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Research Paper

A NOTE ON σ-IDEALS OF DISTRIBUTIVE LATTICES

MUKKAMALA SAMBASIVA RAO*

Abstract. Some properties of σ-ideals of distributive lattices are studied. The classes of Boolean algebras, generalized Stone lattices, relatively complemented lattices are characterized with the help of σ-ideals and maximal ideals. Some significant properties of prime σ-ideals are studied with the help of a congruence.

1. Introduction

In 1970, the theory of relative annihilators was introduced in lattices by Mark Mandelker [6] and he characterized distributive lattices in terms of their relative annihilators. Later many authors introduced the concept of annihilators in the structures of rings as well as lattices and characterized several algebraic structures in terms of annihilators. T.P. Speed [10] and W.H. Cornish [4] made an extensive study of annihilators in distributive lattices. The class of annulets played a vital role in characterizing many algebraic structures like normal lattices [3]

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*Corresponding author

and quasi-complemented lattices [4]. In [4], the author investigated thoroughly the properties of annihilator ideals and α-ideals of distributive lattices. In [5], the author studied the properties of O-ideals and σ-ideals of distributive lattices and characterized the O-ideals with the help of congruences. In 2015, the author introduced e notion of disjunctive ideals [8] in distributive lattices and characterized normal lattices in terms of annihilators. Recently in 2022, the author introduced the notion of ω-filters [9] of distributive lattices and then characterized quasi-complemented lattices with the help of ω-filters.

Motivated by the characterizations given by Cornish in [3] who studied distributive lattices with 0 in which each prime ideal contains a unique minimal prime ideal under the name *normal lattices*. He characterized generalized Stone lattices by means of normal and quasi-complemented distributive lattices with 0. The notion of α-ideals is introduced in [4] and proved that a distributive lattice is a generalized Stone lattice if and only if each prime ideal contains a unique prime α-ideal. This work is greatly motivated by the above works and a desire to extend these investigations to σ-ideals of distributive lattices.

The main aim of this paper is to study some further properties of annihilators in the form of σ-ideals of distributive lattices. In this note, some more properties of σ-ideal of a distributive lattice are studied with the help of prime ideals, maximal ideals, α-ideals, O-ideals and congruences. It is observed that every σ-ideal of a distributive lattice is an α-ideal but not the converse in general. A set of equivalent conditions is derived for every α-ideal of a distributive lattice to become a σ-ideal, which leads to a characterization of generalized Stone lattices. Boolean algebras are characterized in terms of maximal ideals and σ-ideals of a distributive lattices. It is also observed that a maximal ideal need not be a σ-ideal. Some necessary and sufficient conditions are derived for a maximal ideal of a distributive lattice. A set of equivalent conditions is derived for every ideal of a distributive lattice to become a σ-ideal which leads to a characterization of relatively complemented lattices. A congruence ψ is considered on a distributive lattice, and then derived a one-to-one correspondence between the set of all prime σ-ideals of the lattice L and the set of all prime σ-deals of the quotient algebra.

The theory of posets and lattices has many practical applications in distributed computing. Besides the applications in predicate detection, lattice theory is also useful in predicate control [7, 11]. We believe that the future will bring even more applications of the theory of order to distributed computing. We also expect, enrichment of the poset and lattice theory from distributed computing applications. The concepts of linear predicates, efficient advancement property, algorithms for computing slices etc. can be viewed as computational lattice theory.

2. **Preliminaries**

The reader is referred to [1], [2], [3], [4], [5] and [10] for the elementary notions and notations of distributive lattices. However some of the preliminary definitions and results are presented for the ready reference of the reader.

Definition 2.1. [1] An algebra $(L, \wedge, \vee)$ of type $(2, 2)$ is called a distributive lattice if for all $x, y, z \in L$, it satisfies the following properties $(1), (2), (3)$ and (4) along with (5) or $(5')$

(1) $x \wedge x = x$, $x \vee x = x$

(2) $x \wedge y = y \wedge x$, $x \vee y = y \vee x$

(3) $(x \wedge y) \wedge z = x \wedge (y \wedge z)$, $(x \vee y) \vee z = x \vee (y \vee z)$

(4) $(x \wedge y) \vee x = x$, $(x \vee y) \wedge x = x$

(5) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

$(5')$ $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

A non-empty subset A of a lattice L is called an ideal(filter) of L if $a \vee b \in A (a \wedge b \in A)$ and $a \wedge x \in A (a \vee x \in A)$ whenever $a, b \in A$ and $x \in L$. The set $(a] = \{x \in L \mid x \leq a\}$ (resp. $[a) = \{x \in L \mid a \leq x\}$) is called a principal ideal (resp. principal filter) generated by a. The set $\mathcal{I}(L)$ of all ideals of a distributive lattice L with 0 forms a complete distributive lattice. The set $\mathcal{F}(L)$ of all filters of a distributive lattice L with 1 forms a complete distributive lattice. A proper ideal P of a distributive lattice L is called *prime* if for any $x, y \in L$, $x \wedge y \in P$ implies $x \in P$ or $y \in P$. A proper ideal M of a lattice is called *maximal* if there exists no proper ideal N such that $M \subset N$. A prime ideal P of a lattice L is called *minimal* if there exists no proper ideal Q such that $Q \subset P$.

A distributive lattice L with 0 is called *normal* if every prime ideal of L contains a unique minimal prime ideal of L. A distributive lattice L is called *relatively complemented* if each interval of L is complemented. A proper filter P of a distributive lattice L is said to be *prime* if for any $x, y \in L$, $x \vee y \in P$ implies $x \in P$ or $y \in P$. A proper filter P of a lattice L is called *maximal* if there exists no proper filter Q such that $P \subset Q$. A proper filter P of a distributive lattice is maximal if and only if $L - P$ is a minimal prime ideal.

Theorem 2.2. [3] *A prime ideal I of a distributive lattice L, with 0, is minimal if and only if to each $x \in P$ there exists $y \notin P$ such that $x \wedge y = 0$.*

For any non-empty subset A of a distributive lattice L with 0, the annihilator of A is define as the set $A^* = \{x \in L \mid x \wedge a = 0 \text{ for all } a \in A\}$. For any non-empty subset A of L, A^* is an ideal of L with $A \cap A^* = \{0\}$.

