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Research Paper

# LAPLACIAN SPECTRAL CHARACTERIZATION OF SETOSA GRAPHS

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Abstract. A setosa graph $SG(e, f, g, h, d; b_1, b_2, \dots, b_s)$ is a graph consisting of five cycles and $s(\geq 1)$ paths $P_{b_1+1}, P_{b_2+1}, \dots, P_{b_s+1}$ intersecting in a single vertex that all meet in one vertex, where $b_i \geq 1$ (for $i = 1, \dots, s$) and $e, f, g, h, d \geq 3$ denote the length of the cycles $C_e$, $C_f$, $C_g$, $C_h$ and $C_d$, respectively. Two graphs $G$ and $H$ are *L-cospectral* if they have the same Laplacian spectrum. A graph $G$ is said to be determined by the spectrum of its Laplacian matrix (DLS, for short) if every graph with the same Laplacian spectrum is isomorphic to $G$. In this paper we prove that if $H$ is a $L$-cospectral graph with a setosa graph $G$, then $H$ is also a setosa graph and the degree sequence of $G$ and $H$ are the same. We conjecture that all setosa graphs are DLS.

## 1. Introduction

In this paper all graphs are simple. Let $G = (V, E)$ be a graph. By $n(G)$ and $m(G)$ we mean the number of vertices and the number of edges of $G$, respectively. The *order* of $G$ denotes the

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number of vertices of $G$. For two graphs $G$ and $H$, $G+H$ (or $G \cup H$) denote the disjoint union of $G$ and $H$. Especially the *disjoint union* of $r$ copies of $G$ are denoted by $rG$. The *join* of graphs $G$ and $H$, $G \bigtriangledown H$, is the graph that is obtained from $G+H$ by joining every vertex of $G$ to every vertex of $H$. In spectral graph theory they are some well-known associated matrices to graphs such as adjacency matrix , Laplacian matrix and signless Laplacian matrix. The Laplacian matrix of $G$ denoted by $L(G)$ is $A(G) - D(G)$, where $A(G)$ is the adjacency matrix and $D(G)$ is the diagonal matrix $diag(r_1, r_2, \ldots, r_n)$ in which $r_i$ is the degree of the vertex $v_i$ and $\{v_1, \ldots, v_n\}$ is the vertex set of $G$. As usual we let $d_1 \geq d_2 \geq \cdots \geq d_n$ as the degree sequence of the vertices of $G$. The *Laplacian spectrum* or $L$*-spectrum* of $G$ is the multi-set of the eigenvalues of $L(G)$ usually written in non-increasing order $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$.

One of the important sections of spectral graph theory is devoted to determining whether given graphs or classes of graphs are determined by their spectra or not. Thus finding and constructing every class of graphs that are determined by their spectra is interesting and attractive. As usual we say that two graphs $G$ and $H$ are $L$*-cospectral* if they have the same $L$-spectrum. In addition, a graph $G$ is called *determined by its Laplacian spectrum* (DLS), if no other graphs are $L$-cospectral with $G$. In [17] van Dam and Haemers conjectured that almost all graphs are determined by their Laplacian spectrum, that is, they are the only graph (up to isomorphism) with that spectrum. However, very few graphs are known to have that property, and thus finding new classes of such graphs is a nice problem. Due to some applications (for instance in randomized algorithms and machine learning) finding graphs that are determined by their spectrum have received much more attention. See [1]-[18] and the references therein.

A setosa graph $SG(e, f, g, h, d; b_1, b_2, \cdots, b_s)$ is a graph consisting of five cycles and $s(\geq 1)$ paths $P_{b_1+1}, P_{b_2+1}, \ldots, P_{b_s+1}$ intersecting in a single vertex that all meet in one vertex, where $b_i \geq 1$ (for $i = 1, \ldots, s$) and $e, f, g, h, d \geq 3$ denote the length of the cycles $C_e$, $C_f$, $C_g$, $C_h$ and $C_d$, respectively. In this paper we study the L-cospectral graphs with setosa graphs. We show that if $H$ is a $L$-cospectral graph with a setosa graph $G$, then $H$ is also a setosa graph and the degree sequence of $G$ and $H$ are the same. Finally we conjecture that all setosa graphs are DLS.

## 2. Known Results

In this section we recall some known results.

