorem 6.1] *Let $\sigma, \tau \in \mathcal{T}(T)$. Then $\sigma \sim_c \tau$ in $\mathcal{T}(T)$ if and only if the following conditions hold:*

(1) $cs(\sigma) = cs(\tau)$.

(2) σ and τ have double ray but no cycles.

(3) All connected components of σ and τ have rro type.

(a) *For every c-component κ of σ there is a c-component δ of τ so that $< \eta_p^\delta >$ dominates $< \mu_p^\kappa >$ for some maximal right ray μ in κ and some maximal right ray η in δ.*

(b) *For every c-component δ of τ there is a c-component κ of σ such that $< \mu_p^\kappa >$ dominates $< \eta_p^\delta >$ for some maximal right ray η in δ and some maximal right ray μ in κ.*

A c-component of $\sigma \in \mathcal{T}(T)$ cannot have a maximal left ray or a maximal chain. Due to that fact we have the following theorem in $\mathcal{T}(T)$.

Theorem 4.36. *Let $\sigma, \tau \in \mathcal{T}(T)$. Then $\sigma \sim_n \tau$ in $\mathcal{T}(T)$ if and only if the following conditions hold:*
(1) $cs(\sigma) = cs(\tau)$.
(2) σ and τ have double ray but no cycles.
(3) All connected components of σ and τ have rro type and

(a) *for every c-component κ of σ there is a c-component δ of τ so that $< \eta_p^\delta >$ dominates $< \mu_p^\kappa >$ for some maximal right ray μ in κ and some maximal right ray η in δ, and*

(b) *for every c-component δ of τ there is a c-component κ of σ such that $< \mu_p^\kappa >$ dominates $< \eta_p^\delta >$ for some maximal right ray η in δ and some maximal right ray μ in κ.*

(4) There are $\alpha, \beta \in \mathcal{T}(T)$ such that $q\alpha\beta = q$ for any non-initial vertex q of $\Gamma(\sigma)$ and $k\beta\alpha = k$ for every non-initial vertex k of $\Gamma(\tau)$.

Proof. Let $\sigma, \tau \in \mathcal{T}(T)$ and let $\sigma \sim_n \tau$ then by Corollary 3.6 there are $\alpha, \beta \in \mathcal{T}(T)$ such that $q\alpha\beta = q$ for any non-initial vertex q of $\kappa(\alpha)$ and $k\beta\alpha = k$ for every non-initial vertex q of $\Gamma(\sigma)$ and $k\beta\alpha = k$ for every non-initial vertex k of $\Gamma(\tau)$. Since $\sim_n \subseteq \sim_c$, so by Theorem 4.35, conditions (1), (2) and (3) holds.

The converse follows on the similar lines as of Theorem 4.32. □

In case the set T is finite, then $\sigma \in \mathcal{T}(T)$ have no rays, so we have the following corollary.

Corollary 4.37. *Let $\sigma, \tau \in \mathcal{T}(T)$, where T is finite. Then $\sigma \sim_n \tau$ if and only if $cs(\sigma) = cs(\tau)$ and there are $\alpha, \beta \in \mathcal{T}(T)$ such that $q\alpha\beta = q$ for any non-initial vertex q of $\kappa(\alpha)$ and $k\beta\alpha = k$ for every non-initial vertex k of $\kappa(\beta)$.*

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