Lemma 2.3. [10] *Let L be a distributive lattice with 0. For any subsets A and B of L,*

(1) $A \subseteq B$ *implies* $B^* \subseteq A^*$,

(2) $A \subseteq A^{**}$,

(3) $A^{***} = A^*$.

(4) $A^* = L$ *if and only if* $A = \{0\}$.

In case of ideals, we have the following result.

Proposition 2.4. [10] *Let L be a distributive lattice with 0. For any ideals I and J of L,*

(1) $I \cap J = \{0\}$ *implies* $I \subseteq J^*$,

(2) $(I \vee J)^* = I^* \cap J^*$,

(3) $(I \cap J)^{**} = I^{**} \cap J^{**}$.

It is clear that $((x])^* = (x)^*$ and is simply denoted by $(x)^*$. Then clearly $(1)^* = \{0\}$. An element $x \in L$ is called dense if $(x)^* = \{0\}$.

Corollary 2.5. [10] *Let L be a distributive lattice with 0. For any $a, b, c \in L$,*

(1) $a \leq b$ *implies* $(b)^* \subseteq (a)^*$,

(2) $(a \vee b)^* = (a)^* \cap (b)^*$,

(3) $(a \wedge b)^{**} = (a)^{**} \cap (b)^{**}$,

(4) $(a)^* \cap (b)^* = \{0\}$ *if and only if* $(a)^* \subseteq (b)^{**}$,

(5) $(a)^* = L$ *if and only if* $a = 0$.

An ideal I of a distributive lattice L with 0 is called an *annihilator ideal* [4] if $I = I^{**}$. An ideal I of a distributive lattice L with 0 is called a α-ideal [4] of L if $x \in I$ implies $(x)^{**} \subseteq I$ for all $x \in L$. An ideal I of a distributive lattice L is an α-ideal if and only if for all $x, y \in L$, $(x)^* = (y)^*$ and $x \in I$ implies that $y \in I$. Every annihilator ideal of a distributive lattice is an α-ideal. An ideal I of a distributive lattice L is called an O-ideal [5] if $I = O(F)$ for some filter F of L, where $O(F) = \{x \in L \mid x \wedge a = 0 \text{ for some } a \in F\}$.

Definition 2.6. [5] For any ideal I of a lattice L, define $\sigma(I) = \{x \in L \mid (x)^* \vee I = L\}$.

Lemma 2.7. [5] *For any two ideals I, J of a distributive lattice L, we have*

(1) $\sigma(I) \subseteq I$,

(2) $I \subseteq J$ *implies* $\sigma(I) \subseteq \sigma(J)$,

(3) $\sigma(I \cap J) = \sigma(I) \cap \sigma(J)$.

A distributive lattice L is called a *generalized Stone lattice* if $(x)^* \vee (x)^{**} = L$ for all $x \in L$. For any ideal I of a distributive lattice L, $\sigma(I)$ is an ideal of L. An ideal I of a distributive lattice L is called a σ-ideal if $I = \sigma(I)$. Every σ-ideal [5] of a distributive lattice is an α-ideal. Throughout this note, all lattices are bounded distributive lattice unless otherwise mentioned.

3. Characterization theorems

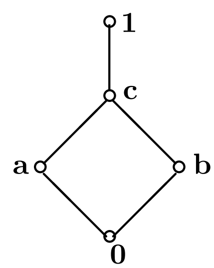
In this section, the algebraic structures like generalized Stone lattices, Boolean algebras, and relatively complemented lattices are characterized with the help of σ-ideals, α-ideals, minimal prime ideals and maximal ideals of lattices.

Proposition 3.1. *Every prime σ-ideal of a lattice is a minimal prime ideal.*

Proof. Let P be a prime σ-ideal of a lattice L. Let $x \in P$. Then $x \in \sigma(P)$ and hence $(x)^* \vee P = L$. Thus there exist $a \in (x)^*$ and $b \in P$ such that $a \vee b = 1$. Since $a \in (x)^*$, we get $a \wedge x = 0$. Suppose $a \in P$. Since $b \in P$, we get $1 = a \vee b \in P$. Hence $a \notin P$. Thus, for any $x \in P$, there exists $a \notin P$ such that $a \wedge x = 0$. Therefore P is minimal. □

The converse of the above proposition is not true. That is every minimal prime ideal of a lattice need not be a σ-ideal. For, consider the following example:

Example 3.2. Consider the distributive lattice $L = \{0, a, b, c, 1\}$ whose Hasse diagram is given in the following figure:



Consider the ideal $P = \{0, a\}$. Clearly P is a prime ideal of L. Since $\{0\}$ is not prime ideal, the prime ideal P is minimal. For $a \in P$, observe that $(a)^* \vee P = \{0, b\} \vee P = \{0, a, b, c\} \neq L$. Hence $a \notin \sigma(P)$. Therefore P is not a σ-ideal.

Theorem 3.3. *The following assertions are equivalent in a lattice L:*

(1) *L is a generalized Stone lattice;*

(2) *every α-ideal is a σ-ideal;*

(3) *every prime α-ideal is a σ-ideal;*

(4) *every minimal prime ideal is a σ-ideal.*

Proof. (1) $\Rightarrow$ (2): Assume that L is a generalized Stone lattice. Let I be an α-ideal of L. Clearly $\sigma(I) \subseteq I$. Conversely, let $x \in I$. Since I is an α-ideal, we get $(x)^{**} \subseteq I$. Hence $L = (x)^* \vee (x)^{**} \subseteq (x)^* \vee I$. Thus $x \in \sigma(I)$. Therefore I is a σ-ideal.

(2) $\Rightarrow$ (3): is clear.

(3) $\Rightarrow$ (4): Since every minimal prime ideal is a prime α-ideal, it is obvious.