**Theorem 2.1.** [12, 17, 18] *Using the Laplacian spectrum of a graph one can obtain the following:*

(1) *the number of vertices,*
(2) *the number of edges,*
(3) *the number of spanning trees,*

(4) *the number of connected components,*

(5) *the summation of the squares of the degrees of the vertices.*

**Theorem 2.2.** [8] *Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$ and $\overline{\mu}_1 \geq \overline{\mu}_2 \geq \cdots \geq \overline{\mu}_n = 0$ be the Laplacian spectra of the graphs $G$ and $\overline{G}$ (the complement of $G$), respectively. Then for $i = 1, 2, \ldots, n-1$, $\overline{\mu}_i = n - \mu_{n-i}$.*

Let $G$ and $H$ be two graphs. By $N_G(H)$ we mean the number of subgraphs of $G$ that are isomorphic to $H$. Further, let $W_G(i)$ be the number of closed walks of length $i$ in $G$ and $W'_H(i)$ be the number of closed walks of length $i$ in $H$ that cover all edges of $H$. Hence $W_G(i) = \sum N_G(H)W'_H(i)$, where the summation is taken over all connected subgraphs $H$ of $G$ for which $W'_H(i) \neq 0$. Using this equality one can obtain some facts about the number of closed walks of a graph.

**Theorem 2.3.** [16] *Let $G$ be a graph. Then the number of closed walks of lengths 2, 3, and 4 in $G$ can be computed by the following formulas, respectively:*

(1) $W_G(2) = 2m$,

(2) $W_G(3) = \mathrm{tr}(A^3(G)) = 6N_G(C_3)$,

(3) $W_G(4) = 2m + 4N_G(P_3) + 8N_G(C_4)$, *where $m$ is the number of edges of $G$ and $tr(B)$ is the trace of a matrix $B$.*

For a vertex $v$ of a graph $G$ by $m(v)$ me mean the average of the degrees of the vertices adjacent to $v$.

**Theorem 2.4.** [10, 12] *If $G$ is a connected graph, then $\mu_1(G) \leq \max_v(\deg(v) + m(v))$. Moreover, equality holds if and only if $G$ is a regular or a semi-regular bipartite graph.*

**Theorem 2.5.** [6, 10] *If $G$ is a nontrivial graph, then $\mu_1(G) \leq d_1 + d_2$; and if $G$ is connected, then $\mu_2(G) \geq d_2(G)$.*

**Theorem 2.6.** [15] *Let $G$ be a graph with $m$ edges. Then the first four coefficients in the Laplacian characteristic polynomial of $G$, $\varphi(G) = \Sigma l_i x^i$ are*

$$l_0 = 1,\ l_1 = -2m,\ l_2 = 2m^2 - m - \frac{1}{2}\sum_{i=1}^{n} d_i^2,$$

*and*

$$l_3 = \frac{1}{3}(-4m^3 + 6m^2 + 3m\sum_{i=1}^{n} d_i^2 - \sum_{i=1}^{n} d_i^3 - 3\sum_{i=1}^{n} d_i^2 + 6N_G(C_3)).$$

The following result is an immediate consequence of Theorem 2.6.

**Corollary 2.7.** *If $G$ and $H$ are L-cospectral graphs with the same degrees, then they have the same number of triangles.*

It follows from Theorems 2.1 and 2.6 that if $G$ and $G'$ are $L$-cospectral graphs with degrees $d_1, d_2, \ldots, d_n$ and $d'_1, d'_2, \ldots, d'_n$ respectively, then

$$\operatorname{tr}(A^3(G)) - \sum_{i=1}^{n} d_i^3 = \operatorname{tr}(A^3(G')) - \sum_{i=1}^{n} d_i'^3.$$

Based on this, Liu and Huang [11] defined the following graph invariant for a graph $G$:

$$\varepsilon(G) = \operatorname{tr}(A^3(G)) - \sum_{i=1}^{n} (d_i - 2)^3.$$

**Theorem 2.8.** [11] *If $G$ and $H$ are $L$-cospectral, then $\varepsilon(G) = \varepsilon(H)$.*

**Theorem 2.9.** [7] *If $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ are the eigenvalues of a symmetric $n \times n$ matrix $M$, and if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1}$ are the eigenvalues of a principal submatrix of $M$, then $\mu_1 \geq \lambda_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \lambda_{n-1} \geq \mu_n$.*

Turning to the degrees of the vertices in graphs, as before, let $d_i$ denote the degree of vertex $v_i$ in a graph $G$, and assume that $d_1 \geq d_2 \geq \cdots \geq d_n$. By $deg(G)$ we mean the degree sequence of $G$ in decreasing order. In addition, the eigenvalues of $G$ are $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$.

**Theorem 2.10.** [9] *If $G$ is a graph with at least one edge, then $\mu_1 \geq d_1 + 1$. Moreover, if $G$ is connected, then the equality holds if and only if $d_1 = n - 1$.*

## 3. L-cospectral graphs with setosa graphs

In this section we study the L-cospectral graphs with setosa graphs. First we find some bounds for the first and the second Laplacian eigenvalue of setosa graphs.

**Lemma 3.1.** *Let $\Gamma$ be the setosa graph $SG(e, f, g, h, d; b_1, b_2, \cdots, b_s)$. Then*

- *(i)* $s + 11 \leq \mu_1(\Gamma) \leq s + 12$.
- *(ii)* $\mu_2(\Gamma) < 4$.

*Proof.* *(i)* Since $d_1 = s + 10$, by Theorems 2.4 and 2.10 we find that, $s + 11 \leq \mu_1(\Gamma) \leq s + 10 + \dfrac{2s + 10(2)}{s + 10} = s + 12$.

*(ii)* Let $n$ be the order of $\Gamma$. Suppose that $v$ is the vertex with maximum degree in $\Gamma$, and let $M_v$ be the $(n-1) \times (n-1)$ principal submatrix of $L(\Gamma)$ formed by removing the row and column corresponding to $v$. Since $M_v$ contains negative entries, we consider $|M_v|$ which is obtained by taking the absolute value of the entries of $M_v$. Now $M_v$ is reducible, but it has $s + 5$ irreducible submatrices that correspond to the components of $\Gamma \setminus v$. On the other hand, each of these components has spectral radius (largest Laplacian eigenvalue) strictly less than 4. So one can conclude that the largest eigenvalue of $|M_v|$ is less than 4, and so is that of $M_v$. By Theorem 2.9, $\mu_2(\Gamma) < 4$, as desired.

The proof is complete. □

Let $G$ be a connected graph with $n$ vertices and $m$ edges. If $m = n+k-1$, then $\Gamma$ is called a *$k$-cyclic graph*. Consider the setosa graph $\Gamma = SG(e, f, g, h, d; b_1, b_2, \cdots, b_s)$. Thus the number of vertices and the number of edges of $\Gamma$ are $n = e + f + g + h + d + b_1 + \cdots + b_s - 4$ and $m = e + f + g + h + d + b_1 + \cdots + b_s$, respectively. Therefore $m = n + 4$, which means that $\Gamma$ is a 5-cyclic graph.

**Theorem 3.2.** *If $H$ is a $L$-cospectral graph with $\Gamma = SG(e, f, g, h, d; b_1, b_2, \ldots, b_s)$, then $deg(\Gamma) = deg(H)$.*

*Proof.* It follows from Lemma 3.1 that $\mu_2(H) < 4$, and so by Theorem 2.5 that $d_2(H) \leq 3$. Since $H$ and $\Gamma$ are $L$-cospectral, by Theorem 2.1, $H$ is also connected. Let $n_i$ denote the number of vertices of degree $i$ in $H$, for $i = 1, 2, \ldots, d_1(H)$. Using Theorem 2.1 we find the following:

$$\sum_{i=1}^{d_1(H)} n_i = n(\Gamma), \tag{1}$$

$$\sum_{i=1}^{d_1(H)} i n_i = 2m(\Gamma), \tag{2}$$

$$\sum_{i=1}^{d_1(H)} i^2 n_i = n_1^{'} + 4n_2^{'} + d_1^2(\Gamma), \tag{3}$$

where $n_i^{'}$ denote the number of vertices of degree $i$ $(i = 1, 2)$ belonging to $\Gamma$.

Let $n = n(\Gamma)$. Clearly, $m(\Gamma) = n+4$, $n_1^{'} = s$, $n_2^{'} = n - s - 1$ and $d_1(\Gamma) = s+10$. By adding up Eqs. (1), (2) and (3) with coefficients 2, $-3$, 1, respectively we get:

$$\sum_{i=1}^{d_1(H)} (i^2 - 3i + 2)n_i = (s + 8)(s + 9). \tag{4}$$

By Lemma 3.1, $s + 11 \leq \mu_1(\Gamma) \leq s + 12$. It follows from Theorem 2.10 that $d_1(H) + 1 \leq \mu_1(H) = \mu_1(\Gamma) \leq s+12$. Therefore, $d_1(H) \leq s+11$. In addition, by Lemma 3.1 and Theorem 2.5, $s + 11 \leq \mu_1(\Gamma) = \mu_1(H) \leq d_1(H) + d_2(H) \leq d_1(H) + 3$, which yields $d_1(H) \geq s + 8$. Therefore, we have $s+8 \leq d_1(H) \leq s+11$. Suppose that $\Gamma$ has $0 \leq t \leq 5$ triangles. It follows from Theorem 2.8 that

$$6N_H(C_3) - \sum_{i=1}^{n} (d_i(H) - 2)^3 = 6t - \big((s + 8)^3 - s\big). \tag{5}$$