(4) $\Rightarrow$ (1): Assume that every minimal prime ideal of L is a σ-ideal. Let $x \in L$. Suppose $(x)^* \vee (x)^{**} \neq L$. Then there exists a prime filter P such that $\{(x)^* \vee (x)^{**}\} \cap P = \emptyset$. Since every prime filter is contained in a maximal filter, there exists a maximal filter M such that $P \subseteq M$. Then $L - M$ is a minimal prime ideal of L. By the assumption, $L - M$ is a σ-ideal. Suppose $x \in M$. Then there exists $y \notin M$ such that $x \vee y = 1 \in P$. Since $y \notin M$, we must have $y \notin P$. Since P is prime, we get $x \in P$. Clearly $x \in (x)^{**} \subseteq (x)^* \vee (x)^{**}$. Hence $x \in \{(x)^* \vee (x)^{**}\} \cap P$, which is a contradiction. Thus $x \notin M$ and therefore $x \in L - M = \sigma(L - M)$. Hence $(x)^* \vee (L - M) = L$, which gives that $1 \in (x)^* \vee (L - M)$. Then $a \vee b = 1 \in P$ for some $a \in (x)^*$ and $b \in L - M$. Since $b \notin M$, we must have $b \notin P$. Since P is prime, we get $a \in P$. Hence $a \in \{(x)^* \vee (x)^{**}\} \cap P$, which is a contradiction. Therefore $(x)^* \vee (x)^{**} = L$ for all $x \in L$. □

Proposition 3.4. *For any maximal ideal M of a lattice L, the set $O(M) = \{x \in L \mid (x)^* \nsubseteq M\}$ is an ideal of L such that $O(M) \subseteq M$.*

Proof. Since M is proper, we get $(0)^* \nsubseteq M$. Hence $0 \in O(M)$. Suppose $x, y \in O(M)$. Then $(x)^* \nsubseteq M$ and $(y)^* \nsubseteq M$. Hence $M \subset M \vee (x)^*$ and $M \subset M \vee (y)^*$. Since M is maximal, we get $M \vee (x)^* = L$ and $M \vee (y)^* = L$. Thus, we get

$$M \vee (x \vee y)^* = M \vee \{(x)^* \cap (y)^*\} = \{M \vee (x)^*\} \cap \{M \vee (y)^*\} = L \cap L = L$$

If $(x \vee y)^* \subseteq M$, then $M = L$ which is a contradiction. Hence $(x \vee y)^* \nsubseteq M$. Thus $x \vee y \in O(M)$. Again, let $x \in O(M)$ and $y \leq x$. Then $(x)^* \nsubseteq M$ and $y \leq x$. Since $y \leq x$, we get $(x)^* \subseteq (y)^*$. Hence $(y)^* \nsubseteq M$. Thus $y \in O(M)$. Therefore $O(M)$ is an ideal of L. Now, let $x \in O(M)$. Then $(x)^* \nsubseteq M$. Hence, there exists $a \in (x)^*$ such that $a \notin M$. Since $a \in (x)^*$, we get $a \wedge x = 0$. Suppose $x \notin M$. Then $M \vee (x] = L$. Since $a \notin M$, we get $M \vee (a] = L$. Hence $L = \{M \vee (a]\} \cap \{M \vee (x]\} = M \vee \{(a] \cap (x]\} = M \vee (x \wedge a] = M \vee (0] = M$, which is a contradiction. Hence $x \in M$. Therefore $O(M) \subseteq M$. □

Proposition 3.5. *Let M be a proper ideal of a lattice L. Then we have*

(1) *if M is prime, then $\sigma(M) \subseteq O(M)$,*

(2) *if M is maximal, then $\sigma(M) = O(M)$.*

Proof. (1) Let $x \in \sigma(M)$. Then $(x)^* \vee M = L$. Suppose $(x)^* \subseteq M$. Then $M = L$, which is a contradiction. Hence $(x)^* \nsubseteq M$. Thus $x \in O(M)$. Therefore $\sigma(M) \subseteq O(M)$.

(2) Since M is prime, we get $\sigma(M) \subseteq O(M)$. Conversely, let $x \in O(M)$. Then $(x)^* \nsubseteq M$. Since M is maximal, we get $(x)^* \vee M = L$. Thus $x \in \sigma(M)$. Therefore $O(M) = \sigma(M)$. □

Let us denote that μ is the set of all maximal ideals of a lattice L. For any ideal I of a lattice L, we also denote $\mu(I) = \{M \in \mu \mid I \subseteq M\}$. Since every maximal ideal of a lattice is prime, by Proposition 3.4, we conclude that $O(M)$ is an ideal such that $O(M) \subseteq M$ for every $M \in \mu$. Then we have the following result.

Theorem 3.6. *For any ideal I of a lattice L, $\sigma(I) = \bigcap_{M \in \mu(I)} O(M)$.*

Proof. Let $x \in \sigma(I)$ and $I \subseteq M$ where $M \in \mu$. Then $L = (x)^* \vee I \subseteq (x)^* \vee M$. Suppose $(x)^* \subseteq M$, then $M = L$, which is a contradiction. Hence $(x)^* \nsubseteq M$. Thus $x \in O(M)$ for all $M \in \mu(I)$. Therefore $\sigma(I) \subseteq \bigcap_{M \in \mu(I)} O(M)$. Conversely, let $x \in \bigcap_{M \in \mu(I)} O(M)$. Then $x \in O(M)$ for all $M \in \mu(I)$. Suppose $(x)^* \vee I \neq L$. Then there exists a maximal ideal M_0 such that $(x)^* \vee I \subseteq M_0$. Hence $(x)^* \subseteq M_0$ and $I \subseteq M$. Since $I \subseteq M_0$, by hypothesis, we get $x \in O(M_0)$. Hence $(x)^* \nsubseteq M_0$, which is a contradiction. Therefore $(x)^* \vee I = L$. Thus $x \in \sigma(I)$. Hence $\bigcap_{M \in \mu(I)} O(M) \subseteq \sigma(I)$. □

From the above theorem, it can be easily observed that $\sigma(I) \subseteq O(M)$ for every $M \in \mu(I)$. Now, in the following, a set of equivalent conditions is derived for the set of all ideals of the form $\sigma(I)$ to become a sublattice to the ideal lattice $\mathcal{I}(L)$ which leads to a characterization of a Boolean algebra.

Theorem 3.7. *Let L be a lattice. Then the following assertions are equivalent:*

(1) *L is a Boolean algebra;*

(2) *for any $M \in \mu$, $O(M)$ is maximal;*

(3) *for any $I, J \in \mathcal{I}(L)$, $I \vee J = L$ implies $\sigma(I) \vee \sigma(J) = L$;*

(4) *for any $I, J \in \mathcal{I}(L)$, $\sigma(I) \vee \sigma(J) = \sigma(I \vee J)$;*

(5) *for any two distinct maximal ideals M and N, $O(M) \vee O(N) = L$;*

(6) *for any $M \in \mu$, M is the unique member of μ such that $O(M) \subseteq M$.*

Proof. (1) $\Rightarrow$ (2) : Assume that L is a Boolean algebra. Let $M \in \mu$. Clearly $O(M) \subseteq M$. Conversely, let $x \in M$. Then there exists $x' \in L$ such that $x \wedge x' = 0$ and $x \vee x' = 1$. If $x' \in M$, then $1 = x \vee x' \in M$ which is a contradiction. Hence $x' \notin M$. Thus $x' \in (x)^*$ and $x' \notin M$, which gives $(x)^* \nsubseteq M$. Thus $x \in O(M)$. Therefore $M \subseteq O(M)$.

(2) $\Rightarrow$ (3) : Assume the condition (2). Then clearly $O(M) = M$ for all $M \in \mu$. Let $I, J \in \mathcal{I}(L)$ be such that $I \vee J = L$. Suppose $\sigma(I) \vee \sigma(J) \neq L$. Then there exists a maximal ideal M such

that $\sigma(I) \vee \sigma(J) \subseteq M$. Hence $\sigma(I) \subseteq M$ and $\sigma(J) \subseteq M$. Now

$$\begin{aligned} \sigma(I) \subseteq M &\Rightarrow \bigcap_{M_i \in \mu(I)} O(M_i) \subseteq M \\ &\Rightarrow O(M_i) \subseteq M \text{ for some } M_i \in \mu(I) \text{ (since } M \text{ is prime)} \\ &\Rightarrow M_i \subseteq M \quad \text{By condition (2)} \\ &\Rightarrow I \subseteq M \quad \text{since } I \subseteq M_i \end{aligned}$$

Similarly, we can obtain that $J \subseteq M$. Hence $L = I \vee J \subseteq M$, which is a contradiction to the maximality of M. Therefore $\sigma(I) \vee \sigma(J) = L$.

(3) $\Rightarrow$ (4) : Assume the condition (3). Let $I, J \in \mathcal{I}(L)$. Clearly $\sigma(I) \vee \sigma(J) \subseteq \sigma(I \vee J)$. Conversely, let $x \in \sigma(I \vee J)$. Then $\{(x)^* \vee I\} \vee \{(x)^* \vee J\} = (x)^* \vee I \vee J = L$. Hence by condition (3), we get $\sigma((x)^* \vee I) \vee \sigma((x)^* \vee J) = L$. Thus $x \in \sigma((x)^* \vee I) \vee \sigma((x)^* \vee J)$. Hence $x = r \vee s$ for some $r \in \sigma((x)^* \vee I)$ and $s \in \sigma((x)^* \vee J)$. Now

$$\begin{aligned} r \in \sigma((x)^* \vee I) &\Rightarrow (r)^* \vee \{(x)^* \vee I\} = L \\ &\Rightarrow L = \{(r)^* \vee (x)^*\} \vee I \subseteq (r \wedge x)^* \vee I \\ &\Rightarrow (r \wedge x)^* \vee I = L \\ &\Rightarrow r \wedge x \in \sigma(I) \end{aligned}$$

Similarly, we can get $s \wedge x \in \sigma(J)$. Now, we have the following consequence:

$$\begin{aligned} x &= x \wedge x \\ &= (r \vee s) \wedge x \\ &= (r \wedge x) \vee (s \wedge x) \end{aligned}$$

where $r \wedge x \in \sigma(I)$ and $s \wedge x \in \sigma(J)$. Hence $x \in \sigma(I) \vee \sigma(J)$. Thus $\sigma(I \vee J) \subseteq \sigma(I) \vee \sigma(J)$. Therefore $\sigma(I) \vee \sigma(J) = \sigma(I \vee J)$.

(4) $\Rightarrow$ (5) : Assume the condition (4). Let M, N be two distinct maximal ideals of L. Choose $x \in M - N$ and $y \in N - M$. Since $x \notin N$, we get $N \vee (x] = L$. Since $y \notin M$, we get

$M \vee (y] = L$. Now, we get

$$\begin{aligned} L &= \sigma(L) \\ &= \sigma(L \vee L) \\ &= \sigma\big(\{N \vee (x]\} \vee \{M \vee (y]\}\big) \\ &= \sigma\big(\{M \vee (x]\} \vee \{N \vee (y]\}\big) \\ &= \sigma(M \vee N) && \text{since } x \in M \text{ and } y \in N \\ &= \sigma(M) \vee \sigma(N) && \text{By condition (4)} \\ &\subseteq O(M) \vee O(N) && \text{By Proposition 3.5(1)} \end{aligned}$$

Therefore $O(M) \vee O(N) = L$.

(5) $\Rightarrow$ (6) : Assume the condition (5). Let $M \in \mu$. Suppose $N \in \mu$ such that $N \neq M$ and $O(N) \subseteq M$. Since $O(M) \subseteq M$, by hypothesis, we get $L = O(M) \vee O(N) = M$, which is a contradiction. Hence M is the unique maximal ideal such that $O(M) \subseteq M$.