Hence

$$N_H(C_3)=\frac{6t-\big((s+8)^3-s\big)+\sum_{i=1}^{n}(d_i(H)-2)^3}{6}. \tag{6}$$

Consider the following cases:

(*i*) $d_1(H)=s+11$. Since $d_1(H)=s+11>3\geq d_2(H)$, $n_{s+11}=1$. By (4) we get

$$(s+11)^2-3(s+11)+2+2n_3=(s+8)(s+9). \tag{7}$$

This implies that $n_3+s=-9<0$, a contradiction.

(*ii*) $d_1(H)=s+10$. Thus $n_{s+10}=1$, since $d_1(H)=s+10>3\geq d_2(H)$. By (4) we obtain that:

$$(s+10)^2-3(s+10)+2+2n_3=(s+8)(s+9). \tag{8}$$

This shows that $n_3=0$. By (1) and (2) we obtain that $n_1=s$ and $n_2=n-1-s$. Therefore the degree sequence of $\Gamma$ and $H$ are the same. In this case, by (6) we have $N_H(C_3)=t$.

(*iii*) $d_1(H)=s+9$. Similar to the previous cases we obtain that $n_{s+9}=1$. Using (4) we find that:

$$(s+9)^2-3(s+9)+2+2n_3=(s+8)(s+9). \tag{9}$$

Hence $n_3=s+8$. By (1) and (2) one can find that $n_1=2s+7$ and $n_2=-3s+n-16$. It follows from (6) that

$$N_H(C_3)=t-\frac{(s+7)(s+8)}{2}. \tag{10}$$

This means that $N_H(C_3)<0$ (since $t\leq 5$), a contradiction.

(*iv*) $d_1(H)=s+8$. Similar to the above cases we obtain that $n_{s+8}=1$. By (4) we conclude that:

$$(s+8)^2-3(s+8)+2+2n_3=(s+8)(s+9), \tag{11}$$

Therefore $n_3=2s+15$. It follows from (1) and (2) that $n_1=3s+13$ and $n_2=-5s+n-29$. Consequently, by (6) we get:

$$N_H(C_3)=t-(s+7)^2. \tag{12}$$

This implies that $N_H(C_3)<0$, a contradiction.

The proof is complete. □

Now we prove one of the main results of this paper.

**Theorem 3.3.** *Let* $\Gamma$ *be the setosa graph* $SG(e, f, g, h, d; b_1, b_2, \ldots, b_s)$*. If* $H$ *is a* $L$*-cospectral graph with* $\Gamma$*, then* $H$ *is a setosa graph.*

*Proof.* Assume that $H$ is $L$-cospectral with $\Gamma$. By Theorem 2.1, $H$ is connected. Using Theorem 3.2 we find that $H$ has exactly one vertex, say $v$, of degree greater than two. More precisely, $deg_H(v) = s + 10$ and the other vertices of $H$ have degree one or two. Thus the degree of every vertex of $H \setminus v$ is at most two. Hence every connected component of $H \setminus v$ is path or cycle. In fact $H \setminus v$ has no cycle as a connected component (because if it does, then since $H$ is connected, $H$ shoud has another vertex except $v$ of degree grater than two, a contradiction). We note that the neighbors of $v$ (in $H$) belong to the set of vertices of degree at most one in $H \setminus v$. Therefore $H$ is a graph consisting of $p$ cycles and $q$ paths intersecting in a single vertex that all meet in vertex $v$. Thus $deg_H(v) = 2p + q$ and the number of pendant vertices of $H$ is $q$. By Theorem 3.2, $q = s$ and $deg_H(v) = 2p + q = s + 10$. Hence $p = 5$ and so $H$ is a setosa graph. □

We finish the paper by the following conjecture.

**Conjecture 3.4.** *All setosa graphs are DLS.*

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