(6) $\Rightarrow$ (1) : Let $M \in \mu$. Assume that M is the unique maximal ideal such that $O(M) \subseteq M$. Let $x \in L$. Suppose $1 \notin (x] \vee (x)^*$. Then there exist a maximal ideal M such that $(x] \vee (x)^* \subseteq M$. Then $x \in M$ and $(x)^* \subseteq M$. Hence $x \in M$ and $x \notin O(M)$. Since $x \notin O(M)$, there exists a maximal ideal M_0 such that $x \notin M_0$ and $O(M) \subseteq M_0$. By the uniqueness of M, we get $M = M_0$. Hence $x \notin M_0 = M$, which is a contradiction. Thus $1 \in (x] \vee (x)^*$, which gives $1 = x \vee a$ for some $a \in (x)^*$. Hence $x \wedge a = 0$ and $x \vee a = 1$. Thus a is the complement of x in L. Therefore L is a Boolean algebra. □

Every maximal ideal of a lattice need not be a σ-ideal. For consider the lattice L given in Example 3.2. The ideal $M = \{0, a, b, c\}$ is a maximal ideal but not a σ-ideal in L because of $(c)^* \vee M = \{0\} \vee M \neq L$. In the following, a set of equivalent conditions is established for every maximal ideal of a lattice to become a σ-ideal.

Theorem 3.8. *The following assertions are equivalent in a lattice L:*

(1) *L is a Boolean algebra;*

(2) *every maximal ideal is a σ-ideal;*

(3) *every maximal ideal is a minimal prime ideal.*

Proof. (1) $\Rightarrow$ (2): Assume that L is a Boolean algebra. Let M be a maximal ideal of L. By the above theorem, $O(M) = M$. By Proposition 3.5(2), $\sigma(M) = O(M) = M$. Therefore M is a σ-ideal of L.

(2) $\Rightarrow$ (3): Assume that every maximal ideal of L is a σ-ideal. Then every maximal ideal of L

is a prime σ-ideal. By Proposition 3.1, every maximal ideal is a minimal prime ideal.

(3) $\Rightarrow$ (1): Assume that every maximal ideal of L is a minimal prime ideal. Let $x \in L$. Suppose $1 \notin (x] \vee (x)^*$. Then there exist a maximal ideal M such that $(x] \vee (x)^* \subseteq M$. Hence $x \in M$ and $(x)^* \subseteq M$. By (3), M is a minimal prime ideal. Since M is minimal, and $(x)^* \subseteq M$, we get $x \notin M$ that leads to a contradiction. Thus $1 \in (x] \vee (x)^*$. Then there exists $a \in (x)^*$ such that $1 = x \vee a$. Hence $x \wedge a = 0$ and $x \vee a = 1$. Thus a is the complement of x in L. Therefore L is a Boolean algebra. □

Theorem 3.9. *Let L be a lattice. Then the following assertions are equivalent in L:*

(1) *L is relatively complemented;*

(2) *every principal ideal is a σ-ideal;*

(3) *every ideal is a σ-ideal;*

(4) *every prime ideal is a σ-ideal;*

(5) *every prime ideal is minimal.*

Proof. (1) $\Rightarrow$ (2): Assume that L is relatively complemented. Let $(x]$ be a principal ideal of L. Clearly $\sigma((x]) \subseteq (x]$. Let $a \in L$. Consider the interval $[0, a]$. Clearly $a \wedge x \in [0, a]$. Since L is relatively complemented, there exists $b \in [0, a]$ such that $b \vee (a \wedge x) = a$ and $b \wedge (a \wedge x) = 0$. Since $b \wedge a \wedge x = 0$, we get $a \wedge b \in (x)^*$. Since $b \in [0, a]$, we get $b \leq a$. Now

$$\begin{aligned} a &= b \vee (a \wedge x) \\ &= (b \vee a) \wedge (b \vee x) \\ &= a \wedge (b \vee x) \\ &= (a \wedge b) \vee (a \wedge x) \in (x)^* \vee (x] \qquad \text{since } a \wedge x \in (x] \end{aligned}$$

Hence $L \subseteq (x)^* \vee (x]$. Thus $x \in \sigma((x])$. Therefore $(x]$ is a σ-ideal of L.

(2) $\Rightarrow$ (3): Assume that every principal ideal of L is a σ-ideal. Let I be an ideal of L. Clearly $\sigma(I) \subseteq I$. Conversely, let $x \in I$. By (2), we get $\sigma((x]) = (x]$. Since $(x] \subseteq I$, we get $x \in (x] = \sigma((x]) \subseteq \sigma(I)$. Hence $I \subseteq \sigma(I)$ and therefore I is a σ-ideal of L.

(3) $\Rightarrow$ (4): It is clear.

(4) $\Rightarrow$ (5): Assume the condition (4). Let P be a prime ideal of L. By (4), P is a σ-ideal of L. Hence $\sigma(P) = P$. Let $x \in P = \sigma(P)$. Then $(x)^* \vee P = L$ and thus $1 \in (x)^* \vee P$. Then $1 = a \vee b$ for some $a \in (x)^*$ and $b \in P$. Since $a \in (x)^*$, we get $x \wedge a = 0$. Suppose $a \in P$. Since P is prime and $b \in P$, we get $1 = a \vee b \in P$ which is a contradiction. Thus $a \notin P$. Hence $x \wedge a = 0$ for some $a \notin P$. Therefore P is a minimal prime ideal in L.

(5) $\Rightarrow$ (1): Assume that every prime ideal is minimal. Let $a, b \in L$ and consider $a \in [0, b]$.

Suppose $b \notin (a] \vee (a)^*$. Then there exists a prime ideal P of L such that $(a] \vee (a)^* \subseteq P$. Thus $a \in P$ and $(a)^* \subseteq P$. Since P is minimal, we get $a \notin P$ which is a contradiction. Hence $b \in (a] \vee (a)^*$. Then $b = s \vee t$ for some $s \in (a]$ and $t \in (a)^*$. Since $t \in (a)^*$, we get $a \wedge t = 0$. Hence $a = a \wedge b = a \wedge (s \vee t) = (a \wedge s) \vee (a \wedge t) = (a \wedge s) \vee 0 = a \wedge s = s$ because of $s \in (a]$. Hence $b = s \vee t = a \vee t$. Also, $t \leq a \vee t = b$. Hence $t \in [0, b]$. Thus t is the complement of a in $[0, b]$. Therefore L is relatively complemented. □

4. Congruences and σ-ideals

In this section, a one-to-one correspondence between the set of all prime σ-ideals of a lattice L and the set of all prime σ-ideals of the corresponding quotient lattice with respect Glivenko type congruence.

Proposition 4.1. *Let L be a lattice and $x, y \in L$. Define a binary relation ψ on L by*

$$(x, y) \in \psi \text{ if and only if } (x)^* = (y)^*.$$

Then ψ is a congruence on L with $Ker\ \psi$ as the smallest congruence class modulo ψ.

For any distributive lattice L, it can be shown that the quotient algebra $L_{/\psi}$ is also a distributive lattice with respect to the following operations:

$$[x]_\psi \wedge [y]_\psi = [x \wedge y]_\psi \text{ and } [x]_\psi \vee [y]_\psi = [x \vee y]_\psi$$

where $[x]_\psi$ is the congruence class of x modulo ψ. It can be routinely verified that the mapping $\Psi : L \longrightarrow L_{/\psi}$ defined by $\Psi(x) = [x]_\psi$ is a homomorphism. For any $x, y \in L$, it is clear that $x \leq y$ implies $[x]_\psi \subseteq [y]_\psi$. Hence $\left(L_{/\psi}, \cap, \vee\right)$ is a complete lattice in which $[0]_\psi$ is the smallest element and $[1]_\psi$ is the greatest element.

Example 4.2. Consider the distributive lattice $L = \{0, a, b, c, 1\}$ given in Example 3.2. Then the congruence ψ indicates the relation $\{(c, 1)\}$ because of $(c)^* = (1)^*$. Hence $[c]_\psi = [1]_\psi = \{1, c\}$ and $[a]_\psi = [b]_\psi = [0]_\psi = \emptyset$. Therefore $L_{/\psi} = \{[0]_\psi, [a]_\psi, [b]_\psi, [c]_\psi, [1]_\psi\} = \{\emptyset, (c, 1)\}$ which is a distributive lattice particularly a totally ordered set.

Lemma 4.3. *Let L be a lattice and ψ be the congruence on L. For any $x \in L$,*

(1) *$[x]_\psi = [0]_\psi$ if and only if $x = 0$,*
(2) *$[x]_\psi = [1]_\psi$ if and only if x is dense.*

Proof. Routine verification. □

Definition 4.4. For any ideal I of a lattice L, define $\bar{I} = \{[x]_\psi \in L_{/\psi} \mid x \in I\}$.

Theorem 4.5. *Let I be an α-ideal of a lattice L. Then the following properties hold:*

(1) *$x \in I$ if and only if $[x]_\psi \in \bar{I}$,*

(2) *if I is an ideal of L, then $\bar{I}$ is an ideal of $L_{/\psi}$,*

(3) *if I is a prime ideal of L, then $\bar{I}$ is a prime ideal of $L_{/\psi}$.*

Proof. (1) Let $x \in I$. Then clearly $[x]_\psi \in \bar{I}$. Conversely, let $[x]_\psi \in \bar{I}$. Then, we get $[x]_\psi = [y]_\psi$ for some $y \in I$. Hence $(x, y) \in \psi$. Thus $(x)^* = (y)^*$ for some $y \in I$. Since $y \in I$ and I is an α-ideal of L, we get $x \in I$.

(2) Let I be an ideal of L. Clearly $[0]_\psi \in \bar{I}$. Let $[x]_\psi, [y]_\psi \in \bar{I}$. By (1), we get $x \in I$ and $y \in I$. Since I is an ideal of L, we get $x \vee y \in I$. Hence $[x]_\psi \vee [y]_\psi = [x \vee y]_\psi \in \bar{I}$. Again, let $[x]_\psi \in \bar{I}$ and $[y]_\psi \in L_{/\psi}$. Then $x \in I$ and $y \in L$. Since I is an ideal, we get $x \wedge y \in I$. Hence $[x]_\psi \wedge [y]_\psi = [x \wedge y]_\psi \in \bar{I}$. Therefore $\bar{I}$ is an ideal of $L_{/\psi}$.

(3) Since I is an ideal of L, by (2), $\bar{I}$ is an ideal of $L_{/\psi}$. Since I is a proper ideal of L, by (1), we get that $\bar{I}$ is a proper ideal in $L_{/\psi}$. Let $[x]_\psi, [y]_\psi \in L_{/\psi}$. Then

$$\begin{aligned} [x]_\psi \wedge [y]_\psi \in \bar{I} &\Rightarrow [x \wedge y]_\psi \in \bar{I} \\ &\Rightarrow x \wedge y \in I \qquad \text{from (1)} \\ &\Rightarrow x \in I \text{ or } y \in I \\ &\Rightarrow [x]_\psi \in \bar{I} \text{ or } [y]_\psi \in \bar{I} \end{aligned}$$

Therefore $\bar{I}$ is a prime ideal in $L_{/\psi}$. □

In the above theorem, the condition for I being an α-ideal is necessary to prove condition (1) which plays a significance role in the later text. For, consider the ideal $I = \{0, a, b, c\}$ of the distributive lattice given in Example 3.2. Clearly, it is not an α-ideal. We have clearly $(c)^* = (1)^* = \{0\}$. Hence $[1]_\psi = [c]_\psi \in \bar{I}$ but $1 \notin I$.

Corollary 4.6. *Let I and J be two α-ideals of a lattice of L. Then $I \subseteq J$ if and only if $\bar{I} \subseteq \bar{J}$.*

Proof. From Theorem 4.5(1), it is clear. □

Definition 4.7. Let L be a lattice and $a \in L$. Define

$$(a)^\Delta = \big\{[x]_\psi \in L_{/\psi} \mid [a]_\psi \wedge [x]_\psi = [0]_\psi\big\}.$$

From Lemma 4.3(1), it is clear that $(a)^\Delta = \big\{[x]_\psi \in L_{/\psi} \mid a \wedge x = 0\big\}$. Obviously $[0]_\psi \in (a)^\Delta$. It can be routinely verified that $(a)^\Delta$ is an ideal of $L_{/\psi}$. Furthermore, we note that $(0)^\Delta = L_{/\psi}$ and $(1)^\Delta = [0]_\psi$.

Lemma 4.8. *Let L be a lattice. For any $x, y \in L$, we have*

(1) $y \in (x)^*$ *if and only if* $[y]_\psi \in (x)^\Delta$,

(2) $(x)^* = (y)^*$ *if and only if* $(x)^\Delta = (y)^\Delta$,

(3) $[x]_\psi \subseteq [y]_\psi$ *implies* $(y)^\Delta \subseteq (x)^\Delta$,

(4) $(x)^\Delta \cap (y)^\Delta = (x \vee y)^\Delta$.

Proof. Routine verification. □

Definition 4.9. For any ideal I of $L_{/\psi}$, define $\sigma(I) = \{[x]_\psi \in L_{/\psi} \mid (x)^\Delta \vee I = L_{/\psi}\}$, where $(x)^\Delta \vee I$ is the supremum of the ideals $(x)^\Delta$ and I in $L_{/\psi}$.

Lemma 4.10. *Let L be a lattice. For any ideal I of the lattice $L_{/\psi}$, we have*

(1) $\sigma(I) \subseteq I$,

(2) $\sigma(I)$ *is an ideal of* $L_{/\psi}$.

Proof. Routine verification. □

Theorem 4.11. *Let I be an ideal of a lattice L and ψ be the congruence defined on L. Then the following properties hold:*

(1) *if I is an α-ideal of L, then $\bar{I}$ is an α-ideal of $L_{/\psi}$,*

(2) *if I is a σ-ideal of L, then $\bar{I}$ is a σ-ideal of $L_{/\psi}$.*

Proof. (1) Let I be an α-ideal of L. Clearly $\bar{I}$ is an ideal of $L_{/\psi}$. Let $x, y \in L$ be such that $(x)^\Delta = (y)^\Delta$ and $[x]_\psi \in \bar{I}$. Hence $(x)^* = (y)^*$. Since I is an α-ideal, by Theorem 4.5(1), we get $x \in I$ and thus $y \in I$. Hence $[y]_\psi \in \bar{I}$. Therefore $\bar{I}$ is an α-ideal of $L_{/\psi}$.
(2) Suppose that I is a σ-ideal of L. Then I is an α-ideal of L. By Theorem 4.5(2), $\bar{I}$ is an ideal of $L_{/\psi}$. Clearly $\sigma(\bar{I}) \subseteq \bar{I}$. Let $[x]_\psi \in \bar{I}$. Since I is an α-ideal, by Theorem 4.5(1), we get $x \in I = \sigma(I)$. Hence $(x)^* \vee I = L$. Let $[a]_\psi \in L_{/\psi}$ be arbitrary. For this $a \in L$, we get $a = b \vee c$ for some $b \in (x)^*$ and $c \in I$. Since $c \in I$, we get $[c]_\psi \in \bar{I}$. Since $b \in (x)^*$, we get $[b]_\psi \in (x)^\Delta$. Hence $[a]_\psi = [b \vee c]_\psi = [b]_\psi \vee [c]_\psi \in (x)^\Delta \vee \bar{I}$. Hence $L_{/\psi} \subseteq (x)^\Delta \vee \bar{I}$. Therefore $\bar{I}$ is a σ-ideal of $L_{/\psi}$. □

Theorem 4.12. *Let L be a lattice and ψ be the congruence defined on L. Then the mapping $I \mapsto \bar{I}$ is an order isomorphism of the set of all prime α-ideals of L onto the set of all prime α-ideals of $L_{/\psi}$.*

Proof. Let I and J be two prime α-ideals of L. Then by Theorem 4.5(1), we get that $I \subseteq J \Leftrightarrow \bar{I} \subseteq \bar{J}$. Let I be a prime α-ideal of L. Then by Theorem 4.5(3) and Theorem 4.11(1), we get that $\bar{I}$ is a prime α-ideal of $L_{/\psi}$. Let R be a prime α-ideal of $L_{/\psi}$. Consider $I = \{x \in L \mid [x]_\psi \in R\}$. Since R is an ideal of $L_{/\psi}$, we get that I is an ideal of L. Let $x, y \in L$ be such that $(x)^* = (y)^*$. Suppose $x \in I$. Then $[x]_\psi \in R$. Since $(x)^* = (y)^*$, we get $(x)^\Delta = (y)^\Delta$. Since R is an α-ideal of $L_{/\psi}$, we get $[y]_\psi \in R$. Hence $y \in I$. Therefore I is an α-ideal of L. Clearly $\bar{I} = R$.

It is enough to derive the order isomorphism between primeness of ideals. Let $x, y \in L$ be such that $x \wedge y \in I$. Then $[x]_\psi \wedge [y]_\psi = [x \vee y]_\psi \in R$. Since R is prime, we get either $[x]_\psi \in R$ or $[y]_\psi \in R$. Hence either $x \in I$ or $y \in I$. Therefore I is a prime ideal of L. □

Lemma 4.13. *The following properties hold in a lattice L:*

(1) *A proper σ-ideal contains no dense element,*

(2) *A non-zero σ-ideal is non-dense,*

(3) *A non-dense prime ideal is an α-ideal.*

Proof. (1) Let I be a proper σ-ideal of L. Suppose I contains a dense element, say d. Then $d \in \sigma(I)$. Hence $L = (d)^* \vee I = \{0\} \vee I = I$, which is a contradiction.

(2) Let $I \neq \{0\}$ be a σ-ideal of L. Suppose $I^* = \{0\}$. Let $0 \neq x \in I = \sigma(I)$. Then $(x)^* \vee I = L$. Hence $(x)^{**} \cap I^* = L^* = \{0\}$. Thus $(x)^{**} = \{0\}$, which gives $(x)^* = L$. Hence $x = 0$, which is a contradiction. Therefore I is non-dense.

(3) Let P be a non-dense prime ideal. Then there exists $0 \neq x \in L$ such that $P = (x)^*$. If $t \in P = (x)^*$. Then $(t)^{**} \subseteq (x)^* = P$. Therefore P is an α-ideal of L. □

Theorem 4.14. *Let L be a lattice and ψ be the congruence defined on L. If every prime ideal of L is non-dense, then the mapping $I \mapsto \overline{I}$ is an order isomorphism of the set of all prime σ-ideals of L onto the set of all prime σ-ideals of $L_{/\psi}$.*

Proof. Let I be a prime σ-ideal of L. Then by Theorem 4.5(3) and Theorem 4.11(2), we get that $\bar{I}$ is a prime σ-ideal of $L_{/\psi}$ and hence a prime α-ideal of $L_{/\psi}$.

Let R be a prime σ-ideal of $L_{/\psi}$. Hence R is a prime α-ideal of $L_{/\psi}$. Consider $I = \{x \in L \mid [x]_\psi \in R\}$. Since R is a prime α-ideal of $L_{/\psi}$, by Theorem 4.12, I is a prime α-ideal of L. Clearly $\bar{I} = R$. We always have $\sigma(I) \subseteq I$. Conversely, let $x \in I$. Then $[x]_\psi \in \bar{I} = R$. Since R is a σ-ideal of $L_{/\psi}$, $(x)^\Delta \vee R = L_{/\psi}$. Suppose $(x)^* \vee I \neq L$. Then there exists a prime ideal P of L such that $(x)^* \vee I \subseteq P$. By the hypothesis, P is non-dense. By Lemma 4.13(3), P is an

α-ideal of L. By Theorem 4.5(3), $\bar{P}$ is a prime ideal of $L_{/\psi}$. Now

$$\begin{aligned}(x)^* \vee I \subseteq P &\Rightarrow (x)^* \subseteq P \text{ and } I \subseteq P\\ &\Rightarrow (x)^{\Delta} \subseteq \bar{P} \text{ and } \bar{I} \subseteq \bar{P}\\ &\Rightarrow (x)^{\Delta} \subseteq \bar{P} \text{ and } R \subseteq \bar{P}\\ &\Rightarrow L_{/\psi} = (x)^{\Delta} \vee R \subseteq \bar{P}\end{aligned}$$

which is a contradiction to that $\bar{P}$ is proper in $L_{/\psi}$. Hence $(x)^* \vee I = L$ and thus $x \in \sigma(I)$. Therefore I is a prime σ-ideal of L. □

The following corollaries are direct consequence of the above theorems.

Corollary 4.15. *Let L be a lattice and ψ be the congruence defined on L. Then the mapping $P \mapsto \bar{P}$ induces a one-to-one correspondence between the set of all prime α-ideals of L and the set of all prime α-ideals of $L_{/\psi}$.*

Corollary 4.16. *Let L be a lattice and ψ be the congruence defined on L. If every prime ideal of L is non-dense, then the mapping $P \mapsto \bar{P}$ induces a one-to-one correspondence between the set of all prime σ-ideals of L and the set of all prime σ-ideals of $L_{/\psi}$.*

Theorem 4.17. *Let ψ be the congruence on the lattice L in which every prime ideal is non-dense. Every α-ideal of L is a σ-ideal if and only if every α-ideal of $L_{/\psi}$ is a σ-ideal.*

Proof. Assume that every α-ideal of L is a σ-ideal. Let R be a α-ideal of $L_{/\psi}$. By Theorem 4.12, there exists an α-ideal P of L such that $\bar{P} = R$. By the assumption, P is a σ-ideal of L. By Theorem 4.11(2), $\bar{P} = R$ is a σ-ideal of $L_{/\psi}$.

Conversely, assume that every α-ideal of $L_{/\psi}$ is a σ-ideal. Let I be an α-ideal of L. Then by Theorem 4.11(1), $\bar{I}$ is an α-ideal of $L_{/\psi_I}$. By the assumption, $\bar{I}$ is a σ-ideal in $L_{/\psi_I}$. Clearly $\sigma(I) \subseteq I$. Let $x \in I$. Then $[x]_\psi \in \bar{I}$. Suppose $(x)^* \vee I \neq L$. Then there exists a prime ideal P of L such that $(x)^* \vee I \subseteq P$. By the hypothesis, P is non-dense and hence P is an α-ideal of L. By Theorem 4.5(3), $\bar{P}$ is a prime ideal of $L_{/\psi}$. Now

$$\begin{aligned}(x)^* \vee I \subseteq P &\Rightarrow (x)^* \subseteq P \text{ and } I \subseteq P\\ &\Rightarrow (x)^{\Delta} \subseteq \bar{P} \text{ and } \bar{I} \subseteq \bar{P}\\ &\Rightarrow L_{/\psi} = (x)^{\Delta} \vee \bar{I} \subseteq \bar{P}\end{aligned}$$

which is a contradiction. Hence $(x)^* \vee I = L$. Therefore I is a σ-ideal of L. □

The condition of possessing non-dense prime ideals by a lattice is essential. For, consider the distributive lattice given in Example 3.2. Clearly L is possessing three prime ideals $P_1 = \{0, a\}$, $P_2 = \{0, b\}$, and $P_3 = \{0, a, b, c\}$ in which P_3 is dense. It can also see that the α-ideals P_1 and P_2 are not σ-ideals. Note that L is not a generalized Stone lattice. By Theorem 3.3, the following corollary is an immediate consequence.

Corollary 4.18. *Let L be a lattice in which every prime ideal is non-dense. Then L is a generalized Stone lattice if and only if $L_{/\psi}$ is a generalized Stone lattice.*

5. Conclusion

In this article, as an extension of the results of W.H. Cornish, further significant properties of σ-ideals are investigated. Using these observations, characterizations of the algebraic structures like generalized Stone lattices, Boolean algebras, and relatively complemented lattices are given. In association with a Glivenko type congruences, some properties of σ-ideals are investigated. A one-to-one correspondence is obtained between the set of all prime σ-ideals of distributive lattice and the set of all prime σ-ideals of the corresponding quotient lattice generated from this congruence. Finally, the class of generalized Stone lattices is characterized with help of these congruences.

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Mukkamala Sambasiva Rao

Department of Mathematics,

MVGR College of Engineering Chintalavalasa, Vizianagaram,

Andhra Pradesh India-535005.

mssraomaths35@rediffmail.com